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ON ONE CONSTRUCTIVE METHOD OF THE BOUNDARY–VALUE PROBLEM INVESTIGATION FOR THE DIFFERENTIAL EQUATIONS OF THE HYPERBOLIC TYPE

We investigate one constructive modification of the two–sided method of the approximate integration of the boundary–value problem (BVP) for a system of second order non–linear differential equations of the hyperbolic type on the plain, when a bound of the domain of the independent variables consists of a pair of "free" curves and characteristics of the given system.

Досліджується одна конструктивна модифікація двостороннього методу наближеного інтегрування крайової задачі для систем нелінійних диференціальних рівнянь другого порядку гіперболічного типу на площині, коли край області зміни незалежних змінних складається із пари "вільних"кривих та характеристик заданої системи.

The results given below continue our investigations from [1, 2].

Let us observe the domain $D = D_1 \cup D_2 \subset \mathbb{R}^2$, where

$$D_1 = \{(x, y)\,|x \in (x_0, x_1]\,, y \in (y_0, g_1(x))\}\,,$$

$$D_2 = \{(x, y)\,|x \in [x_1, x_2]\,, y \in (g_2(x), y_1]\}\,,$$

and $x_0 < x_1 < x_2$, $y_0 < y_1 < y_2$, $y = g_r(x)$ $(x = k_r(y))$, $x \in [x_{r-1}, x_r]$, $r = 1, 2$, are "free" curves [3], $g_r'(x) > 0$, $g_1(x_{r-1}) = y_r$, $g_2(x_r) = y_{r-1}$.

Let us investigate the problem [4]: to find the solution of the system of differential equations

$$L_2U(x, y) = f\,(x, y, U(x, y)) := f\,[U(x, y)]\,, \tag{1}$$

$$L_2U(x, y) := U_{xy}(x, y) + A_1(x, y)U_x(x, y) + A_2(x, y)U_y(x, y),$$

$U(x, y) := (u_i(x, y))$, $f[U(x, y)] := (f_i[U(x, y)])$, $i = \overline{1, n}$ — vector–functions, $A_r(x, y) := \left(\delta_{i,j}a_{i,j}^{(r)}(x, y)\right)$, $r = 1, 2$, $j = \overline{1, n}$, are given matrices, $\delta_{i,j}$ is the Kronecker symbol, that satisfies boundary restrictions of the form

$$\begin{gathered} U(x_0, y) = \Psi(y), \Psi(y) \in C^1[y_0, y_1], \\ U(x, y_0) = \Phi(x), \Phi(x) \in C^1[x_0, x_1], \\ \Psi(y_0) = \Phi(x_0), \end{gathered} \tag{2}$$

$$\begin{gathered} U(x, g_r(x)) = \Omega_r(x), x \in [x_{r-1}, x_r], \Omega_r(x) \in C^1[x_{r-1}, x_r], r = 1, 2, \\ \Omega_1(x_0) = \Psi(y_1), \Omega_2(x_1) = \Phi(x_1), \end{gathered} \tag{3}$$

where $\Psi(y) := (\psi_i(y))$, $\Phi(x) := (\phi_i(x))$, $\Omega_r(x) := (\omega_{i,r}(x))$, $i = \overline{1, n}, r = 1, 2$ — given functions, in the space of vector–functions

$$C^*(\bar{D}) := C^{(1.1)}(D) \cap C(\bar{D}).$$

Let us devide the domain D_1 by characteristic of the system (1) $y = y_1$ on two subdomains $D_{1,r}, r = 1, 2$, where

$$D_{1,1} = \{(x, y)|x \in (x_0, x_1], y \in (y_0, y_1]\}\,,$$

$$D_{1,2} = \{(x,y)|x \in [x_0,x_1], y \in [y_1, g_1(x))\}.$$

It is clear that the solution of the BVP (1)–(3)

$$U(x,y) = \begin{cases} U_1(x,y), (x,y) \in \bar{D_{1,1}}, \\ U_2(x,y), (x,y) \in \bar{D_{1,2}}, \\ U_3(x,y), (x,y) \in \bar{D}_2, \end{cases}$$

where $U_1(x,y)$ is the solution the the Goursat problem (1), (2) for $(x,y) \in \bar{D_{1,1}}$ and $U_s(x,y), s=1,2$ are the solutions of the Darboux problems (1), (3) for $(x,y) \in \bar{D_{1,2}}$ and $(x,y) \in \bar{D}_2$; $U_2(x,y_1) = U_1(x,y_1)$ and $U_3(x_1,y) = U_1(x_1,y)$, $U_s(x,y) := (u_{s,i}(x,y))$ are unknown vector–functions.

Forehead we suppose that $A_1(x,y) \in C(D) \cap C^{(1.0)}(D_{1,1} \cup D_2)$, $A_2(x,y) \in C(D) \cap C^{(0.1)}(D_1)$, $f[U(x,y)] \in C(\bar{B})$, $f : \bar{B} \to \mathbb{R}^n$, $\bar{B} \subset \mathbb{R}^{n+2}$.

Using the notations from [1] let's write the Goursat problem (1), (2) and the Darboux problem (1), (3) in the equivalent integral form

$$U_s(x,y) = \Gamma_s(x,y) + \epsilon_s T_{1,s} F[U_1(\xi,\eta) + T_s F[U_s(\xi,\eta)]], \qquad (4)$$

for $(x,y) \in \bar{D_{1,s}}, s = 1,2$, and $(x,y) \in \bar{D}_2, s = 3$, where

$$\epsilon_s = \begin{cases} 0, s=1, \\ 1, s=2,3, \end{cases} \quad F[U(x,y)] := \begin{cases} F^*[U(x,y)], (x,y) \in \bar{D}_1, \\ F^{**}[U(x,y)], (x,y) \in \bar{D}_2, \end{cases}$$

$$F^*[U(x,y)] := f[U(x,y)] + [A_{2y}(x,y) + A_1(x,y)A_2(x,y)]U(x,y),$$

$$F^{**}[U(x,y)] := F^*[U(x,y)] + [A_{1x}(x,y) - A_{2y}(x,y)]U(x,y),$$

$$T_1 F[U_1(\xi,\eta)] := \int\limits_{x_0}^{x} \int\limits_{y_0}^{y} K(x,y;\xi,\eta) F[U_1(\xi,\eta)] d\eta d\xi, (x,y) \in \bar{D}_{1,1},$$

$$T_2 F[U_2(\xi,\eta)] := \int\limits_{k_1(y)}^{x} \int\limits_{y_1}^{y} K(x,y;\xi,\eta) F[U_2(\xi,\eta)] d\eta d\xi, (x,y) \in \bar{D}_{1,2},$$

$$T_3 F[U_3(\xi,\eta)] := \int\limits_{g_2(x)}^{y} \int\limits_{x_1}^{x} K^{-1}(\xi,\eta;x,y) F[U_3(\xi,\eta)] d\xi d\eta, (x,y) \in \bar{D}_2,$$

$K(x,y;\xi,\eta) = (\delta_{i,j} k_{i,j})(x,y;\xi,\eta)$, $K^{-1}(\xi,\eta;x,y) = (\delta_{i,j} k_{i,j}^{-1})(\xi,\eta;x,y)$ — matrixes,

$$k_{i,i}(x,y;\xi,\eta) := exp\left(\int\limits_{x}^{\xi} a_{i,i}^{(2)}(\tau,y)d\tau + \int\limits_{y}^{\eta} a_{i,i}^{(1)}(\xi,\tau)d\tau\right),$$

$\Gamma_s(x,y) = (\gamma_{s,i}(x,y))$, $s = 1,2,3$ — vector–functions,

$$\gamma_{1,i} := \psi_i(y) exp\left(\int\limits_{x}^{x_0} a_{i,i}^{(2)}(\xi,y)d\xi\right) +$$

$$+ \int\limits_{x_0}^{x} k_{i,i}(x,y;\xi,y_0)[\phi_i'(\xi) + a_{i,i}^{(2)}(\xi,y_0)\phi_i(\xi)]d\xi, (x,y) \in \bar{D}_{1,1},$$

$$\gamma_{2,i} := \omega_{i,1}(k_1(y))exp\left(\int\limits_{x}^{k_1(y)} a_{i,i}^{(2)}(\xi,y)d\xi\right) + \\ + \int\limits_{k_1(y)}^{x} k_{i,i}(x,y;\xi,y_0)[\phi_i'(\xi) + a_{i,i}^{(2)}(\xi,y_0)\phi_i(\xi)]d\xi, (x,y) \in \bar{D}_{1,2},$$

$$\gamma_{3,i} := \omega_{i,2}(x)exp\left(\int\limits_{y}^{g_2(x)} a_{i,i}^{(1)}(x,\eta)d\eta\right) + \\ + \int\limits_{g_2(x)}^{y} k_{i,i}^{-1}(x_0,\eta;x,y)[\psi_i'(\eta) + a_{i,i}^{(1)}(x_0,\eta)\psi_i(\eta)]d\eta, (x,y) \in \bar{D}_2,$$

$$T_{1,2}F[U_1(\xi,\eta)] := \int\limits_{k_1(y)}^{x}\int\limits_{y_0}^{y_1} K(x,y;\xi,\eta)F[U_1(\xi,\eta)]d\eta d\xi, (x,y) \in \bar{D}_{1,2},$$

$$T_{1,3}F[U_1(\xi,\eta)] := \int\limits_{g_2(x)}^{y}\int\limits_{x_0}^{x_1} K^{-1}(\xi,\eta;x,y)F[U_1(\xi,\eta)]d\xi d\eta, (x,y) \in \bar{D}_2,$$

Remark 1. *If $A_{1x}(x,y) = A_{2y}(x,y)$, $(x,y) \in D$, then $F^*[U(x,y)] \equiv F^{**}[U(x,y)]$ and $K(x,y;\xi,\eta) \equiv K^{-1}(\xi,\eta;x,y)$.*

According to the problem setting $U_{1x}(x,y_1) = U_{2x}(x,y_1)$ and $U_{1y}(x_1,y) = U_{3y}(x_1,y)$ for $x \in [x_0,x_1]$ and $y \in [y_0,y_1]$,

$$\begin{aligned} u_{1,iy}(x,y_1) - u_{2,iy}(x,y_1) &= \rho_{1,i}exp\left(\int\limits_{x}^{x_0} a_{i,i}^{(2)}(\xi,y_1)d\xi\right), x \in [x_0,x_1], \\ u_{1,ix}(x_1,y) - u_{3,ix}(x_1,y) &= \rho_{2,i}exp\left(\int\limits_{y}^{y_0} a_{i,i}^{(1)}(x_1,\eta)d\eta\right), y \in [y_0,y_1], \end{aligned} \tag{5}$$

where

$$\rho_{1,i} := \psi'(y_1) - k_1'(y_1)\Bigg\{\omega_{i,1}'(x_0) + a_{i,i}^{(2)}(x_0,y_1)\omega_{i,1}(x_0) - \Big[\phi_i'(x_0) + a_{i,i}^{(2)}(x_0,y_0)\phi_i(x_0)\Big] \times \\ \times \exp\left(\int\limits_{y_1}^{y_0} a_{i,i}^{(1)}(x_0,\eta)d\eta\right) - \int\limits_{y_0}^{y_1}[f_i(x_0,\eta,\psi_1(\eta),\ldots,\psi_n(\eta)) + \\ + \left(a_{i,i\eta}^{(2)}(x_0,\eta) + a_{i,i}^{(1)}(x_0,\eta)a_{i,i}^{(2)}a_{i,i}^{(2)}(x_0,\eta)\right) \times \psi_i(\eta)\Big] \exp\left(\int\limits_{y_1}^{\eta} a_{i,i}^{(1)}(x_0,\tau)d\tau\right) d\eta\Bigg\},$$

$$\rho_{2,i} := \phi'(x_1) - \omega'_{i,2}(x_1) - g'_2(x_1)\Big\{ a^{(1)}_{i,i}(x_1, y_0)\omega_{i,2}(x_1) - \Big[\phi'_i(y_0) + a^{(1)}_{i,i}(x_0, y_0)\phi_i(y_0)\Big] \times$$

$$\times \exp\left(\int\limits_{x_1}^{x_0} a^{(2)}_{i,i}(\xi, y_0)d\xi \right) - \int\limits_{x_0}^{x_1} [f_i(\xi, y_0, \phi_1(\xi), \dots, \phi_n(\xi))+$$

$$+a^{(1)}_{i,i\xi}(\xi, y_0) + a^{(1)}_{i,i}(\xi, y_0)a^{(2)}_{i,i}(\xi, y_0)\phi_i(\xi)\Big] \exp\left(\int\limits_{x_1}^{\xi} a^{(2)}_{i,i}(\tau, y_0)d\tau \right) d\xi \Big\}.$$

Lemma 1. *Suppose that* $f[U(x,y)] \in C(\bar{B})$, $A_r(x,y) \in C(D)$, $r = 1,2$, $A_1(x,y) \in C^{(1.0)}(D_1 \cup D_3)$, $A_2(x,y) \in C^{(0.1)}(D_1 \cup D_2)$ *and the BVP (1)–(3) has a solution.*

The solution of the BVP (1)–(3) is regular (i.e. $U(x,y) \in C^*(\bar{D})$*) if and only if the equality* $\rho_{r,i} = 0$ *is true for all* $r = 1,2$ *and* $i = \overline{1,n}$.

In the opposite case the equalities (5) take place and the solution of the BVP (1)–(3) is irregular.

Definition 1. *We say that* $F[U(x,y)] \in C_1^*(\bar{B})$, *if the vector–function* $F[U(x,y)]$ *satisfies following conditions [5]:*
1) $F[U(x,y)] \in C(\bar{B})$,
2) in the space of vector–functions $C(\bar{B}_1)$, $\bar{B}_1 \subset \mathbb{R}^{2(n+1)}$, $Pr_{xOy}\bar{B}_1 = \bar{D}$, *there exists vector–function* $H(x,y,U(x,y);V(x,y)) := H[U(x,y);V(x,y)]$, *such that*
a) $H[U(x,y);U(x,y)] \equiv F[U(x,y)]$,
b) for any pair of vector–functions $U(x,y)$, $V(x,y) \in \bar{B}_1$, *from* $C(\bar{D})$ *such that* $U(x,y) \geq V(x,y)$, $(x,y) \in \bar{D}$, *in* $\bar{B}_1$ *the inequality*

$$H[U(x,y);V(x,y)] - H[V(x,y);U(x,y)] \leq 0, \tag{6}$$

holds
3) the vector–function $H[U(x,y);V(x,y)]$ *in* $\bar{B}_1$ *satisfies the Lipschitz condition, i.e. for all vector–functions* $U_r(x,y)$, $V_r(x,y) \in \bar{B}_1$, $r = 1,2$, *from* $C(\bar{D})$ *the condition*

$$|H[U_1(x,y);U_2(x,y)] - H[V_1(x,y);V_2(x,y)]| \leq L(|W_1(x,y)| + |W_2(x,y)|),$$

holds, where $W_r(x,y) := U_r(x,y) - V_r(x,y)$, $r = 1,2$, $L = (\delta_{i,j}l_{i,j})$ *— is the Lipschitz matrix,* $l_{i,j} \geq 0$, $i,j = \overline{1,n}$.

It is clear that if the vector–function $f[U(x,y)] \in C(\bar{B})$ and has limited first order partial derivatives on all its arguments, starting with the third one, then $F[U(x,y)]$ always belongs to the space $C_1^*(\bar{B})$. The opposite statement isn't true.

Let's give the sufficient conditions of existence and uniqueness of regular or irregular solution of (1)–(3) for $(x,y) \in \bar{D}$.

Let vector–functions $Z_{s,p}(x,y) := (z_{s,i,p}(x,y)), V_{s,p}(x,y) := (v_{s,i,p}(x,y)) \in C(\bar{D})$ belong to the domain $\bar{B}_1$, $s = 1,2,3$, $p \in \mathbb{N}$.

We will put:

$$W_{s,p}(x,y) := Z_{s,p}(x,y) - V_{s,p}(x,y),$$

$$(x,y) \in \bar{D}_s, \bar{D}_s := \begin{cases} \bar{D}_{1,s}, s = 1,2, \\ \bar{D}_2, s = 3, \end{cases}$$

$$f_s^p(x,y) := H[Z_{s,p}(x,y);V_{s,p}(x,y)],$$

$$\begin{aligned} f_{s,p}(x,y) &:= H[V_{s,p}(x,y); Z_{s,p}(x,y)], \\ \alpha_{s,p}(x,y) &:= Z_{s,p}(x,y) - \Gamma_s(x,y) - \epsilon_s T_{1,s} f_1^p(\xi,\eta) - T_s f_s^p(\xi,\eta), \\ \beta_{s,p}(x,y) &:= V_{s,p}(x,y) - \Gamma_s(x,y) - \epsilon_s T_{1,s} f_{1,p}(\xi,\eta) - T_s f_{s,p}(\xi,\eta), \\ R_s^p(x,y) &:= \Gamma_s(x,y) + \epsilon_s T_{1,s} f_1^p(\xi,\eta) + T_s f_s^p(\xi,\eta), \\ R_{s,p}(x,y) &:= \Gamma_s(x,y) + \epsilon_s T_{1,s} f_{1,p}(\xi,\eta) + T_s f_{s,p}(\xi,\eta), \\ \bar{Z}_{s,p}(x,y) &:= Z_{s,p}(x,y) - Q_{s,p}(x,y) W_{s,p}(x,y), \\ \bar{V}_{s,p}(x,y) &:= V_{s,p}(x,y) + C_{s,p}(x,y) W_{s,p}(x,y), p \in \mathbb{N}, \\ \bar{\alpha}_{s,p}(x,y) &:= \bar{Z}_{s,p} - \Gamma_s(x,y) - \epsilon_s T_{1,s} F_1^p(\xi,\eta) - T_s F_s^p(\xi,\eta), \\ \bar{\beta}_{s,p}(x,y) &:= \bar{V}_{s,p} - \Gamma_s(x,y) - \epsilon_s T_{1,s} F_{1,p}(\xi,\eta) - T_s F_{s,p}(\xi,\eta), \end{aligned} \tag{7}$$

$Q_{s,p}(x,y) := (\delta_{i,j} q_{s,i,p}(x,y))$, $C_{s,p}(x,y) := (\delta_{i,j} c_{s,i,p}(x,y))$, $i,j = \overline{1,n}$ are some arbitrary matrixes with non–negative continuous components for $(x,y) \in \bar{D}_s$, $s = 1,2,3$, satisfying the following conditions

$$0 \le c_{s,i,p}(x,y) \le 0,5, 0 \le q_{s,i,p}(x,y) \le 0,5, \tag{8}$$

for all $p \in \mathbb{N}$, $i = \overline{1,n}$ ($c_{s,i,0} = q_{s,i,0} = 0$) and $(x,y) \in \bar{D}_s$,

$$F_s^p(x,y) := H[\bar{Z}_{s,p}(x,y); \bar{V}_{s,p}(x,y)],$$

$$F_{s,p}(x,y) := H[\bar{V}_{s,p}(x,y); \bar{Z}_{s,p}(x,y)],$$

$R_s^p(x,y) = (r_{s,i}^p(x,y))$, $R_{s,p}(x,y) = (r_{s,i,p}(x,y))$ are vector–functions.

Lets built the sequences of functions $\{Z_{s,p}(x,y)\}$ and $\{V_{s,p}(x,y)\}$ according to [?, 5].

$$\begin{gathered} Z_{s,p+1}(x,y) = \Gamma_s(x,y) + \epsilon_s T_{1,s} F_1^p(\xi,\eta) + T_s F_s^p(\xi,\eta), \\ V_{s,p+1}(x,y) = \Gamma_s(x,y) + \epsilon_s T_{1,s} F_{1,p}(\xi,\eta) + T_s F_{s,p}(\xi,\eta) \\ (x,y) \in \bar{D}_s, \end{gathered} \tag{9}$$

where as the zero approximation $Z_{s,0}(x,y), V_{s,0}(x,y) \in \bar{B}_1$ can be taken any functions from $C(\bar{D}_s)$, satisfying (2), (3) and the inequalities

$$\begin{gathered} W_{s,0}(x,y) \ge 0, \alpha_{s,0}(x,y) \ge 0, \beta_{s,0}(x,y) \ge 0, \\ (x,y) \in \bar{D}_s, s = 1,2,3. \end{gathered} \tag{10}$$

Forehead vector–functions $Z_{s,0}(x,y)$, $V_{s,0}(x,y) \in C(\bar{B}_1)$, such that satisfy conditions (2), (3), inequalities (10) and belong to the domain $\bar{B}_1$ we will call *comparison functions* of the BVP (1)– (3).

The following lemma takes place.

Lemma 2. *Let $F[U(x,y)] \in C_1^*(\bar{B})$ and the integral equations (4) in $C(\bar{D}_r)$, $r = 1,2$, have solutions such that for $(x,y) \in C(\bar{D}_r)$ satisfy inequalities*

$$V_{s,0}(x,y) \le U_s(x,y) \le Z_{s,0}(x,y), (x,y) \in \bar{D}_s, s = 1,2,3. \tag{11}$$

Then in $\bar{B}_1$ the inequalities (10) are true.

Lemma 3. *If $F[U(x,y)] \in C_1^*(\bar{B})$, then the set of comparison functions of the BVP (1)– (3) is non-empty.*

Proof. Let

$$u^*(x,y) = \Gamma_s(x,y) + \epsilon_s T_{1,s} F[u_1^*(\xi,\eta)] + T_s F[h(\xi,\eta)],$$

where $h(x,y) \in C(\bar{D})$ — is some function from $\bar{B}$. As $u_s(x,y) \in \bar{B}_1$, lets put

$$\alpha_s^*(x,y) = u_s^*(x,y) - \Gamma_s(x,y) - \epsilon_s T_{1,s} F[u_1^*(\xi,\eta)] - T_s F[u_s^*(\xi,\eta)].$$

Then the vector–functions

$$Z_{s,0}(x,y) = u_s^*(x,y) + |\alpha_s^*(x,y)|,$$
$$V_{s,0}(x,y) = u_s^*(x,y) - |\alpha_s^*(x,y)|$$

if $Z_{s,0}(x,y), V_{s,0}(x,y) \in \bar{B}_1$ are comparison functions of the BVP (1)– (3).

Indeed $W_{s,0}(x,y) \geq 0$, and as $K(x,y;\xi,\eta) > 0$, than taking into account condition (6), we have that

$$\alpha_{s,0}(x,y) = |\alpha_s^*(x,y)| + \alpha_s^*(x,y) + \epsilon_s T_{1,s} \left\{F[U_1^*(\xi,\eta)] - f_1^0(\xi,\eta)\right\} + \\ + T_s \left\{F[U_s^*(\xi,\eta)] - f_s^0(\xi,\eta)\right\} \geq 0,$$

$$\beta_{s,0}(x,y) = -|\alpha_s^*(x,y)| + \alpha_s^*(x,y) + \epsilon_s T_{1,s} \left\{F[U_1^*(\xi,\eta)] - f_{1,0}(\xi,\eta)\right\} + \\ + T_s \left\{F[U_s^*(\xi,\eta)] - f_{s,0}(\xi,\eta)\right\} \leq 0,$$

$s = 1,2,3, (x,y) \in \bar{D}_s$.

From (7) and (9) we get

$$Z_{s,0}(x,y) - Z_{s,1}(x,y) = \alpha_{s,0}(x,y) \geq 0,$$

$$V_{s,0}(x,y) - V_{s,1}(x,y) = \beta_{s,0}(x,y) \leq 0,$$

$$W_{s,1}(x,y) = \epsilon_s T_{1,s}(f_1^0(\xi,\eta) - f_{1,0}(\xi,\eta)) + T_s(f_s^0(\xi,\eta) - f_{s,0}(\xi,\eta)) \leq 0,$$

$$\alpha_{s,1}(x,y) = \epsilon_s T_{1,s}(f_1^0(\xi,\eta) - f_1^1(\xi,\eta)) + T_s(f_s^0(\xi,\eta) - f_s^1(\xi,\eta)) \leq 0, (x,y) \in \bar{D}_s,$$

$$\beta_{s,1}(x,y) = \epsilon_s T_{1,s}(f_{1,0}(\xi,\eta) - f_{1,1}(\xi,\eta)) + T_s(f_{s,0}(\xi,\eta) - f_{s,1}(\xi,\eta)) \geq 0.$$

Let the inequalities

$$V_{s,0}(x,y) \leq Z_{s,1}(x,y), Z_{s,0}(x,y) \geq V_{s,1}(x,y) \tag{12}$$

are true for $(x,y) \in \bar{D}_s$.

Using the upper inequalities for $(x,y) \in \bar{D}_s$ we get

$$V_{s,0}(x,y) \leq Z_{s,1}(x,y) \leq V_{s,1}(x,y) \leq Z_{s,0}(x,y), (x,y) \in \bar{D}_s, s = 1,2,3.$$

It means that if $Z_{s,0}(x,y), V_{s,0}(x,y) \in \bar{B}_1$, then $Z_{s,1}(x,y), V_{s,1}(x,y) \in \bar{B}_1$, $s = 1,2,3$.

Let us put

$$\Omega_s^p(x,y) := \Gamma_s(x,y) + \epsilon_s T_{1,s} F_1^p(\xi,\eta) + T_s F_s^p(\xi,\eta),$$

$$\Omega_{s,p}(x,y) := \Gamma_s(x,y) + \epsilon_s T_{1,s} F_{1,p}(\xi,\eta) + T_s F_{s,p}(\xi,\eta),$$

then the iterations (9) can be written as

$$Z_{s,p+1}(x,y) = \Omega_s^p(x,y), V_{s,p+1}(x,y) = \Omega_{s,p}(x,y), \tag{13}$$

$(x,y) \in \bar{D}_s$, $s = 1,2,3$, $p \in \mathbb{N}$, $\Omega_s^p(x,y) = (\omega_{s,i}^p(x,y))$, $\Omega_{s,p}(x,y) = (\omega_{s,i,p}(x,y))$ are vector–functions.

From (7) and (13) we have

$$\begin{aligned} Z_{s,p+1}(x,y) - \bar{V}_{s,p}(x,y) = \Omega_s^p(x,y) - \Omega_{s,p-1}(x,y) - C_{s,p}(x,y)W_{s,p}(x,y), \\ V_{s,p+1}(x,y) - \bar{Z}_{s,p}(x,y) = \Omega_{s,p}(x,y) - \Omega_s^{p-1}(x,y) + Q_{s,p}(x,y)W_{s,p}(x,y), \end{aligned} \tag{14}$$

$$W_{s,p+1}(x,y) = \Omega_s^p(x,y) - \Omega_{s,p}(x,y), (x,y) \in \bar{D}_s, s = 1,2,3, \tag{15}$$

$$\begin{aligned} \bar{\alpha}_{s,p+1}(x,y) = \Omega_s^p(x,y) - \Omega_s^{p+1}(x,y) - Q_{s,p+1}(x,y)W_{s,p+1}(x,y) = \\ = Z_{s,p+1}(x,y) - Z_{s,p+2}(x,y) - Q_{s,p+1}(x,y)W_{s,p+1}(x,y), \\ \bar{\beta}_{s,p+1}(x,y) = \Omega_{s,p}(x,y) - \Omega_{s,p+1}(x,y) + C_{s,p+1}(x,y)W_{s,p+1}(x,y) = \\ = V_{s,p+1}(x,y) - V_{s,p+2}(x,y) + C_{s,p+1}(x,y)W_{s,p+1}(x,y). \end{aligned} \tag{16}$$

In [7] was shown that the vector–functions $\bar{Z}_{s,p}(x,y)$, $\bar{V}_{s,p}(x,y) \in \bar{B}_1$ will be the comparison functions of the BVP (1)–(3) if and only if the inequalities

$$\bar{W}_{s,p}(x,y) \geq (\leq)0, \bar{\alpha}_{s,p}(x,y) \geq (\leq)0, \bar{\beta}_{s,p}(x,y) \leq (\geq)0$$

for even (odd) p, $(x,y) \in \bar{D}_s, s = 1,2,3, p \in \mathbb{N}$.

Note that according to (8)

$$Z_{s,1}(x,y) \leq \bar{Z}_{s,1}(x,y) \leq \bar{V}_{s,1}(x,y) \leq V_{s,1}(x,y),$$

i.e., $\bar{Z}_{s,1}(x,y), \bar{V}_{s,1}(x,y) \in \bar{B}_1$. But then using (6) from (15) for $p = 1$ $W_{s,2}(x,y) \geq 0$.

Choosing the components of matrixes $Q_{s,1}(x,y)$, $C_{s,1}(x,y)$ such that for $(x,y) \in \bar{D}_s$, $s = 1,2,3$ the conditions

$$\Omega_s^1(x,y) - \Omega_{s,0}(x,y) - C_{s,1}(x,y)W_{s,1}(x,y) \leq 0,$$

$$\Omega_{s,1}(x,y) - \Omega_s^0(x,y) + Q_{s,1}(x,y)W_{s,1}(x,y) \geq 0.$$

hold.

According to (14) for $p = 1$

$$Z_{s,2}(x,y) \leq \bar{V}_{s,1}(x,y), V_{s,2}(x,y) \geq \bar{Z}_{s,1}(x,y),$$

$$Z_{s,2}(x,y) - Z_{s,1}(x,y) + Q_{s,1}(x,y)W_{s,1}(x,y) \geq 0, (x,y) \in \bar{D}_s, s = 1,2,3,$$

$$V_{s,2}(x,y) - V_{s,1}(x,y) - C_{s,1}(x,y)W_{s,1}(x,y) \leq 0,$$

and so from (16) for $p = 0$ $\bar{\alpha}_{s,1}(x,y) \leq 0$, $\bar{\beta}_{s,1}(x,y) \geq 0$.

It follows that

$$\begin{aligned} V_{s,0}(x,y) \leq Z_{s,1}(x,y) \leq V_{s,2}(x,y) \leq Z_{s,2}(x,y) \leq \bar{V}_{s,1}(x,y) \leq \\ \leq V_{s,1}(x,y) \leq Z_{s,0}(x,y), (x,y) \in \bar{D}_s, s = 1,2,3, \end{aligned}$$

and $\alpha_{s,2}(x,y) \geq 0$, $\beta_{s,2}(x,y) \leq 0$.

But in this case, taking the components of $Q_{s,2}(x,y)$, $C_{s,2}(x,y)$ such that

$$\Omega_s^2(x,y) - \Omega_{s,1}(x,y) - C_{s,2}(x,y)W_{s,2}(x,y) \geq 0,$$

$$\Omega_{s,2}(x,y) - \Omega_s^1(x,y) + Q_{s,2}(x,y)W_{s,2}(x,y) \leq 0,$$

from (14), (15) for $p = 2$ we get

$$Z_{s,3}(x,y) \geq \bar{V}_{s,2}(x,y), V_{s,3}(x,y) \leq \bar{Z}_{s,2}(x,y),$$

$$W_{s,3}(x,y) \leq 0, \bar{\alpha}_{s,2}(x,y) \geq 0, \bar{\beta}_{s,2}(x,y) \leq 0, (x,y) \in \bar{D}_s, s = 1,2,3,$$

i.e., the inequalities

$$\begin{aligned} V_{s,0}(x,y) \leq \bar{Z}_{s,1}(x,y) \leq \bar{V}_{s,2}(x,y) \leq \bar{Z}_{s,3}(x,y) \leq \bar{V}_{s,1}(x,y) \leq \\ \leq \bar{V}_{s,3}(x,y) \leq \bar{Z}_{s,2}(x,y) \leq \bar{V}_{s,1}(x,y) \leq Z_{s,0}(x,y), \end{aligned}$$

and $\alpha_{s,3}(x,y) \leq 0$, $\beta_{s,3}(x,y) \geq 0$, take place.

Repeating upper thoughts and choosing on each step of iterations (13) the components of matrixes $Q_{s,p}(x,y)$ and $C_{s,p}(x,y)$ in such way that conditions

$$\begin{gathered} \Omega_s^p(x,y) - \Omega_{s,p-1}(x,y) - C_{s,p}(x,y)W_{s,p}(x,y) \geq (\leq)0, \\ \Omega_{s,p}(x,y) - \Omega_s^{p-1}(x,y) + Q_{s,p}(x,y)W_{s,p}(x,y) \leq (\geq)0, \\ (x,y) \in \bar{D}_s, s = 1,2,3, \end{gathered} \tag{17}$$

are true, for even (odd) p with the help of the method of mathematical induction we improve that functions of the sequences $\{Z_{s,p}(x,y)\}$, $\{V_{s,p}(x,y)\}$ built according to (9), (10), (8), (17) satisfy the inequalities

$$\begin{aligned} V_{s,2p}(x,y) \leq Z_{s,2p+1}(x,y) \leq V_{s,2p+2}(x,y) \leq Z_{s,2p+3}(x,y) \leq \\ \leq V_{s,2p+3}(x,y) \leq Z_{s,2p+2}(x,y) \leq V_{s,2p+1}(x,y) \leq Z_{s,2p}(x,y), \end{aligned} \tag{18}$$

and $\alpha_{s,2p}(x,y) \geq 0$, $\alpha_{s,2p+1}(x,y) \leq 0$, $\beta_{s,2p}(x,y) \leq 0$, $\beta_{s,2p+1}(x,y) \geq 0$, $(x,y) \in \bar{D}_s$, $s = 1,2,3$, $p = 0,1,2,\ldots$.

Let us put $\tau_{s,p}(x,y) := \alpha_{s,p}(x,y) - \beta_{s,p}(x,y) + W_{s,p}(x,y)$.

Lemma 4. *If $F[U(x,y)] \in C_1^*(\bar{B})$ then a set of functional matrixes $Q_{s,p}(x,y)$, $C_{s,p}(x,y)$, which elements satisfy conditions (8), such that inequalities (17) hold, is non–empty.*

Proof. Really, let us put

$$\begin{gathered} q_{s,i,p}(x,y) = \begin{cases} \frac{w_{s,i,p(x,y)} + \beta_{s,i,p}(x,y)}{\tau_{s,i,p}(x,y)}, w_{s,i,p}(x,y) \neq 0, \\ 0, w_{s,i,p}(x,y) = 0, \end{cases} \\ c_{s,i,p}(x,y) = \begin{cases} \frac{w_{s,i,p(x,y)} - \alpha_{s,i,p}(x,y)}{\tau_{s,i,p}(x,y)}, w_{s,i,p}(x,y) \neq 0, \\ 0, w_{s,i,p}(x,y) = 0, \\ (x,y) \in \bar{D}_s, p \in \mathbb{N}, \end{cases} \end{gathered} \tag{19}$$

$\tau_{s,p}(x,y) = (\tau_{s,i,p}(x,y))$, $W_{s,p}(x,y) = (w_{s,i,p}(x,y))$, $i = \overline{1,n}$ are vector–functions.

As

$$w_{s,i,p}(x,y)+\beta_{s,i,p}(x,y)=\omega^{p-1}_{s,i}(x,y)-r_{s,i,p}(x,y),$$
$$w_{s,i,p}(x,y)-\alpha_{s,i,p}(x,y)=r^{p}_{s,i}(x,y)-\omega_{s,i,p-1}(x,y),$$

then components of the matrixes $Q_{s,p}(x,y)$ and $C_{s,p}(x,y)$ given by (19) satisfy conditions (8) for all $(x,y)\in\bar{D}_s$, $s=1,2,3$, $i=\overline{1,n}$, $p\in\mathbb{N}$ and the substitution of the last ones into (17) gives that

$$\omega^{p}_{s,i}(x,y)-\omega_{s,i,p-1}(x,y)-c_{s,i,p}(x,y)w_{s,i,p}(x,y)=$$
$$=\omega^{p}_{s,i}(x,y)-r^{p}_{s,i}(x,y)+(r^{p}_{s,i}(x,y)-\omega_{s,i,p-1}(x,y))\left(1-\tfrac{w_{s,i,p}(x,y)}{\tau_{s,i,p}(x,y)}\right)\geq(\leq)0,$$

$$\omega^{p}_{s,i}(x,y)-\omega^{p-1}_{s,i}(x,y)+q_{s,i,p}(x,y)w_{s,i,p}(x,y)=$$
$$=\omega_{s,i,p}(x,y)-r_{s,i,p}(x,y)+(r_{s,i,p}(x,y)-\omega^{p-1}_{s,i}(x,y))\left(1-\tfrac{w_{s,i,p}(x,y)}{\tau_{s,i,p}(x,y)}\right)\leq(\geq)0,$$

for all even (odd) p, $(x,y)\in\bar{D}_s$, $s=1,2,3$, $i=\overline{1,n}$, $p\in\mathbb{N}$.

Theorem 1. *Let the vector–function*

$$F[U(x,y)]\in C^{*}_{1}(\bar{B}),A_1(x,y)\in C(D)\cap C^{(1.0)}(D_{1,1}\cup D_2),$$
$$A_2(x,y)\in C(D)\cap C^{(0.1)}(D_1).$$

Then the vector–functions $Z_{s,p}(x,y)$, $V_{s,p}(x,y)$ build according to (9), (10), (8), (17), satisfy inequalities (17) in the domain $\bar{B}_1$.

Let us show that the sequences of functions $\{Z_{s,p}(x,y)\}$, $\{V_{s,p}(x,y)\}$ converge uniformly in direct domains $\bar{D}_s$, $s=1,2,3$ to the unique solution of the system of integral equations (4). On behalf of the inequalities (18) in $\bar{B}_1$ it is sufficient to show that

$$\lim_{p\to\infty}W_{s,p}(x,y)=0,(x,y)\in\bar{D}_s,s=1,2,3.$$

We put

$$\max_{s,i}\sup_{\bar{D}_s}|w_{s,i,p}(x,y)|=d,||L||=l,$$
$$\max_{s,i,p}\sup_{\bar{D}_s}(1-c_{s,i,p}(x,y)-q_{s,i,p}(x,y))\leq q,$$
$$\max_{s,i}\sup_{\bar{D}_1\cup\bar{D}_s}\{k_{i,i}(x,y;\xi,\eta),k^{-1}_{i,i}(xi,\eta;x,y)\}\leq 0,5k,s=2,3,$$
$$\max\{1,\sup_{\bar{D}}(y-y_0+x-x_0)\}=\gamma.$$

Then from (15) on the basis of the method of mathematical induction we insure that the inequalities

$$\max_{s,i}\sup_{\bar{D}_s}|w_{s,i,p}(x,y)|:=||W_{s,p}(x,y)||\leq\frac{1}{p!}(lqk\gamma n|y-y_0+x-x_0|)^p\cdot d,\qquad(20)$$

i.e., $\lim\limits_{p\to\infty}W_{s,p}(x,y)=0$, so

$$\lim_{p\to\infty}Z_{s,p}(x,y)=\lim_{p\to\infty}V_{s,p}(x,y)=U_s(x,y),(x,y)\in\bar{D}_s,s=1,2,3.$$

are true.

Passing in (9) to limit for $p\to\infty$ we get that the limit function $U_s(x,y)$ is the solution of the system of integral equations (4) for $(x,y)\in\bar{D}_s,s=1,2,3$.

Theorem 2. *Let the conditions of Theorem 1 hold.*

Then the sequences of vector–functions $\{Z_{s,p}(x,y)\}$, $\{V_{s,p}(x,y)\}$, *built according to (9), (10), (8), (17), where as* $Z_{s,0}(x,y)$, $V_{s,0}(x,y) \in \bar{B}_1$ *are chosen the comparison functions of the BVP (1)–(3):*
1) are uniform convergent to the unique solution of the system of integral equations (4) for $(x,y) \in \bar{D}_s, s = 1,2,3,$
2) the estimations (21) are true,
3) the inequalities

$$\begin{gathered} V_{s,2p}(x,y) \le Z_{s,2p+1}(x,y) \le V_{s,2p+2}(x,y) \le Z_{s,2p+3}(x,y) \le \\ \le U_s(x,y) \le V_{s,2p+3}(x,y) \le Z_{s,2p+2}(x,y) \le V_{s,2p+1}(x,y) \le Z_{s,2p}(x,y), \\ (x,y) \in \bar{D}_s, s = 1,2,3, p \in \mathbb{N} \end{gathered} \tag{21}$$

are true in the domain $\bar{B}_1$*;*
4) the convergence of the iteration method (9), (10), (8), (17) isn't slower than the convergence of the two–sided method when $Q_{s,p}(x,y) = C_{s,p} = 0$.

To prove this theorem it is sufficient to repeat steps given in [1, 7].

References

1. *V. V. Marynets, K. V. Marynets.* On Goursat–Darboux boundary–value problem for systems of non–linear differential equations of hyperbolic type// Miskolc Mathematical Notes. — 2013. — **14**, No 3. — P. 1009–1020.
2. *В. В. Маринець, К. В. Маринець.* Крайова задача Гурса–Дарбу для нелінійного рівняння гіперболічного типу// Доповіді НАНУ. – 2013. – No10. – С. 23–28.
3. *R. Courant.* Partial differential equations. NEW YORK —LONDON, 1962.
4. *L. Collatz.* Funktionalanalysis und numerische mathematic. Berlin–*G*öttingen–Heidelberg: Springlerverlag, 1964.
5. *Marynets V. V. and Dobryden A. V.* About one characteristic initial value problem// Nonlinear Oscillations. — 2001. — **4**, No4. — P. 487–499.
6. *Маринець В. В., Маринець Т. Й., Добридень А. В.* Про одну некласичну задачу теорії рівнянь гіперболічного типу// Праці міжнародного симпозіуму "Питання оптимізації обчислень". – К.: Інститут кібернетики ім. В. М. Глушкова НАНУ. – 2009. – Т.2. – С. 79–84.
7. *В. В. Маринець, К. В. Маринець, О. Ю. Питьовка.* Про одну крайову задачу теорії ДРЧП гіперболічного типу в області із складною структурою краю // Наук. вісник УжНУ. Сер. матем. і інформ. – 2014. – **25**, №2. – С. 110–117.

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