На основі властивостей матриць толерантності і ядер булевих функцій встановлено критерій реалізованості функцій алгебри логіки одним узагальненим нейронним елементом відносно довільної системи характевів. Отримано ряд необхідних та достатніх умов реалізованості булевих функцій одним узагальненим нейронним елементом і на основі достатніх умов розроблено ефективний алгоритм синтезу цілочислових узагальнених нейронних елементів з великим числом входів

Ключові слова: матриця толерантності, ядро булевої функції, характер групи,спектр булевої функції

На основании матриц толерантности и ядер булевых функций установлен критерий реализуемости функций алгебры логики на одном обобщенном нейронном элементе относительно произвольной системы характеров. Получен ряд необходимых и достаточных условий реализуемости булевых функций на одном обобщенном нейронном элементе, и на основании достаточных условий разработан эффективный алгоритм синтеза целочисленных обобщенных нейронных элементов с большим числом входов

Ключевые слова: матрица толерантности, ядро булевой функции, характер группы, спектр булевой функции

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SYNTHESIS OF GENERALIZED NEURAL ELEMENTS BY MEANS OF THE TOLERANCE MATRICES

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1. Introduction

Intensification of theoretical development and practical applications has been observed recently in the field of information technology and neurocomputers. This is due to the increased interest in information systems and neurolike structures that have found wide application in encription, protection of information, image recognition, forecasting and other fields of human activities.

Solving complex applied problems using neuromorphic structures will become more effective when generalized neural elements (GNE) (which by their functional capabilities exceed classical neural elements) with threshold functions of activation will be used as basic elements. Therefore, information processing in the neurobase will be more effective on condition that generalized neural elements are used. To this end, it is necessary to devise practically suitable methods for the synthesis of neural elements with generalized threshold functions of activation and synthesis of logic circuits from them.

Relevance and practical value of development of new methods for the synthesis of generalized neural elements are evidenced by an increasing volume of investments in software and hardware for artificial intelligence. It should be mentioned that an extremely important requirement to the new methods of synthesis of generalized neural elements is that these methods should be practically suitable for synthesizing GNE with a large number of inputs. This is explained by the fact that the volume of information and the degree of complexity of the tasks that are solved in the neurobase are constantly growing. That is why the studies giving results applicable in synthesizing generalized neural elements with a large number of inputs appear to be topical. The application of generalized neural elements can reduce the number of artificial neurons in the neural networks employed for the tasks on recognition, compression and encoding of discrete signals and images.

2. Literature review and problem statement

Currently, neurolike structures are increasingly used to solve varied applied problems. An indication of this is the increase in the number of scientific publications and new methods of training (synthesis) of neural networks used in various spheres of human activities. Development of new methods for data processing in the neurobase is a relevant and practically important task. For example, [1] introduces the concept of the operational base of neural networks and shows its application in the development of effective data processing methods. Possibility of using artificial real-time neural networks in the problems on digital signal processing is considered in article [2], while paper [3] investigates the feasibility of employment of neuromorphic structures for solving prediction problems in the field of intelligent data analysis.

The field of practical applications of neural network models is vast. These models are effectively used to improve resolution of images based on artificial neural networks [4], for segmentation [5], classification and pattern recognition [6, 7]. On the basis of neural networks, intelligent blocks of various systems for controlling chemical processes [8] and for the classification of diseases [9] are developed. These models are successfully used in diagnostics [10], economic [11] and biological process [12] forecasting and morbidity prediction for the diseases under study [13]. As the studies show, neural network methods are widely applied for the compression of discrete signals and images [14–16] and in the banking sector for credit risk assessment [17].

It should be mentioned that various iterative methods and methods of approximation of various orders form the basis for construction of neural networks for the above spheres of human activities. These methods solve the tasks of training one neural element with varied functions of activation and training neural networks consisting of these elements with a certain accuracy. However, there are problems for which approximate solutions are unacceptable, for example, the problem of feasibility of Boolean and multivalued logic functions by a single neural element with a threshold function of activation or a generalized neural element relative to a specified system of characters and in the synthesis of combinational circuits from the mentioned neural elements. These combinational circuits can be successfully used in the construction of functional blocks of logical devices for controlling technological processes, compression of discrete signals, recognition of discrete images and so on. Disadvantages of approximation methods and iterative methods of training neural elements and neural networks for solving problems on the implementation of Boolean and multiple-valued logic functions by a single neural element (neural network) are as follows:

– instead of an exact solution, one obtains an approximate solution of the problem (for example, a discrete function is implemented by one generalized neural element and the approximation method and iterative methods show its unreliazability relative to the prescribed accuracy (here a problem appears about choosing exactness, the order of approximation and convergence of the process of training the generalized neural element relative to the prescribed accuracy));

– ability of applying methods of approximation and iterative methods of training artificial neurons with just a small number of inputs (up to 50) whereas biological neurons can have thousands of entries.

Given this, a development of methods for checking realizability of Boolean functions by one generalized neural element relative to an arbitrary system of characters should be recognized as promising. Solutions on the synthesis of corresponding generalized neural elements under certain restrictions on their nuclei can be used in a case when application of approximation and iterative methods is inexpedient or practically impossible.

3. The aim and objectives of the study

The study objective was to develop effective methods for verifying realizeability of the logic algebra functions by one generalized neural element and methods for the synthesis of generalized neural elements with integer structural vectors. On the basis of these elements, one can develop logical blocks of different devices for solving problems of practical importance in the field of compression and transmission of discrete signals, recognition of discrete images, diagnosis of technical devices.

To achieve this goal, it was necessary to implement the following:

– to establish a criterion of realizability of Boolean functions by one generalized neural element;

– to ensure such necessary conditions for realizeability of the logic algebra functions by one neuron element with a generalized threshold activation function which would be easily verified;

– to obtain sufficient conditions for realizeability of the logic algebra functions by one generalized neural element by establishing which algorithm of the synthesis of integer generalized neural elements is constructed.

4. Mathematical model of neural elements with a generalized threshold activation function and their application when implementing Boolean functions

4. 1. The criterion for implementing Boolean functions by one generalized neural element

Let H_2={–1,1} be a cyclic group of second order, $G_n = H_2 \otimes \ldots \otimes H_2$ is the direct product of n cyclic groups H_2, and $\chi(G_n)$ is the group of characters [18] of the group G_n over the field of real numbers R. Use the set $R\backslash\{0\}$ to define the function:

$$\operatorname{Rsign} x = \begin{cases} 1, & if\ x > 0, \\ -1, & if\ x < 0. \end{cases} \tag{1}$$

Let Z_2={0,1}, $i\in\{0, 1,..., 2^n-1\}$ and $(i_1,...,i_n)$ is its binary code, i. e.

$$i = i_1 2^{n-1} + i_2 2^{n-2} + \ldots + i_n,\ \ i_j \in \{0,1\}.$$

Values of character χ_i on the element

$$\mathbf{g} = ((-1)^{\alpha_1}, \ldots, (-1)^{\alpha_n}) \in G_n$$

($(\alpha_1, \ldots, \alpha_n) \in Z_2^n$ – n-and the Cartesian degree Z_2) are assigned as follows:

$$\chi_i(\mathbf{g}) = (-1)^{\alpha_1 i_1 + \alpha_2 i_2 + \ldots + \alpha_n i_n}. \tag{2}$$

Consider the 2^n-dimensional vector space

$$V_R = \{\phi \mid \phi : G_n \to R\}$$

over the field R. Elements

$$\chi_i\ (i = 0, 1, 2, \ldots, 2^n - 1)$$

of the group $X(G_n)$ form an orthogonal basis of the space V_R [18]. The Boolean function in alphabet {–1,1} specifies a single-valued transformation $f : G_n \to H_2$, i. e. $f\in V_R$. Consequently, the arbitrary Boolean function $f\in V_R$ can be uniquely written in the form:

$$f(\mathbf{g}) = s_0\chi_0(\mathbf{g}) + s_1\chi_1(\mathbf{g}) + \ldots + s_{2^n-1}\chi_{2^n-1}(\mathbf{g}). \tag{3}$$

Vector $\mathbf{s}_f = (s_0, s_1, \ldots, s_{2^n-1})$ is called a spectrum of the Boolean function f in the system of characters $\chi(G_n)$ (in the system of Walsh-Hadamard basis functions [19]).

Taking various characters exept the main one, construct an m-element set $\chi=\{\chi_{i_1},\ldots,\chi_{i_m}\}$ and consider the mathematical model of the neural element with the generalized threshold function of activation relative to the chosen system of characters:

$$f(x_1(\mathbf{g}),\ldots,x_n(\mathbf{g}))=\mathrm{Rsign}\left(\sum_{j=1}^{m}\omega_j\chi_{i_j}(\mathbf{g})+\omega_0\right), \qquad (4)$$

where vector $\mathbf{w}=(\omega_1,\ldots,\omega_m;\omega_0)$ is called the vector of the structure of the generalized neural element with respect to the system of characters χ and $\mathbf{g}\in G_n$.

If there exists a vector satisfying equation (4) for the given function $f:G_n\to H_2$ and the system $\chi=\{\chi_{i_1},\ldots,\chi_{i_m}\}$, then it is said that function f is realized by one generalized neural element relative to χ.

It is obvious that the neural element in relation to the system of characters $\chi=\{\chi_1,\chi_2,\chi_4,\ldots,\chi_{2^{n-1}}\}$ coincides with the neural element with a threshold activation function (with a threshold element [20]). For an arbitrary Boolean function $f:G_n\to H_2$, one can always choose such a system χ that the generalized neural element realizes the function f relative to χ. Indeed, if no limitations are imposed on the number of entries of generalized neural elements, then a system of characters $\chi(G_n)\setminus\chi_0$. can be chosen as χ. Then it follows from (3) and (4) that an arbitrary Boolean function is realized by one generalized neural element with a structure vector coinciding with the spectrum of the function in the system of Walsh-Hadamard basis functions. Further, in addition to this trivial case, in order to reduce the number of entries to the GNE, consider systems χ that do not coincide with $\chi(G_n)\setminus\chi_0$. Obviously, the less elements in the χ system, the more efficiently these elements can be used in neural networks for compressing, transmitting and recognizing discrete signals and images.

Let

$$f(x_1,\ldots,x_n)$$

be the Boolean function in alphabet {–1,1}, that is $f:G_n\to H_2$. Consider the problem whether function $f(x_1,\ldots,x_n)$ is realized by one GNE relative to the system of characters

$$\chi=\left\{\chi_{i_1},\chi_{i_2},\ldots,\chi_{i_m}\right\}\subset X(G_n),$$

and if so, then how its structure vector can be found?

Using transformation

$$\mathbf{x}'=\frac{1}{2}(\mathbf{x}+1),$$

realize mapping $\{-1,1\}\to\{0,1\}$ and consider the system

$$\chi'=\left\{\chi'_{i_1}=\frac{1}{2}\left(\chi_{i_1}+1\right),\chi'_{i_2}=\frac{1}{2}\left(\chi_{i_2}+1\right),\ldots,\chi'_{i_m}=\frac{1}{2}\left(\chi_{i_m}+1\right)\right\}.$$

Let

$$f^{-1}(-1)=\left\{\mathbf{g}\in G_n\mid f(\mathbf{g})=-1\right\}$$

and

$$f^{-1}(1)=\left\{\mathbf{g}\in G_n\mid f(\mathbf{g})=1\right\}.$$

With the help of the system χ', find:

$$f_\chi^{-1}(-1)=\bigcup_{\mathbf{g}\in f^{-1}(-1)}\left\{\left(\chi'_{i_1}(\mathbf{g}),\ldots,\chi'_{i_m}(\mathbf{g})\right)\right\},$$

$$f_\chi^{-1}(1)=\bigcup_{\mathbf{g}\in f^{-1}(1)}\left\{\left(\chi'_{i_1}(\mathbf{g}),\ldots,\chi'_{i_m}(\mathbf{g})\right)\right\}.$$

The nucleus of the Boolean function $f(x_1,\ldots,x_n)$ with respect to the system of characters χ of the group G_n is defined as follows:

$$K(f_\chi)=\begin{cases}f_\chi^{-1}(1), & if\ \left|f_\chi^{-1}(1)\right|\le\left|f_\chi^{-1}(-1)\right|,\\ f_\chi^{-1}(-1), & if\ \left|f_\chi^{-1}(1)\right|>\left|f_\chi^{(-1)}(-1)\right|,\end{cases}$$

if

$$f_\chi^{-1}(1)\cap f_\chi^{-1}(-1)=\varnothing,$$

where $\left|f_\chi^{-1}(i)\right|$ is the number of elements of the set

$$f_\chi^{-1}(i)\ \ (i\in\{-1,1\}).$$

If $f_\chi^{-1}(1)\cap f_\chi^{-1}(-1)\ne\varnothing$, than nucleus $K(f_\chi)$ does not exist and this means that function f is not realized by one GNE relative to the system χ.

Let

$$K(f_\chi)=\{\mathbf{a}_1=(\alpha_1^1,\ldots,\alpha_n^1),\ldots,\mathbf{a}_q=(\alpha_1^q,\ldots,\alpha_n^q)\}.$$

Construct the reduced nucleus $K(f_\chi)_i$ relative to the element

$$\mathbf{a}_i=(\alpha_1^i,\ldots,\alpha_n^i)\in K(f_\chi)$$

and a set of reduced nuclei $T(f_\chi)$ as follows:

$$K(f_\chi)_i=\mathbf{a}_iK(f_\chi)=$$
$$=\{(\alpha_1^i\oplus\alpha_1,\ldots,\alpha_n^i\oplus\alpha_n)\mid(\alpha_1,\ldots,\alpha_n)\in K(f_\chi)\},$$

$$T(f_\chi)=\left\{K(f_\chi)_i=\mathbf{a}_iK(f_\chi)\mid i=1,2,\ldots,q\right\},$$

where $\oplus$ is the sum by modulus 2.

Let $Z_2=\{0,1\}$ and Z_2^n be the n-th Cartesian power of the Z_2 set. If the following conditions are fulfilled for the function $f:G_n\to H_2$ and the system $\chi=\{\chi_{i_1},\ldots,\chi_{i_m}\}$

$$f_\chi^{-1}(1)\cap f_\chi^{-1}(-1)=\varnothing \text{ and } Z_2^m\ne f_\chi^{-1}(1)\cup f_\chi^{-1}(-1),$$

then function f is a partially defined function in the group G_m and a concept of a extended nucleus with respect to the system of characters χ is introduced for such functions as follows: let $K(f_\chi)=\{\mathbf{a}_1,\ldots,\mathbf{a}_q\}$ be a nucleus of the Boolean function f with respect to $\chi=\{\chi_{i_1},\ldots,\chi_{i_m}\}$ and $f_\chi^{-1}(*)$ is the sets of those nests with Z_2^m for which function was not defined, then the following will be implied: under the extended nucleus of function f relative to the system χ:

$$K(f_\chi,s)=\{\mathbf{a}_1,\ldots,\mathbf{a}_q,\mathbf{b}_1,\ldots,\mathbf{b}_s\},$$

where $\mathbf{b}_1,\ldots,\mathbf{b}_s$ are the arbitrary elements of the set $f_\chi^{-1}(*)$ and $q+s\le 2^{m-1}$. It should be mentioned [20] that the Boolean function f is simultaneously realized or is not simultaneously realized by one neural element in different alphabets $\{-1,1\}$ $(\{0,1\})$.

If notation $\mathbf{b}_1=\mathbf{a}_{q+1},...,\mathbf{b}_s=\mathbf{a}_{q+s}$, is introduced, then the set of reduced nuclei of the Boolean function f with respect to the system $\chi=\{\chi_{i_1},\ldots,\chi_{i_m}\}$ is specified as follows:

$$T(f_\chi,s)=\{K(f_\chi,s)_i= \\ =\mathbf{a}_i K(f_\chi,s)\,|\,i=1,2,\ldots,q+s\}.$$

Let E_m be the set of tolerance matrices [21] and

$$K(f_\chi)=\{\mathbf{a}_1,\ldots,\mathbf{a}_q\}$$

is the nucleus of the Boolean function f:$G_n\to H_2$ with respect to the system of characters $\chi=\{\chi_{i_1},\ldots,\chi_{i_m}\}$ of the group G_n over the field R. Use elements of the nucleus $K(f_\chi)$ to construct matrix $K_\xi(f_\chi)$ as follows: the first row of the matrix will be the vector

$$\mathbf{a}_{\xi(1)}=\left(\alpha_{\xi(1)1},\ldots,\alpha_{\xi(1)n}\right)$$

with $K(f_\chi)$, and the second row of the matrix will be vector

$$\mathbf{a}_{\xi(2)}=\left(\alpha_{\xi(2)1},\ldots,\alpha_{\xi(2)n}\right),$$

the last row of $K_\xi(f_\chi)$ will be

$$\mathbf{a}_{\xi(q)}=\left(\alpha_{\xi(q)1},\ldots,\alpha_{\xi(q)n}\right),$$

where $\xi(i)$ is the effect of substitution of $\xi\in S_q$ for i. Denote the first r lines of the matrix $L\in E_m$ by $L(r)$ and enter the concept of representation of the nucleus $K(f_\chi)$ (extended nucleus $K(f_\chi,s)$) with the matrices of tolerance with E_m as follows: if there is such an element $\xi\in S_q$ and such a matrix $L\in E_m$ that $K_\xi(f_\chi)=L(q)$ $(K_\xi(f_\chi s)=L(q))$, then the nucleus $K(f_\chi)$ (extended nucleus $K(f_\chi,s)$) admits representation by the matrices of tolerance with E_m.

Theorem 1. *The Boolean function f:$G_n\to H_2$ is realized by one generalized neural element with respect to the system of characters $\chi=\{\chi_{i_1},\ldots,\chi_{i_m}\}$ of the group Gn over the field R, if and only if one of the conditions exists and is fulfilled*:

1) the nucleus $K(f_\chi)$ admits representation by the matrices of tolerance with E_m and

$$Z_2^m=f_\chi^{-1}(1)\cup f_\chi^{-1}(-1);$$

2) the nucleus $K(f_\chi)$ or extended nucleus $K(f_\chi,s)$ admits representation by the matrices of tolerance with E_m and

$$Z_2^m\neq f_\chi^{-1}(1)\cup f_\chi^{-1}(-1).$$

Proving. In the case

$$Z_2^m=f_\chi^{-1}(1)\cup f_\chi^{-1}(-1)$$

provided that $K(f_\chi)$ exists, the theorem is proved analogous to Theorem 1 in [21].

Let

$$Z_2^m\neq f_\chi^{-1}(1)\cup f_\chi^{-1}(-1),$$

Ω_m be the set of all m-dimensional real vectors $\mathbf{w}$ such that for all different $\mathbf{x}_1,\mathbf{x}_2\in Z_2^m$, numbers $(\mathbf{x}_1,\mathbf{w})$ and $(\mathbf{x}_2,\mathbf{w})$ are different ($(\mathbf{x}_i,\mathbf{w})$ is the scalar product of the vectors $\mathbf{x}_i$, $\mathbf{w}$) and $L\in E_m$. Denote by $h(L(q))$ the set of m-dimensional Boolean vectors constructed from the rows of the matrix $L(q)$. Obviously,

$$K(f_\chi)=h(K_\xi(f_\chi))$$

for all $\xi\in S_q$.

If the function f:$G_n\to H_2$ is realized by one GNE with respect to the system

$$\chi=\{\chi_{i_1},\ldots,\chi_{i_m}\},$$

then in accordance with [22], this function is realized in alphabet Z_2 as well. This means that there exists such vector

$$\mathbf{w}=(\omega_1,\ldots,\omega_m)\in\Omega_m,$$

that satisfies one of the following conditions: if

$$K(f_\chi,s)=f_\chi^{-1}(1),$$

then

$$\forall\mathbf{x}\in K(f_\chi,s)$$

and

$$\forall\mathbf{y}\in Z_2^m\setminus K(f_\chi,s)\ \ (\mathbf{x},\mathbf{w})>(\mathbf{y},\mathbf{w}),\tag{5}$$

in the opposite case, $(K(f_\chi,s)=f^{-1}(-1))$

$$\forall\mathbf{x}\in K(f_\chi,s)$$

and

$$\forall\mathbf{y}\in Z_2^m\setminus K(f_\chi,s)\ \ (\mathbf{x},\mathbf{w})<(\mathbf{y},\mathbf{w}).\tag{6}$$

On the basis of (5) and the properties of the matrices of tolerance [21], it can be asserted that there exists such matrix of tolerance $L_\mathbf{w}\in E_m$, that

$$K(f_\chi,s)=h(L_\mathbf{w}(q+s)).$$

In the case of (6), we have

$$K(f_\chi,s)=h(L_{\mathbf{w}_1}(q+s)),$$

where $\mathbf{w}_1=-\mathbf{w}$. Thus, $K(f_\chi,s)$ admits representation by ноу matrices of tolerance with E_m, so the necessity is proved.

Let $K(f_\chi s)$ admits representation by the matrices of tolerance with E_m, that is, there is such matrix

$$L=(\alpha_{ij})\in E_m\,(i=1,2,\ldots,2^{m-1};j=1,2,\ldots,m),$$

that

$$K(f_\chi s)=h(L(q+s)).$$

Make the matrix of tolerance $L^*=(\alpha_{sj})$ to correspond to the matrix of tolerance $\underline{L}=(\alpha_{ij})$ in the following way: $s=2^{n-1}-i+1$ and $\alpha_{sj}=\overline{\alpha}_{ij}$ ($\overline{\alpha}_{ij}$ is an inverted value of α_{ij}). Define operation v for the matrices of tolerance L and L^* as follows:

$$L\nabla L^* = \begin{pmatrix} L \\ L^* \end{pmatrix}.$$

According to the construction of the set E_m [21], there exists such vector

$$\mathbf{w} = (\omega_1,...,\omega_m) \in \Omega_m,$$

for an arbitrary matrix $L \in E_m$ that

$$\left(L\nabla L^*\right)\cdot \mathbf{w}^T = \mathbf{c}_{\mathbf{w}}^T, \quad (7)$$

where

$$\mathbf{c}_{\mathbf{w}} = (c_1, c_2, c_3 ..., c_t)\ (c_1 > c_2 > c_3 > ... > c_t; t = 2^m).$$

It follows immediately from equation (7) and equality

$$K(f_\chi, s) = h(L(q+s))$$

that there exists such vector $\mathbf{w}$ in the set Ω_m which satisfies inequality (5). This means that the generalized neural element relative to the system of characters χ' with the weight vector $\mathbf{w}$ in alphabet Z_2 realizes function f or $\overline{f}$. These functions, either are simultaneously realized or simultaneously not realized by one generalized neural element. Sufficiency is proven. Consequently, the theorem is proved completely.

Let

$$\Omega_m^- = \{\mathbf{w} = (\omega_1,\ldots,\omega_n) \in \Omega_m \mid 0 > \omega_1 > \ldots > \omega_m\}$$

and

$$E_m^- = \bigcup_{\mathbf{w}\in\Omega_m^-} L_{\mathbf{w}},$$

where

$$\left(L_{\mathbf{w}} \nabla L_{\mathbf{w}}^*\right)\cdot \mathbf{w}^T = \mathbf{c}_{\mathbf{w}}^T.$$

Use the notion of reduced nuclei of the Boolean function $f{:}G_n \to H_2$ with respect to the system of characters

$$\chi = \{\chi_{i_1},\ldots,\chi_{i_m}\}$$

of the G_n group and obtain the following on the basis of Theorem 1 and equality

$$E_m = \{(\mathbf{g}L)^\sigma \mid L \in E_m^-, \mathbf{g} \in G_m, \sigma \in S_m\} \ \text{[21]:}$$

Theorem 2. *The Boolean function $f{:}G_n \to H_2$ is realized by one generalized neuron element relative to the system of characters $\chi = \{\chi_{i_1},\ldots,\chi_{i_m}\}$ of the group G_n over the field R if and only if there is $K(f_\chi)$ and one of the following conditions is fulfilled*:

1) there is at least one such element $K(f_\chi)_i$, in the set of reduced nuclei $T(f_\chi)$ that admits representation by the matrices of tolerance with E_m^- and

$$Z_2^m = f_\chi^{-1}(1) \cup f_\chi^{-1}(-1);$$

2) there is at least one such element $K(f_\chi)_i$ in the set of reduced nuclei $T(f_\chi)$ or such element $K(f_\chi,s)_i$, in a set of extended reduced nuclei $T(f_\chi,s)$ which admits representation by matrices of tolerance with E_m^- and E_m^-.

4. 2. Conditions necessary for the implementation of Boolean functions by one generalized neural element

Verification of the conditions necessary for realization of Boolean functions by one generalized neural element is an important step in the GNE synthesis. With the help of these conditions, functions of the logic algebra can be identified at the initial stage of synthesis of generalized neural elements which cannot be realized by one GNE in relation to a specified system of characters.

Theorem 3. *If the boolean function $f{:}G_n \to H_2$ is realized by one generalized neural element with respect to the system of characters $\chi = \{\chi_{i_1},\ldots,\chi_{i_m}\}$ of the group G_n over the field R, then there is a nucleus $K(f_\chi)$ and the following takes place*:

$$\mathbf{a} = (\alpha_1,\ldots,\alpha_m) \in K(f_\chi) \Rightarrow \overline{\mathbf{a}} = (\overline{\alpha}_1,\ldots,\overline{\alpha}_m) \notin K(f_\chi), \quad (8)$$

where $\overline{\alpha}_i$ is the inverted value α_i.

Proving. As it follows from Theorem 1, there is a nucleus $K(f_\chi)$ or an extended nucleus $K(f_\chi,s)$ that admits representation by the matrices of tolerance with E_m. Consequently, there exists such a matrix of tolerance $L \in E_m$, and such an element $\xi \in S_q$ or an element $\xi \in S_{q+s}$, that one of equalities

$$K_\xi(f_\chi) = L(q) \ \text{ or } \ K_\xi(f_\chi,s) = L(q)$$

is satisfied. Rows of the matrix L are the elements of a certain class of tolerance relative to

$$\tau : (\alpha_1,\ldots,\alpha_n)\tau(\beta_1,\ldots,\beta_n) \Leftrightarrow \exists i(\alpha_i = \beta_i).$$

It follows from this that the prematrix of tolerance $L(q)$ does not concurrently contain vectors

$$\mathbf{a} = (\alpha_1,\ldots,\alpha_m), \ \ \overline{\mathbf{a}} = (\overline{\alpha}_1,\ldots,\overline{\alpha}_m),$$

where $\mathbf{a} = (\alpha_1,\ldots,\alpha_m)$ is an arbitrary row of the matrix $L(q)$, so the theorem is proved.

The vector

$$\mathbf{a} = (\alpha_1,\ldots,\alpha_m) \in \mathrm{Z}_2^m$$

precedes the vector

$$\mathbf{b} = (\beta_1,\ldots,\beta_m) \in \mathrm{Z}_2^m \ \ (\mathbf{a} \prec \mathbf{b}), \text{ if } \alpha_i \le \beta_i \ (i = 1,2,\ldots,m).$$

Denote by $M_{\mathbf{a}}$ the set of all such vectors with Z_2^m, that precede the vector **a.**

Theorem 4. *If the Boolean function $f{:}G_n \to H_2$ is realized by one generalized neural element with respect to the system of characters $\chi = \{\chi_{i_1},\ldots,\chi_{i_m}\}$ of the group G_n over the field R, then there is a nucleus $K(f_\chi)$ and one of the following conditions is fulfilled:*

1) if $Z_2^m = f_\chi^{-1}(1) \cup f_\chi^{-1}(-1)$, then there is such an element $K(f_\chi)_i$, in the set of reduced nuclei $T(f_\chi)$ that

$$\forall \mathbf{a} \in K(f_\chi)_i \Rightarrow M_{\mathbf{a}} \subset K(f_\chi)_i; \quad (9)$$

2) if $Z_2^m \ne f_\chi^{-1}(1) \cup f_\chi^{-1}(-1)$, then either there is such an element $K(f_\chi)_i$, in the set of reduced nuclei $T(f_\chi)$ which satisfies condition (8) or it is possible to construct a set of extended reduced nuclei $T(f_\chi,s)$, which contains such an element $K(f_\chi,s)_i$ that

$$\forall \mathbf{a} \in K(f_\chi,s)_i \Rightarrow M_{\mathbf{a}} \subset K(f_\chi,s)_i. \quad (10)$$

The proving of this theorem follows directly from the rules of constructing the set of extended reduced nuclei $T(f_\chi,s)$, of Theorem 2 and Theorem 3 in [23].

Suppose $B=(\beta_{kr})$ is a rectangular $q\times m$ matrix over $Z_2, A\subseteq Z_2^m$, $\mathbf{e}_i$ is a unit vector the i-th coordinate of which is equal to 1,

$$n(A)=\{i\,|\,\mathbf{e}_i\in A\},\ \ k(B)=\sum_{r=1}^{n}\beta_{kr}$$

and $|A|$ is the number of elements of the set A.

Theorem 5. *If the Boolean function $f{:}G_n\to H_2$ is realized by one generalized neural element with respect to the system of characters $\chi=\{\chi_{i_1},\ldots,\chi_{i_m}\}$ of the group G_n over the field R, then there is a nucleus $K(f_\chi)=\{\mathbf{a}_1,\ldots,\mathbf{a}_q\}$ and the folowing takes place:*

1) if

$$Z_2^m=f_\chi^{-1}(1)\cup f_\chi^{-1}(-1)$$

and

$$2^j<q\le 2^{j+1}\ (j\in\{1,2,\ldots,m-2\}),$$

then there is such element $K(f_\chi)_i$ in the set of reduced nuclei $T(f_\chi)$ that

$$1)\ \ \forall k\in\{1,2\ldots,\mathrm{q}\}\ k(K(f_\chi)_i)\le j+1\,; \tag{11}$$

$$2)\ \ |\,n(K(f_\chi)_i\,|\ge j+1\,; \tag{12}$$

2) if $Z_2^m\ne f_\chi^{-1}(1)\cup f_\chi^{-1}(-1)$, then either there is an element $K(f_\chi)_i$, in the set of reduced nuclei $T(f_\chi)$ which satisfies conditions (10), (11), or it is possible to construct a set of extended reduced nuclei $T(f_\chi,s)$, containing such an element $K(f_\chi,s)_i$, that

$$3)\ \ 2^j<q+\mathrm{s}\le 2^{j+1}\ (j\in\{1,2,\ldots,m-2\}); \tag{13}$$

$$4)\ \ \forall k\in\{1,2\ldots,q+s\}\ k(K(f_\chi s)_i)\le j+1; \tag{14}$$

$$5)\ \ |\,n(K(f_\chi,s)_i\,|\ge j+1. \tag{15}$$

Proving. It is stated that function $f{:}G_n\to H_2$ is realized by one generalized neural element with respect to the system of characters $\chi=\{\chi_{i_1},\ldots,\chi_{i_m}\}$ of the group G_n over the field R. Let

$$Z_2^m=f_\chi^{-1}(1)\cup f_\chi^{-1}(-1),$$

then it follows from Theorem 1 that there exists a nucleus

$$K(f_\chi)=\{\mathbf{a}_1,\ldots,\mathbf{a}_q\},$$

admitting representation by matrices of tolerance with E_m. Consequently, there is such matrix of tolerance $H\in E_m$ and such element $\xi\in S_q$, that $K_\xi(f_\chi)=H(q)$. If the first row of the matrix $K_\xi(f_\chi)$, is denoted by $\mathbf{a}_i$ $(\xi(i)=1)$, then

$$K_\xi(f_\chi)_i=H_1(q),$$

where $H_1=\mathbf{a}_iH\in E_m^-$, follows from the equality of matrices $K_\xi(f_\chi)=H(q)$.

Consider vector $\mathbf{w}=(\omega_1,\ldots,\omega_m)\in\Omega_m^-$, with coordinates satisfying conditions

$$\omega_1=-1,\ \ \omega_i=\sum_{j=1}^{i-1}\omega_i-1\,(i=2,3,\ldots,m)$$

and a system of matrices of tolerance

$$L_1=(0_1),\ L_2=\begin{pmatrix}L_1 & 0_1\\ L_1^* & 0_1\end{pmatrix},\ldots,\ L_m=\begin{pmatrix}L_{m-1} & 0_{m-1}\\ L_{m-1}^* & 0_{m-1}\end{pmatrix}, \tag{16}$$

where 0_t is the zero column with size $2^{t-1}\times 1$.

It is easy to see that

$$(L_\mathbf{w}\nabla L_\mathbf{w}^*)\cdot\mathbf{w}^T=\mathbf{c}_\mathbf{w}^T. \tag{17}$$

It follows from construction of the vector $\mathbf{w}$ and (16) that any matrix $V\in E_m^-$ satisfies the condition:

$$\forall k\in\{1,2,\ldots,q\}\ k(V)\le k(L_n)\le j{+}1.$$

Then, on the basis of equality

$$K_\xi(f_\chi)_i=H_1(q)\,(\mathrm{H}_1\in E_m^-)$$

and

$$2^j<q\le 2^{j+1}\ \ (j\in\{1,2,\ldots,\ \ldots,m-2\}),$$

it follows that

$$\forall k\in\{1,2,\ldots,q\}\ \ \ k(K(f_\chi)_i)\le j+1.$$

The order number of row $\mathbf{e}_i$ in any matrix of tolerance $V\in E_m^-$ does not exceed the order number of row $\mathbf{e}_i$ in the matrix

$$L_m\ (i\in\{1,2,\ldots,m-1\}).$$

Consequently,

$$|\,n(V(q))\,|\ \ge|\,n(L_m(q))\,|=j+1.$$

follows from the inequality $2^j<q\le 2^{j+1}$ and construction of the matrix L_m. In view of arbitrariness of the matrix of tolerance $V\in E_m^-$ and $K_\xi(f_\chi)_i=H_1(q)\,(\mathrm{H}_1\in E_m^-)$, we have that

$$|\,n(\mathbf{a}_iA)\,|\ge|\,n(L_n(q))\,|=j+1$$

and the first part of the theorem is proved when

$$Z_2^m=f_\chi^{-1}(1)\cup f_\chi^{-1}(-1)$$

and

$$2^j<q\le 2^{j+1}\,(j\in\{1,2,\ldots,m-2\}).$$

Let

$$Z_2^m\ne f_\chi^{-1}(1)\cup f_\chi^{-1}(-1)$$

and function $f{:}G_n\to H_2$ be realized by one generalized neural element in relation to the system of characters $\chi=\{\chi_{i_1},\ldots,\chi_{i_m}\}$ of the G_n group. Then, based on Theorem 1, either the nucleus $K(f_\chi)=\{\mathbf{a}_1,\ldots,\mathbf{a}_q\}$, or the extended nucleus

$$K(f_\chi,\mathrm{s})=\{\mathbf{a}_1,\ldots,\mathbf{a}_q,\mathbf{a}_{q+1,\ldots,}\mathbf{a}_{q+s}\}$$

admits representation by the matrices of tolerance with E_m. The number of rows of an arbitrary matrix of tolerance with E_m is 2^{m-1}, therefore, $q+s \le 2^{m-1}$ and inequality (13) takes place for $q+s$. Inequalities (14), (15) are proved in the same way as the inequalities (11) and (12) were proved above. Consequently, this theorem is proved completely.

Let $f{:}G_n \to H_2$ be a Boolean function and $f{:}G_n \to H_2$ is its nucleus with respect to the system of characters

$$\chi = \{\chi_{i_1},\dots,\chi_{i_m}\}$$

of the group G_n over the field R. $s(i;K_\xi(f_\chi))$, Denote the number of units of the i-th column of matrix $K_\xi(f_\chi)$ by $s(i;K_\xi(f_\chi))$ and enter the notation $K(f_\chi) = K(f_\chi,0)$.

Theorem 6. *If the Boolean function $f{:}G_n \to H_2$ is realized by one generalized neural element with respect to the system of characters $\chi = \{\chi_{i_1},\dots,\chi_{i_m}\}$ of the group G_n over the field R, then there is an extended nucleos*

$$K(f_\chi,s) = \{\mathbf{a}_1,\dots,\mathbf{a}_q,\mathbf{a}_{q+1},\dots,\mathbf{a}_{q+s}\}(s \ge 0)$$

with element

$$\mathbf{a}_t \in K(f_\chi,s),$$

elements

$$\xi \in S_{q+s}, \sigma \in S_m$$

than there is the following inequality for the extended reduced nucleus

$$K(f_\chi,s)_t = \mathbf{a}_t K(f_\chi,s)$$

and for all $i \in \{2,3,\dots,m\}$

$$s(i-1;K_\xi^\sigma(f_\chi,s)_t) \ge s(i;K_\xi^\sigma(f_\chi,s)_t). \tag{18}$$

Proving. According to Theorem 2, $K(f_\chi,s)$ admits representation by matrices of tolerance with E_m, that is, there is such matrix of tolerance $L \in E_m$ and such element $\xi \in S_q$, that

$$K_\xi(f_\chi,s) = L(q). \tag{19}$$

Denote the first row of the matrix $K_\xi(f_\chi,s)$ by

$$\mathbf{a}_t(\xi(t)=1)$$

and, transform the equality with the help of this element (19) as follows:

$$\mathbf{a}_u K_\xi(f_\chi s) = \mathbf{a}\ L(q). \tag{20}$$

The matrix $L_\mathbf{w} = \mathbf{a}_t L$ defines vector $\mathbf{w} \in \Omega_m^-$, all coordinates of which are negative since the first coordinate of the vector $\mathbf{c}_\mathbf{w}^T = (L_\mathbf{w} \nabla L_\mathbf{w}^*) \cdot \mathbf{w}^T$ is 0. Choose the element $\sigma \in S_m$ so that the coordinates of the vector $\mathbf{w}_1 = \mathbf{w}^\sigma$ are arranged in a descending order. Then matrix

$$L_{\mathbf{w}_1} = (\mathbf{a}_t L)^\sigma = \mathbf{a}_t^\sigma L^\sigma$$

satisfies condition

$$\left(L_{\mathbf{w}_1} \nabla L_{\mathbf{w}_1}^*\right) \cdot \mathbf{w}_1^T = c_{\mathbf{w}_1}^T,$$

i. e. $L_{\mathbf{w}_1} \in E_m^-$. The following is obtained from (20):

$$\mathbf{a}_u^\sigma K_\xi^\sigma(f_\chi s) = \mathbf{a}^\sigma L^\sigma(q) = L_{\mathbf{w}_1}(q). \tag{21}$$

Let

$$\mathbf{a} = (\alpha_1,\dots,\alpha_{i-2},0,1,\alpha_{i+1},\dots,\alpha_n)$$

be a row of the pre-matrix of tolerance $L_{\mathbf{w}_1}(q)$. Then

$$\mathbf{b} = (\alpha_1,\dots,\alpha_{i-2},1,\ \ 0,\alpha_{i+1},\dots,\alpha_n)$$

will also be a row of the matrix $L_{\mathbf{w}_1}(q)$ and the order number of row $\mathbf{b}$ in the matrix $L_{\mathbf{w}_1}(q)$ will be smaller than the order number $\mathbf{a}$ since $\mathbf{w}_1 \in \Omega_n^-$. This means that the following inequality is fulfilled for any $i \in \{2,3,\dots,m\}$, and for any $k \in \{1,2,\dots,q+s\}$:

$$s\left(i-1;L_{\mathbf{w}_1}(k)\right) \ge s\left(i;L_{\mathbf{w}_1}(k)\right).$$

The last inequality and equality (21) directly result in inequality

$$s(i-1;K_\xi^\sigma(f_\chi,s)_t) \ge s(i;K_\xi^\sigma(f_\chi,s)_t)$$

and the theorem is proved.

4. 3. Conditions sufficient for the implementation of Boolean functions by one generalized neural element

In this section, we consider conditions sufficient for realizeability of Boolean functions by one generalized neural element which can be successfully used in synthesis of neural networks based on the GNE with integer-valued structure vectors.

Let p be a threshold operator [23] with labels $\mathbf{a}_k$, σ_k and $K(f_\chi,s)(s \ge 0)$ is an extended nucleus of the Boolean function $f{:}G_n \to H_2$ with respect to the system of characters $\chi = \{\chi_{i_1},\dots,\chi_{i_m}\}$ of the group G_n over the field R. Assuming

$$p(K(f_\chi,s)) = p\left(\mathbf{a}_k^{\sigma_k} K^{\sigma_k}(f_\chi,s)\right),$$

obtain

$$p(K(f_\chi,s)) = \\ = p_0(K(f_\chi,s)) \nabla p_1(K(f_\chi,s)) \nabla \dots \nabla p_{t_0}(K(f_\chi,s)), \tag{22}$$

where

$$p_0(K(f_\chi s)) = p_0\left(\mathbf{a}_k^{\sigma_k} K^{\sigma_k}(f_\chi,s)\right) = (L_{j_k} 0_{j_k} \dots 0_{j_k});$$

$$p_1(K(f_\chi,s)) = p_1\left(\mathbf{a}_k^{\sigma_k} K(f_\chi,s)\right) = (L_{j_k}^*(q_0^k)\underbrace{0\dots0}_{n-j_k});$$

....

$$p_{t_0}(K(f_\chi,s)) = p_{t_0}\left(\mathbf{a}_k^{\sigma_k} K^{\sigma_k}(f_\chi,s)\right) = (L_{j_k+t_0-1}^*(q_{t_0-1}^k)\underbrace{0\dots0}_{n-(j_k+t_0-1)}),$$

and $q_0^k \ge q_1^k \ge \dots \ge q_{t_0-1}^k > 0$.

Theorem 7. *Let $K(f_\chi,s) = \{\mathbf{a}_1,\dots,\mathbf{a}_q\}$ is the nucleus of the Boolean function $f{:}G_n \to H_2$ relative to the system of characters $\chi = \{\chi_{i_1},\dots,\chi_{i_m}\}$ of the group G_n over the field R. If it is possible to construct such an extended nucleus $K(f_\chi,s)(s \ge 0)$ for which there are such elements $\mathbf{a}_k \in K(f_\chi,s), \xi \in S_{q+s}$ and $\sigma_k \in S_m$, that for the extended reduced nucleus $K(f_\chi,s)_k = \mathbf{a}_k K(f_\chi,s)$, the following takes place:*

$$K_\xi(f_\chi,s)_k = p\left(\mathbf{a}_k^{\sigma_k} K^{\sigma_k}(f_\chi,s)\right), \quad (23)$$

then function f is realized by one generalized neural element in relation to the system of characters χ.

Proving. To prove this theorem, it suffices to show that matrix

$$p\left(\mathbf{a}_k^{\sigma_k} K^{\sigma_k}(f_\chi,\mathrm{s})\right)$$

is a tolerance prematrix of some matrix $L \in E_m^-$. We shall show that there exists such m-dimensional real vector

$$\mathbf{w} = (\omega_1,\ldots,\omega_m)$$

which satisfies the condition

$$\forall \mathbf{x} \in \mathbf{a}_k^{\sigma_k} K^{\sigma_k}(f_\chi,\mathrm{s}), \forall \mathbf{y} \in Z_2^m \setminus \mathbf{a}_k^{\sigma_k} K^{\sigma_k}(f_\chi,s)$$

$$(\mathbf{x},\mathbf{w}) > (\mathbf{y},\mathbf{w}) \quad (24)$$

By the condition (23), matrix $K_\xi(f_\chi,s)_k$ admits representation (22), hence, t_0 is the least positive integer such that $q_{t_0}^k \neq 0$ and

$$q_{t_0+1}^k = \ldots = q_{n-j_k}^k = 0.$$

Denote by $\mathbf{z}_0,\ldots,\mathbf{z}_{t_0-1}$ the last rows of the corresponding matrices

$$p_1(K(f_\chi,s)),\ldots,p_{t_0}(K(f_\chi,s)),$$

when $t_0 \geq 2$, and construct vector $\mathbf{w} = (\omega_1,\ldots,\omega_m)$ as follows:

$$1)\ \omega_1 = -1, \omega_2 = \omega_1 - 1,\ldots,\omega_{j_k} = \sum_{i=1}^{j_k-1}\omega_i - 1; \quad (25)$$

2) Coordinates ω_{j_k+r} are found sequentially from equations:

$$(\mathbf{z}_r,(\omega_1,\ldots,\omega_{j_m},\ldots,\omega_{j_k+r},0,\ldots,0)) =$$
$$= (\mathbf{z}_{r-1},(\omega_1,\ldots,\omega_{j_k+r-1},0,\ldots,0)), r = 1,\ldots,t\ (t = t_0 - 1); \quad (26)$$

$$3)\ \omega_{j_k+t+1} = \ldots = \omega_m =$$
$$= (\mathbf{z}_t,(\omega_1,\ldots,\omega_{j_k}\ldots,\omega_{j_k+t},0,\ldots,0)) - 1. \quad (27)$$

The vector thus constructed satisfies condition (24).

Then existence of such a vector $\mathbf{v} \in \Omega_m^-$ and such a matrix $L_\mathbf{v} \in E_m^-$, follows from [22] that $(L_\mathbf{v} \nabla L_\mathbf{v}^*) \cdot \mathbf{w}^T = \mathbf{c}_\mathbf{v}^T$. Consequently, it is possible to construct a premarix of tolerance $L_\mathbf{v}(q+\mathrm{s})$ from the elements of the extended reduced nucleus $K(f_\chi,s)_k$ and, in accordance with Theorem 2 (in the case when $t_0 \geq 2$), the function f is realized by one generalized neural element with respect to the system of characters χ. If $t_0 = 0$ or $t_0 = 1$, then, as it follows from the properties of the matrices of tolerance of the set E_m^-, the extended reduced nucleus $K_\xi(f_\chi,s)_k$ admits representation by the matrices of tolerance with E_m^- and the theorem is proved.

Use the set of vector coordinates $\{\alpha_1,\ldots,\alpha_m\}$

$$\mathbf{a} = (\alpha_1,\ldots,\alpha_j,\ldots,\alpha_{j+t},\ldots,\alpha_m) \in Z_2^m$$

to sequentially define the system of functions $\varepsilon_j^0,\varepsilon_j^1,\ldots,\varepsilon_j^t$ for fixed t and j ($j \geq 2$) in the same way as in [23] as follows:

$$\varepsilon_j^k(\alpha_i) = \begin{cases} \alpha_i, & \text{if } i \leq j-1; \\ \alpha_i(j - r_k), & \text{if } i = j+k; \\ \alpha_i j, & \text{if } i > j+\mathrm{t}; \end{cases} \quad (28)$$

where

$$k = 0,1,\ldots,t;\ r_0,r_1,\ldots,r_t \in \{1,2,\ldots,j-1\}.$$

Specify the following mapping through functions

$$\varepsilon_j^k\ (k = 0,\ldots,t)$$

for fixed $t \in \{0,\ldots,m-j\}$ and j:

$$\varepsilon_j^t : Z_2^m \to Z_{j+1}^m\ (Z_{j+1} = \{0,1,\ldots,j\}, 2 \leq j \leq m)$$

as follows:

$$\varepsilon_j^t(\mathbf{a}) = (\varepsilon_j^t(\alpha_1),\ldots,\varepsilon_j^t(\alpha_{j-1}),\varepsilon_j^0(\alpha_j),\varepsilon_j^1(\alpha_{j+1}),\ldots,$$
$$\varepsilon_j^t(\alpha_{j+t}),\varepsilon_j^t(\alpha_{j+t+1}),\ldots,\varepsilon_j^t(\alpha_m)) \quad (29)$$

and define functional v_j^t in the set Z_2^m by the formula:

$$\forall \mathbf{a} \in Z_2^m \quad v_j^t(\mathbf{a}) = \sum_{i \in I_t(j)} \varepsilon_j^t(\alpha_i) + \sum_{i=0}^{t} \varepsilon_j^i(\alpha_{j+i}), \quad (30)$$

where

$$I_t(j) = \{1,2,\ldots,m\} \setminus \{j,j+1,\ldots,j+t\}.$$

Using functional v_j^t for each $k \in \{0,1,\ldots,t\}$, construct a set of Boolean vectors as follows:

$$F_{j+k}^{(r_k,t)} = \{\mathbf{a} \in h(L_{j+k}^* 0\ldots0) \mid v_j^t(\mathbf{a}) \leq j-1\}, \quad (31)$$

where $h(L_{j+k}^* 0\ldots0)$ is the set of Boolean vectors which is constructed from rows of the matrix $\left(L_{j+k}^* 0\ldots0\right)$.

Let

$$K(f_\chi) = \{\mathbf{a}_1,\ldots,\mathbf{a}_q\}$$

be nucleus of the Boolean function $f{:}G_n \to H_2$ relative to the system of characters

$$\chi = \{\chi_{i_1},\ldots,\chi_{i_m}\}$$

of the group G_n over the field R. Similarly to Theorem 5 in [23] for generalized neural elements, we have:

Theorem 8. *If in the extended nucleus*

$$K(f_\chi,\mathrm{s}) = \{\mathbf{a}_1,\ldots,\mathbf{a}_{q+s}\}\ (\mathrm{s} \geq 0,\ \mathrm{q+s} \leq 2^{\mathrm{m}-1})$$

of the Boolean function $f{:}G_n \to H_2$ relative to the system of characters

$$\chi = \{\chi_{i_1},\ldots,\chi_{i_m}\}$$

of the group G_n over the field R, there exist respectively such elements $\mathbf{a},\sigma$ and integers

$$r_0 \geq r_1 \geq \ldots \geq r_t > 0$$

in the group S_m, that

$$\mathbf{a}^{\sigma}K^{\sigma}(f_{\chi},\mathrm{s})=h(L_j\underbrace{0\ldots0}_{m-j})\cup\left(\bigcup_{i=0}^{t}F_{j+i}^{(r_i,t)}\right),\qquad(32)$$

then function f is realized by one generalized neural element in relation to the system of characters χ.

Consider generalization of the system of functions

$$\varepsilon_j^k\ (k=0,1,2,...,t)$$

and functional v_j^t. Let

$$\mathbf{a}=(\alpha_1,\ldots,\alpha_j,\ldots,\alpha_{j+t},\ldots,\alpha_m)\in Z_2^m,$$

$$t\in\{0,1,\ldots,m-j\},\ \ (j\geq 2).$$

Construct a set of vectors $u_j(d)$ for fixed $j\in\{2,3,\ldots,m\}$ and $d\in\{1,2,j-1\}$:

$$\mathbf{u}_j(d)=\{(u_1,\ldots,u_d)\,|\,u_1+\ldots+u_d=$$
$$=j-1,\ u_1,\ldots,u_d\in\{1,2,\ldots,j-d\}\}\qquad(33)$$

and define the following through it:

$$U_j=\bigcup_{d=1}^{j-1}\mathbf{u}_j(d).$$

Admit that $\mathbf{u}=(u_1,\ldots,u_{l_{\mathbf{u}}})\in U_j$, $l_{\mathbf{u}}\geq 2$ ($l_{\mathbf{u}}$ is dimensionality of vector $\mathbf{u}$) and construct a system of functions $\varepsilon_j^{(\mathbf{u},0)},\varepsilon_j^{(\mathbf{u},1)},...,\varepsilon_j^{(\mathbf{u},t)}$:

$$\varepsilon_j^{(\mathbf{u},k)}(\alpha_i)=$$
$$=\begin{cases}\alpha_i, & \text{if } i\leq u_1,\\ \alpha_i 2^{r-1}, & \text{if } \sum_{p=1}^{r-1}u_p< i\leq\sum_{p=1}^{r}u_p,\\ \alpha_i\left(\sum_{p=1}^{l_{\mathbf{u}}}u_p 2^{p-1}-r_k+1\right), & \text{if } i=j+k,\\ \alpha_i\left(\sum_{p=1}^{l_{\mathbf{u}}}u_p 2^{p-1}+1\right), & \text{if } i>j+t,\end{cases}\qquad(34)$$

where $k=0,1,\ldots,t$, $r\in\{2,3,\ldots,l_{\mathbf{u}}\}$.

If $l_{\mathbf{u}}=1$, than $\varepsilon_j^{(\mathbf{u},k)}=\varepsilon_j^k$. For fixed

$$j\in\{2,3,\ldots,m\}, t\in\{0,1,\ldots,m-j\}\ \text{and}\ \mathbf{u}=(u_1,\ldots,u_{l_{\mathbf{u}}})\in U_j$$

specify representation

$$\varepsilon_j^{(\mathbf{u},t)}:Z_2^m\to Z_{c+1}^m\left(\mathrm{c}=\sum_{p=1}^{l_{\mathbf{u}}}u_p 2^{p-1}+1\right)$$

as follows:

$$\varepsilon_j^{(\mathbf{u},t)}(\mathbf{a})=(\varepsilon_j^{(\mathbf{u},t)}(\alpha_1),\ldots$$
$$\ldots,\varepsilon_j^{(\mathbf{u},t)}(\alpha_{j-1}),\varepsilon_j^{(\mathbf{u},0)}(\alpha_j),\varepsilon_j^{(\mathbf{u},1)}(\alpha_{j+1}),\ldots$$
$$\ldots,\varepsilon_j^{(\mathbf{u},t)}(\alpha_{j+t}),\varepsilon_j^{(\mathbf{u},t)}(\alpha_{j+t+1}),\ldots,\varepsilon_j^{(\mathbf{u},t)}(\alpha_m))\qquad(35)$$

and define functional $v_j^{(\mathbf{u},t)}$ on the set Z_2^m with the help of the following formula:

$$\forall\mathbf{a}\in Z_2^m\ \ v_j^{(\mathbf{u},t)}(\mathbf{a})=\sum_{i\in I_t(j)}\varepsilon_j^{(\mathbf{u},t)}(\alpha_i)+\sum_{i=0}^{t}\varepsilon_j^{(\mathbf{u},i)}(\alpha_{j+i}),\qquad(36)$$

where

$$I_t(j)=\{1,2,\ldots,m\}\setminus\{j,j+1,\ldots,j+t\}.$$

Specify a set of Boolean vectors $F_{j+k}^{(\mathbf{u},r_k,t)}$ through the functional $v_j^{(\mathbf{u},t)}$:

$$F_{j+k}^{(\mathbf{u},r_k,t)}=\left\{\mathbf{a}\in h(L_{j+k}^{*}0\ldots0)\,|\,v_j^{(\mathbf{u},t)}(\mathbf{a})\leq\sum_{p=1}^{l_{\mathbf{u}}}u_p 2^{p-1}\right\},\qquad(37)$$

where

$$r_0,r_1,\ldots,r_t\in\left\{1,2,\ldots,\sum_{p=1}^{l_{\mathbf{u}}}u_p 2^{p-1}\right\}, k\in\{0,1,\ldots,t\}.$$

Theorem 9. *If in the extended nucleus*

$$K(f_{\chi},\mathrm{s})=\{\mathbf{a}_1,\ldots,\mathbf{a}_{q+s}\}\ \ (\mathrm{s}\geq 0,\ \mathrm{q+s}\leq 2^{\mathrm{m-1}})$$

of the Boolean function $f:G_n\to H_2$ with respect to the system of characters $\chi=\{\chi_{i_1},\ldots,\chi_{i_m}\}$ of the group G_n over the field R, then, respectively, there are such elements $\mathbf{a},\sigma,\mathbf{u}$ and such integers

$$r_0\geq r_1\geq\ldots r_t>0\left(r_0\leq\sum_{p=1}^{l_{\mathbf{u}}}u_p 2^{p-1}\right),$$

in the group S_m and in the set U_j that

$$\mathbf{a}^{\sigma}K^{\sigma}(f_{\chi},s)=h(L_j\underbrace{0\ldots0}_{m-j})\cup\left(\bigcup_{i=0}^{t}F_{j+i}^{(\mathbf{u},r_i,t)}\right),\qquad(38)$$

then function f is realized by one generalized neural element in relation to the system of characters χ.

Proving. It is given that equality (38) is valid with respect to the elements

$$\mathbf{a}\in K^{\sigma}(f_{\chi},s),\sigma\in S_m\ \text{and}\ \mathbf{u}=(u_1,\ldots,u_{l_{\mathbf{u}}}).$$

Then one can construct a vector $\mathbf{w}=(\omega_1,\ldots,\omega_m)$, satisfying the condition:

$$\forall\mathbf{x}\in\mathbf{a}^{\sigma}K^{\sigma}(f_{\chi},\mathrm{s}),\forall\mathbf{y}\in Z_2^m\setminus\mathbf{a}^{\sigma}K^{\sigma}(f_{\chi},s)$$

$$(\mathbf{x},\mathbf{w})>(\mathbf{y},\mathbf{w}).\qquad(39)$$

Define coordinates ω_i of the vector $\mathbf{w}$ as follows:

$$\omega_1=\ldots=\omega_{u_1}=-1,$$

$$\omega_{u_1+1}=\ldots=\omega_{u_1+u_2}=-2,\ \ \omega_{u_1+u_2+1}=\ldots=\omega_{u_1+u_2+u_3}=-2^2,$$

$$\ldots,\ \ \omega_{u_1+u_2+\ldots+u_{l_{\mathbf{u}}-1}+1}=\ldots=\omega_{u_1+u_2+\ldots+u_{l_{\mathbf{u}}}}=-2^{l_{\mathbf{u}}-1}$$

$$\omega_j=r_0-\left(\sum_{p=1}^{l_{\mathbf{u}}}u_p 2^{p-1}+1\right),\ \ \omega_{j+1}=r_1-\left(\sum_{p=1}^{l_{\mathbf{u}}}u_p 2^{p-1}+1\right),\ \ldots,$$

$$\omega_{j+t}=r_t-\left(\sum_{p=1}^{l_{\mathbf{u}}}u_p 2^{p-1}+1\right),\ \ \omega_{j+t+1}=\ldots=\omega_m=-\sum_{p=1}^{l_{\mathbf{u}}}u_p 2^{p-1}-1.$$

The following is obtained from the vector $\mathbf{w}$ consruction:

$$\min\{(\mathbf{x},\mathbf{w}) \mid \mathbf{x}\in h(L_j 0\ldots 0)\} = -\sum_{p=1}^{l_{\mathbf{u}}} u_p 2^{p-1},$$

$$\forall k\in\{0,\ldots,t\}\ \min\{(\mathbf{x},\mathbf{w}) \mid \mathbf{x}\in F_{j+k}^{(\mathbf{u},r_k,k)}\} =$$

$$= -\sum_{p=1}^{l_{\mathbf{u}}} u_p 2^{p-1} > -\sum_{p=1}^{l_{\mathbf{u}}} u_p 2^{p-1} - 1 =$$

$$= \max\left\{(\mathbf{x},\mathbf{w}) \mid \mathbf{x}\in h\left(L_{j+k}^{*} 0\ldots 0\right) \setminus F_{j+k}^{(\mathbf{u},r_k,k)}\right\},$$

$$\forall z\in\{t+1,\ldots,m\}\ \max\{(\mathbf{x},\mathbf{w}) \mid \mathbf{x}\in h(L_{j+z}^{*} 0\ldots 0)\} = -\sum_{p=1}^{l_{\mathbf{u}}} u_p 2^{p-1} - 1.$$

Then (39) follows immediately from (38) and, according to Theorem 7, we can show existence of such a matrix of tolerance $L_1 \in E_m^{-}$, in which the first $q+s$ rows can be constructed from the elements of the extended nucleus $\mathbf{a}^{\sigma}K^{\sigma}(f_{\chi},s)$. Consequently, there exists such an element $\xi\in S_{q+s}$, that

$$\mathbf{a}^{\sigma}K_{\xi}^{\sigma}(f_{\chi},s) = L_1(q)$$

and the theorem is proved.

Consider synthesis of a neural element with a generalized threshold function of activation relative to the system of characters

$$\chi = \{x_1 = \chi_{2^{10-1}}, x_2 = \chi_{2^{10-2}} \ldots, x_{10} = \chi_1\}.$$

Let n=10 $\mathbf{a}_i = (0,0,0,0,0,1,1,1,1,1)$,

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 10 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \end{pmatrix},$$

$$j=5,\ r_0 = r_1 = 3,\ r_2 = 2,\ r_3 = 1,\ r_4 = r_5 = 0$$

and $\mathbf{u} = (2,1,1)\in U_5$.

Determine $\varepsilon_5^{(\mathbf{u},3)}$ for an arbitrary vector

$$\mathbf{a} = (\alpha_1,\ldots,\alpha_{10})\in Z_2^{10},$$

$$\varepsilon_5^{(\mathbf{u},3)}(\mathbf{a}) = (\varepsilon_5^{(\mathbf{u},3)}(\alpha_1), \varepsilon_5^{(\mathbf{u},3)}(\alpha_2), \varepsilon_5^{(\mathbf{u},3)}(\alpha_3),$$

$$\varepsilon_5^{(\mathbf{u},3)}(\alpha_4), \varepsilon_5^{(\mathbf{u},0)}(\alpha_5), \varepsilon_5^{(\mathbf{u},1)}(\alpha_6), \varepsilon_5^{(\mathbf{u},2)}(\alpha_7), \varepsilon_5^{(\mathbf{u},3)}(\alpha_8), \varepsilon_5^{(\mathbf{u},3)}(\alpha_9), \varepsilon_5^{(\mathbf{u},3)}(\alpha_{10})) =$$
$$= (\alpha_1, \alpha_2, 2\alpha_3, 4\alpha_4, 6\alpha_5, 6\alpha_6, 7\alpha_7, 8\alpha_8, 9\alpha_9, 9\alpha_{10}).$$

Construct sequentially

$$F_5^{(\mathbf{u},3,3)}, F_6^{(\mathbf{u},3,3)}, F_7^{(\mathbf{u},2,3)}, F_8^{(\mathbf{u},1,3)}$$

according to the rule:

$$F_{j+k}^{(\mathbf{u},r_k,s)} = \left\{\mathbf{a}\in h(L_{j+k}^{*} 0\ldots 0) \mid v_j^{(\mathbf{u},s)}(\mathbf{a}) \le 2\cdot 1 + 1\cdot 2 + 1\cdot 2^2\right\},$$

$$F_5^{(\mathbf{u},3,3)} = \{(0,0,0,0,1,0,0,0,0,0),(1,0,0,0,1,0,0,0,0,0),(0,1,0,0,1,0,0,0,0,0),$$

$$(1,1,0,0,1,0,0,0,0,0),(0,0,1,0,1,0,0,0,0,0)\};$$

$$F_6^{(\mathbf{u},3,3)} = \{(0,0,0,0,0,1,0,0,0,0),(1,0,0,0,1,0,0,0,0,0),(0,1,0,0,0,1,0,0,0,0),$$

$$(1,1,0,0,0,1,0,0,0,0),(0,0,1,0,0,1,0,0,0,0)\};$$

$$F_5^{(\mathbf{u},2,3)} = \{(0,0,0,0,0,0,1,0,0,0),(1,0,0,0,0,0,1,0,0,0),(0,1,0,0,0,0,1,0,0,0)\};$$

$$F_8^{(\mathbf{u},1,3)} = \{(0,0,0,0,0,0,0,1,0,0)\}.$$

In accordance with Theorem 9,

$$\omega_1 = \omega_2 = -1, \omega_3 = -2, \omega_4 = -4,$$

$$\omega_5 = \omega_6 = -6, \omega_7 = -7, \omega_8 = -8,$$

$$\omega_9 = \omega_{10} = -9.$$

Thus, if

$$K(f_{\chi},0) = f_{\chi}^{-1}(1),$$

then a neural element with a weight vector

$$\mathbf{w}_1 = \mathbf{a}_i \mathbf{w}^{\sigma^{-1}} = (-9,-9,-8,-7,-6,6,4,2,1,1)$$

and threshold [20]

$$\omega_0 = (\mathbf{a}_i \mathbf{x}^{\sigma^{-1}}, \mathbf{w}_1) = 6$$

($\mathbf{x} = (\mathbf{z},0,0,0,0,0)$, $\mathbf{z}$ is the last row of the matrix of tolerance L_5) realizes function $f(x_1,\ldots,x_{10})$ with respect to the system of characters χ in alphabet $\{0,1\}$. In the opposite case,

$$K(f_{\chi},0) = f_{\chi}^{-1}(-1),$$

the function $f(x_1,\ldots,x_{10})$ is realized by a neural element with a vector of structure

$$[\mathbf{w}_2 = (9,9,8,7,6,-6,-4,-2,-1,-1);-5]$$

relative to the system of characters χ in alphabet $\{0,1\}$.

Remark. The operation $\mathbf{aw}$ of a boolean vector

$$\mathbf{a} = (\alpha_1,\ldots\alpha_m)$$

to a real vector

$$\mathbf{w} = (\omega_1,\ldots,\omega_m)$$

is defined as follows:

$$\mathbf{aw} = ((-1)^{\alpha_1}\omega_1,\ldots,(-1)^{\alpha_m}\omega_m).$$

5. Discussion of results obtained in the study of synthesis of generalized neural elements

On the basis of the obtained necessary and sufficient conditions for realizability of the logic algebra functions by one generalized neural element in relation to the system of characters $\chi = \{\chi_{i_1},\ldots,\chi_{i_m}\}$, construct an algorithm for synthesis of such neural elements.

Algorithm for the synthesis of generalized neural elements

Step 1. Let the Boolean function $f{:}G_n \to H_2$ and the system of characters $\chi = \{\chi_{i_1},\ldots,\chi_{i_m}\}$ of the group G_n be defined over the field R. Search for nucleus $K(f_{\chi})$. If the nucleus exists, then proceed to step 2 and in the opposite case, make a conclusion that function f is not realized by one generalized neural element relative to the system of characters

$$\chi=\{\chi_{i_1},\ldots,\chi_{i_m}\}$$

and the algorithm completes its work.

Step 2. Check the conditions necessary for realizability of the function f by one generalized neural element relative to the system χ (Theorems 3–5). If the conditions of at least one of the theorems are not fulfilled, then the function f is not realized by a single GNE with respect to the system χ and the algorithm completes its work, or, in the opposite case, denote the extended reduced nucleus which satisfies the conditions of all three theorems by

$$K(f_\chi,s)(s\geq 0)$$

and proceed to step 3.

Step 3. Based on the reduced nucleus $K(f_\chi,s)(s\geq 0)$, construct sequentially the elements of the set of the reduced nuclei $T(f_\chi,s)$, apply Theorem 6 to the constructed reduced nucleus and verify equality (23). If equality (23) is fulfilled, then, according to Theorem 7, find the vector of the structure of the generalized neuron element with respect to the system of characters χ, which realizes function f in alphabet {0,1} and the synthesis of the GNE is completed. If no reduced nucleus with $T(f_\chi,s)$ satisfies equality (23), then check conditions of Theorems 8 and 9 relative to

$$K(f_\chi,s)(s\geq 0).$$

If

$$K(f_\chi,s)(s\geq 0)$$

satisfies conditions of Theorem 8, then the vector of structure of the generalized neuron element that realizes function f relative to the system χ in alphabet $\{0,1\}$ is to be found by Theorem 5 [23].

If $K(f_\chi,s)(s\geq 0)$ satisfies conditions of Theorem 9, then vector of the GNE structure which realizes function f in alphabet {0,1} with respect to the system χ, must be found in accordance with this theorem.

If the conditions of any of the three theorems (7, 8, 9) are not fulfilled, then synthesis of the GNE for realization of function f with respect to the system χ is not successful and the algorithm completes its work.

Remark 1. If the number of entries to the GNE is not limited from above (the maximum number of entries for realization of the function $f{:}G_n\to H_2$ in alphabet {0,1} is 2^n-1), then expand the system χ by adding new character(s) of the groups G_n over field R.

Example. Let $f{:}G_n\to H_2$ be a Boolean function,

$$\chi=\{\chi_{i_1},\ldots,\chi_{i_8}\}$$

is a system of characters of the group G_n over the field R (n is a natural integer satisfying the inequality $2^n>8$) and

$$f^{-1}(1)=\{(11100001),\ (11000001),\ (11110001),\ (11010001),\ (11101001),(11001001),(11111001),(01100001),(01000001),(10100001)\}.$$

Let

$$K(f_\chi)=f^{-1}(1),$$

ξ be a unit element of the group

$$S_{10},\mathbf{a}_1=(11100001).$$

Then

$$K_\xi(f_\chi)=\begin{pmatrix}1&1&1&0&0&0&0&1\\1&1&0&0&0&0&0&1\\1&1&1&1&0&0&0&1\\1&1&0&1&0&0&0&1\\1&1&1&0&1&0&0&1\\1&1&0&0&1&0&0&1\\1&1&1&1&1&0&0&1\\0&1&1&0&0&0&0&1\\0&1&0&0&0&0&0&1\\1&0&1&0&0&0&0&1\end{pmatrix}\xrightarrow{\mathbf{a}_1K_\xi(f_\chi)}\begin{pmatrix}0&0&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&1&1&0&0&0&0\\0&0&0&0&1&0&0&0\\0&0&1&0&1&0&0&0\\0&0&0&1&1&0&0&0\\1&0&0&0&0&0&0&0\\1&0&1&0&0&0&0&0\\0&1&0&0&0&0&0&0\end{pmatrix}\xrightarrow{\sigma}$$

$$\xrightarrow{\sigma}\begin{pmatrix}0&0&0&0&0&0&0&0\\1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\1&1&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\1&0&1&0&0&0&0&0\\0&1&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\1&0&0&1&0&0&0&0\\0&0&0&0&1&0&0&0\end{pmatrix}=K_\xi^\sigma(f_\chi)_1,$$

where

$$\sigma=\begin{pmatrix}1&2&3&4&5&6&7&8\\4&5&1&2&3&6&7&8\end{pmatrix}.$$

Element σ is determined by Theorem 6. The reduced nucleus $K(f_\chi)_1$ satisfies conditions of Theorems 3–5 and equality (23) takes place:

$$K_\xi^\sigma(f_\chi)_1=\left(\begin{array}{ll}\left.\begin{array}{c}00000000\\10000000\\01000000\\11000000\end{array}\right\} & L_3\\ \left.\begin{array}{c}00100000\\10100000\\01100000\end{array}\right\} & L_3^*(3)\\ \left.\begin{array}{c}00010000\\10010000\end{array}\right\} & L_4^*(2)\\ \left.\begin{array}{c}00001000\end{array}\right\} & L_5^*(1)\end{array}\right).$$

Therefore,

$$K_\xi^\sigma(f_\chi)_1=$$

$$=\left(L_3 00000\right)\nabla\left(L_3^*(3)00000\right)\nabla\left(L_4^*(2)0000\right)\nabla\left(L_5^*(1)000\right).$$

Denote last rows of the blocks

$$\left(L_3^*(3)00000\right),\ \left(L_4^*(2)0000\right),\ \left(L_5^*(1)000\right)$$

by

$$\mathbf{z}_0=(01100000),\ \ \mathbf{z}_1=(10010000),\ \ \mathbf{z}_2=(00001000)$$

respectively, then vector of the structure of the generalized neural element in alphabet {0,1} which realizes function $f{:}G_n\rightarrow H_2$ with nucleus

$$K(f_\chi)=f^{-1}(1)$$

in respect to the system of characters $\chi=\{\chi_{i_1},\ldots,\chi_{i_8}\}$, will be found in accordance with Theorem 7:

$$\mathbf{w}=\left(\omega_1,\omega_2,\omega_3,\omega_4,\omega_5,\omega_6,\omega_7,\omega_8;\omega_0\right);$$

$$\mathbf{w}_1=\left(\omega_1,\omega_2,\omega_3\right);$$

$$\omega_1=-1,\omega_2=\omega_1-1=-2,\ \omega_3=\omega_1+\omega_2-1=-4;$$

$$\mathbf{w}_1=\left(-1,-2,-4\right);$$

$$\mathbf{w}_2=\left(-1,-2,-4,\omega_4\right);$$

$$\left(\mathbf{z}_0,\mathbf{w}_1\right)=\left(\mathbf{z}_1,\mathbf{w}_2\right)\Rightarrow\omega_4=-5;$$

$$\mathbf{w}_3=\left(-1,-2,-4,-5,\omega_5\right);$$

$$\left(\mathbf{z}_1,\mathbf{w}_2\right)=\left(\mathbf{z}_2,\mathbf{w}_3\right)\Rightarrow\omega_5=-6;$$

$$\mathbf{w}=\left(-1,-2,-4,-5,-6,\omega_6,\omega_7,\omega_8\right);$$

$$\omega_6=\omega_7=\omega_8=\left(\mathbf{z}_2,\mathbf{w}_3\right)-1=-7;$$

$$\mathbf{w}=\left(-1,-2,-4,-5,-6,-7,-7,-7\right);$$

$$\begin{aligned}&\mathbf{w}^*=\mathbf{a}\mathbf{w}^{\sigma^{-1}}=\\&=\left(1,1,1,0,0,0,0,1\right)\left(-5,-6,-1,-2,-4,-7,-7,-7\right)=\\&=(5,6,1,-2,-4,-7,-7,7).\end{aligned}$$

The threshold ω_0 is defined as follows:

$$\omega_0=(\mathbf{a}_1\mathbf{z}_2^{\sigma^{-1}},\mathbf{w}^*)=13.$$

If $K(f)=f^{-1}(1)$, then the generalized neural element with a weight vector

$$\mathbf{w}^*=(5,6,1,-2,-4,-7,-7,7)$$

and threshold $\omega_0=13$ realizes function f in alphabet {0,1} with respect to the system of characters

$$\chi=\{\chi_{i_1},\ldots,\chi_{i_8}\},$$

In the opposite case, that is, when

$$K(f)=f^{-1}(-1),$$

function f is realized by one generalized neural element in alphabet {0,1} with a weight vector

$$\mathbf{w}^{**}=-\mathbf{w}^*=(-5,-6,-1,2,4,7,7,-7)$$

and threshold

$$\omega_0^*=-\omega_0+1=-12.$$

The connection between the vectors of structure of the neural elements which realize the same function in different alphabets {0,1} and {–1,1}, was established in [20].

6. Conclusions

1. Expansion of functional capabilities of neural elements by generalization of activation functions provides for a more efficient use of these elements in the tasks of processing discrete signals and images. However, in order to successfully apply generalized neural elements in the field of compression and transmission of discrete signals, classification and recognition of discrete images, it is necessary to have practically suitable methods for checking realizability of the logic algebra functions on such elements and the methods of synthesis of these elements with a large number of entries.

2. Based on the results presented in the paper on the structure of nuclei and extended nuclei of Boolean functions with respect to the system of characters and properties of the matrices of tolerance, the following was established:

– if there is a nucleus relative to a specified system of characters for a Boolean function, then the function is realized by one generalized neural element with respect to the system of characters if and only if the nucleus or extended nucleus of the function admits representation by the matrices of tolerance;

– efficient conditions are required for checking realizability of Boolean functions by one generalized neural element in relation to the system of characters;

– conditions sufficient for realizability of the logic algebra functions by one generalized neural element in respect to the system of characters on the basis of which an algorithm of synthesis of integer-valued generalized neural elements with a large number of entries has been developed.

3. The results obtained in the work can be used in working out effective methods for synthesizing neural network schemes from integer-valued generelized neural elements with a large number of entries for encoding, classification and recognition of discrete signals and images.

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