УДК 517.9

Rontó M., Varha Y. (University of Miskolc ,Uzhhorod National University)

INTEGRAL BOUNDARY VALUE PROBLEMS AND DIVISION INTO SUBINTERVALS

We show how a suitable interval division and parametrization technique can help to essentially improve the sufficient convergence condition for the successive approximations dealing with solutions of nonlinear non-autonomous systems of ordinary differential equations under integral boundary conditions. The constructivity of a suggested new technique is shown on the example.

1 Introduction. Recently, boundary value problems (BVPs) with integral conditions for non-linear differential equations have attracted much attention, see, e.g. [1], [2], [3], [8], [7], [9] and the references therein.

We study the non-linear integral boundary value problem (BVP)

$$\frac{du(t)}{dt} = f\left(t, u(t)\right), t \in [a, b], \tag{1}$$

$$g\left(u(a), u(b), \int_a^c u(s)ds\right) = d, \tag{2}$$

where $f : [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$, $g : [a, b] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous functions in a certain bounded sets , which will be exactly specified later and $d \in \mathbb{R}^n$ is a given vector, $a < c \leq b$.

Following to the idea used in numerical methods for approximate solution of initial value problems for ordinary differential equations, let us divide the interval $[a, b]$ by the variable mesh-points

$$t_k = t_{k-1} + h_k,\ k = 1, ..., N, t_0 = a,\ t_N = b, \tag{3}$$

into N subintervals

$$[t_0, t_1], [t_1, t_2], [t_2, t_3], ..., [t_{N-1}, t_{1N}].$$

Note, that in (3) can be used also a constant step size

$$h = h_k = \frac{b-a}{N} = \frac{t_N - t_0}{N}, k = 1, ..., N.$$

The aim of this paper to show how an N subinterval divisions of type (3) and parametrization technique can help to improve the sufficient convergence conditions for analytic approximations in the case of integral BVP (1), (2).

The main idea of our approach is the following. First we simplify the integral conditions (2). In order to replace it by certain linear two-point separated parametrized ones, we introduce the vectors of parameters

$$z^{(k)} = col(z_1^{(k)}, z_2^{(k)}, ..., z_n^{(k)}),\ k = 0, 1, 2, ..., N,$$

by formally putting

$$z^{(k)} := u(t_k),\ k = 0, 1, 2, ..., N.$$

Instead of (1), (2) we will study N "model-type" two-point BVPs with separated parameterized conditions

$$\frac{dx^{(k)}}{dt} = f\left(t, x^{(k)}\right), \; t \in [t_{k-1}, t_k], \tag{4}$$

$$x(t_{k-1}) = z^{(k-1)}, \; x\left(t_k\right) = z^{(k)}, k = 1, 2, ..., N. \tag{5}$$

where $z^{(0)}, \; z^{(1)}, ..., z^{(N)} \in \mathbb{R}^n$ are parameters. Note, that the length of the interval in problems (4), (5), which will be studied independently, is equal to step-size h_k in opposite to $b - a$ in the case of original BVP (1), (2).

To study the solutions of BVPs (4) ,(5) we use the special form of parametrized successive approximations $\left\{x_m^{(k)}(t, z^{(k-1)}, z^{(k)})\right\}_{m=0}^{\infty}$, $k = 1, 2, ..., N$ constructed in analytic form and well defined on the intervals

$$t \in [t_{k-1}, t_k], k = 1, 2, ..., N,$$

respectively.

The following notations and definitions are used from [9], which we state again below for the convenience of the readers.

We fix an $n \in \mathbb{N}$ and a bounded closed set $D \subset \mathbb{R}^n$.

1. For vectors $x = col(x_1, ..., x_n) \in \mathbb{R}^n$ the obvious notation $|x| = col(|x_1|, ..., |x_n|)$ is used and the inequalities between vectors are understood componentwise.

The same convention is adopted implicitly for operations $'\max', '\min', '\sup', '\inf'$.

2. 1_n is the unit matrix of dimension n.

3. 0_n is the zero matrix of dimension n.

4. $r(K)$ is the maximal, in modulus, eigenvalue of a matrix K.

5. For a set $D \subset \mathbb{R}^n$, closed interval $[a, b] \subset \mathbb{R}$, continuous function $f : [a, b] \times D \to \mathbb{R}^n$, $n \times n$ matrix K with non-negative entries, we write $f \in Lip(K, D)$ if the inequality $|f(t, u) - f(t, v)| \leq K\,|u - v|$ holds for all $\{u, v\} \subset D$ and $t \in [a, b]$.

Definition 1. *For any non-negative vector $\rho \in \mathbb{R}^n$ under the componentwise $\rho-$neighbourhood of a point $z \in \mathbb{R}^n$ we understand*

$$B(z, \rho) := \{\xi \in \mathbb{R}^n : |\xi - z| \leq \rho\}.$$

Similarly, for the given bounded connected set $\Omega \subset \mathbb{R}^n$, we define its componentwise $\rho-$neighbourhood by putting $B(\Omega, \rho) := \underset{\xi \in \Omega}{\cup} B(\xi, \rho)$.

Definition 2. *For given two bounded connected sets $D_a \subset \mathbb{R}^n$ and $D_b \subset \mathbb{R}^n$, introduce the set*

$$D_{a,b} := (1 - \theta)z + \theta\eta, \; z \in D_a, \eta \in D_b, \theta \in [0, 1] \tag{6}$$

and its componentwise $\rho-$neighbourhood

$$D := B(D_{a,b}, \rho)\ . \tag{7}$$

Finally, on the base of function $f : [a, b] \times D \to \mathbb{R}^n$ we introduce the vector

$$\delta_{[a,b],D}(f) := \frac{1}{2}\left[\max_{(t,x) \in [a,b] \times D} f(t, x) - \min_{(t,x) \in [a,b] \times D} f(t, x)\right]. \tag{8}$$

2 Interval division and successive approximations

Let us fix for the nonlinear integral BVP (1), (2) certain closed bounded sets

$$D^k \subset \mathbb{R}^n, k = 0, 1, 2, ..., N \tag{9}$$

and focus on the continuously differentiable solutions u of problem (1), (2) with the values $u(t_k) \in D^k, k = 0, 1, 2, ..., N$. Without loss of generality, we shall choose $D^k, k = 0, 1, 2, ..., N$ to be convex.

Based on the sets (9) according to (6) we introduce the sets:

$$D_{k-1,k} := (1-\theta)z^{(k-1)} + \theta z^{(k)},\ z^{(k-1)} \in D^{k-1},\ z^{(k)} \in D^k, \theta \in [0,1], k = 1, 2, ..., N$$

according to (6) and respectively its some componentwise $\rho^{(k)}-$vector neighbourhoods:

$$D^{[k]} := B(D_{k-1,k}, \rho^{(k)}), k = 1, 2, ..., N \tag{10}$$

as in (7).

We suppose the fulfillment of the local Lipschitz conditions

$$f \in Lip(K_k, D^{[k]}), t \in [t_{k-1}, t_k],\ k = 1, 2, ..., N$$

and a certain smallness of the eigenvalues $r(Q_k)$

$$r(Q_k) < 1 \tag{11}$$

for the matrixes

$$Q_k := \frac{3h_k}{10} K_k, k = 1, 2, ..., N. \tag{12}$$

Now, instead of boundary value problem (1), (2) using a natural interval division technique (3), we will consider on the subintervals $t \in [t_{k-1}, t_k],\ k = 1, 2, ..., N$ respectively N "model-type" two-point BVPs (4), (5) with linear separated parameterized conditions.

We then go back to the original problem by choosing the values of the introduced parameters $z^{(k)}$, $k = 0, 1, 2, ..., N$ appropriately.

Let us suppose that the domains for the space variables in problems (4), (5) are $D^{[k]}$ respectively defined in (10) with vector $\rho^{(k)}$ is satisfying the inequality

$$\rho^{(k)} \geq \frac{h_k}{2} \delta_{[t_{k-1},t_k], D^{[k]}}(f), \tag{13}$$

where $\delta_{[t_{k-1},t_k],D^{[k]}}(f)$ is defined according to (8).

Let us define for the parametrized problems (4), (5) the recurrence parametrized sequences of functions$x_m^{(k)} : [t_{k-1}, t_k] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $k = 1, 2, ..., N, m = 0, 1, 2, ...,$ by putting

$$x_0^{(k)}\left(t, z^{(k-1)}, z^{(k)}\right) := z^{(k-1)} + \frac{(t - t_{k-1})}{h_k}\left[z^{(k)} - z^{(k-1)}\right] = \tag{14}$$

$$= \left[1 - \frac{t - t_{k-1}}{h_k}\right]\ z + \frac{t - t_{k-1}}{h_k} z^{(k)},\ t \in [t_{k-1}, t_k],\ k = 1, 2, ..., N,$$

$$x_m^{(k)}\left(t,z^{(k-1)},z^{(k)}\right):=z^{(k-1)}+\int_{t_{k-1}}^{t} f\left(s,x_{m-1}^{(k)}\left(s,z^{(k-1)},z^{(k)}\right)\right)ds- \tag{15}$$

$$-\frac{t-t_{k-1}}{h_k}\int_{t_{k-1}}^{t_k} f\left(s,x_{m-1}^{(k)}\left(s,z^{(k-1)},z^{(k)}\right)\right)ds+\frac{t-t_{k-1}}{h_k}\left[z^{(k)}-z^{(k-1)}\right],\ t\in[t_{k-1},t_k],$$

k=1,2,...,N for all $m=1,2,...,z^{(k-1)}\in\mathbb{R}^n$ and $z^{(k)}\in\mathbb{R}^n$.

We note that all members of the sequences (14), (15) satisfy two-point boundary conditions (5) respectively for any $z^{(k-1)}\in\mathbb{R}^n$ and $z^{(k)}\in\mathbb{R}^n$.

3 Convergence of successive approximations

We would like to use the sequences $\left\{x_m^{(k)}\left(t,z^{(k-1)},z^{(k)}\right)\right\}_{m=0}^{\infty}$, $k=1,2,...,N$ from (14) and (15) for the investigation of solutions of the given BVP (1), (2).

The following statement shows that the sequences (15) are uniformly convergent and their limit is a solution of a certain additively perturbed problem for all $\left(z^{(k-1)},z^{(k)}\right)\in D^{k-1}\times D^k$.

Theorem 1. *Let $f\in Lip(K_k,D^{[k]})$, for $t\in[t_{k-1},t_k]$, $k=1,2,...,N$ and there exist the non negative vectors $\rho^{(k)}$ such that inequalities (13) hold and the eigenvalues of matrices Q_k of form (12) satisfy (11) for $k=1,2,...,N$.*

Then, for arbitrary fixed pair of vectors $\left(z^{(k-1)},z^{(k)}\right)\in D^{k-1}\times D^k$:

1. All members of sequences (15) are continuously differentiable functions on the interval $t\in[t_{k-1},t_k]$ satisfying the two-point linear separated parametrized boundary conditions (5).

2. The sequences of functions (15) in $t\in[t_{k-1},t_k]$ converge uniformly as $m\to\infty$ to the limit functions

$$x_\infty^{(k)}(t,z^{(k-1)},z^{(k)})=\lim_{m\to\infty}x_m^{(k)}(t,z^{(k-1)},z^{(k)}),\ k=1,2,...,N$$

3.The limit functions satisfy the conditions

$$x_\infty^{(k)}(t_{k-1},z^{(k-1)},z^{(k)})=z^{(k-1)},\ x_\infty^{(k)}(t_k,z^{(k-1)},z^{(k)})=z^{(k)},k=1,2,...,N\ .$$

4.The functions $x_\infty^{(k)}(t,z^{(k-1)},z^{(k)})$ are the unique continuously differentiable solutions of the integral equations

$$x^{(k)}(t)=z^{(k-1)}+\int_{t_{k-1}}^{t}f(s,x^{(k)}(s))ds-$$

$$-\frac{t-t_{k-1}}{h_k}\int_{t_{k-1}}^{t_k}f(s,x^{(k)}(s))ds+\frac{t-t_{k-1}}{h_k}\left[z^{(k)}-z^{(k-1)}\right],\ t\in[t_{k-1},t_k],\ k=1,2,...,N,$$

in the domain $D^{[k]}$,respectively.

In other words, $x_\infty^{(k)}(t,z^{(k-1)},z^{(k)})$ are the solutions of the following Cauchy problems of the modified system of integro-differential equations:

$$\frac{dx^{(k)}}{dt}=f(t,x^{(k)})\ +\frac{1}{h_k}\Delta^{(k)}(z^{(k-1)},z^{(k)}),\ t\in[t_{k-1},t_k],k=1,2,...,N,$$

$$x\left(t_{k-1}\right)=z^{(k-1)},$$

where $\Delta^{(k)}(z^{(k-1)},z^{(k)}) : D^{k-1}\times D^{k}\to\mathbb{R}^n$ *are the mapping given by formula*

$$\Delta^{(k)}(z^{(k-1)},z^{(k)}) = z^{(k)} - z^{(k-1)} - \int_{t_{k-1}}^{t_k} f(s,x^{(k)}(s))ds, k=1,2,...,N. \tag{16}$$

5.The following estimates hold

$$\left|x_{\infty}^{(k)}(\cdot,z^{(k-1)},z^{(k)}) - x_{m}^{(k)}(\cdot,z^{(k-1)},z^{(k)})\right| \leqslant$$

$$\leqslant \frac{10}{9}\alpha_1(t,t_{k-1},h_k)Q_k^m\left(1_n - Q_k\right)^{-1}\delta_{[t_{k-1},t_k],D^{[k]}}(f),\ t\in[t_{k-1},t_k],m\geq 0,k=1,2,...,N$$

where the vectors $\delta_{[t_{k-1},t_k],D^{[k]}}(f))$ *are given in (8) and*

$$\alpha_1(t,\tau,I) = 2\left(t-\tau\right)\left(1-\frac{t-\tau}{I}\right),\ |\alpha_1(t,\tau,I)|\leq\frac{I}{2},\ t\in[\tau,\tau+I].$$

Proof. The proof can be carried out similarly as in Theorem 1 from [7].

It is natural to expect that the limit functions $x_{\infty}^{(k)}(t,z^{(k-1)},z^{(k)})$ of the iterations (15) on the subintervals $t\in[t_{k-1},t_k]$, $k=1,2,...,N$ will help us to formulate criteria of solvability of the integral BVP (1), (2). It turns out that there are the functions

$$\Delta^{(k)}(z^{(k-1)},z^{(k)}) : D^{k-1}\times D^{k}\to\mathbb{R}^n$$

defined according to equalities (16) that provide such conclusion.

Indeed, Theorem 1 guarantee that under the conditions assumed, the functions

$$x_{\infty}^{(k)}(t,z^{(k-1)},z^{(k)})\ \ :[t_{k-1},t_k]\to\mathbb{R}^n$$

are well defined for all $\left(z^{(k-1)},z^{(k)}\right)\in D^{k-1}\times D^k$. Therefore, by putting

$$u_{\infty}(t,z^{(0)},z^{(1)},...,z^{(N)}) := \begin{cases} x_{\infty}^{(1)}(t,z^{(0)},z^{(1)}),\ \text{if } t\in[t_0,t_1], \\ x_{\infty}^{(2)}(t,z^{(1)},z^{(2)}),\ \text{if } t\in[t_1,t_2], \\ .... \\ x_{\infty}^{(N)}(t,z^{(N-1)},z^{(N)}),\ \text{if } t\in[t_{N-1},t_N] \end{cases} \tag{17}$$

we obtain a function $u_{\infty}(\cdot,z^{(0)},z^{(1)},...,z^{(N)}) : [a,b]\to\mathbb{R}^n$, which is well defined for the values $z^{(k)}\in D^k$, $k=0,1,2,...,N$. This function is obviously continuous, because at the points $t=t_k$ we have

$$x_{\infty}^{(k)}(t_k,z^{(k-1)},z^{(k)}) = x_{\infty}^{(k)}(t_k,z^{(k)},z^{(k+1)}), k=0,1,2,...,N\ .$$

The following theorem establishes a relation of function (17) to the solution of integral BVP (1), (2) in terms of the zeroes of functions $\Delta^{(k)}(z^{(k-1)},z^{(k)}) : D^{k-1}\times D^k\to\mathbb{R}^n$, defined according to (16).

Theorem 2. *Let the conditions of Theorem1 hold. Then :*

1. The function $u_{\infty}(t,z^{(k-1)},z^{(k)}) : [a,b]\to\mathbb{R}^n$ *given by (17) is an continuously differentiable solution of BVP (1), (2) if and only if the vectors*

$$z^{(k)}, k=0,1,2,...,N \tag{18}$$

satisfy the system of $n(N+1)$ *algebraic equations*

$$\Delta^{(k)}(z^{(k-1)}, z^{(k)}) = z^{(k)} - z^{(k-1)} - \int_{t_{k-1}}^{t_k} f(s, x_\infty^{(k)}(s, z^{(k-1)}, z^{(k)}))ds = 0,\ k = 1,2,...,N,$$

$$\Delta^{(N+1)}(z^{(0)}, z^{(1)}, ..., z^{(l)}, z^{(l+1)}) =$$

$$= g\left(z^{(0)}, z^{(N)}, \sum_{j=0}^{l} \int_{t_j}^{t_{j+1}} x_\infty^{(j)}(s, z^{(j-1)}, z^{(j)})ds + \int_{t_l}^{c} x_\infty^{(l)}(s, z^{(l)}, z^{(l+1)})ds\right) - d = 0. \tag{19}$$

2. For every solution $U(\cdot)$ *of problem (1), (2) with* $U(t_k) \in D^k, k = 0,1,2,...,N$ *there exist vectors (18) such that* $U(\cdot) = u_\infty(t, z^{(0)}, z^{(1)}, ..., z^{(N)})$, *where the function* $u_\infty(t, z^{(0)}, z^{(1)}, ..., z^{(N)})$ *is given in (17).*

Proof. The proof can be carried out similarly to the proof of Theorem4 from [4].

Equations (19) are usually referred to as *determining* or *bifurcation* equations because their roots determine solutions of the original problem.

4 Approximate determining equations

Although Theorem 2 provides a theoretical answer to the question on the construction of a solution of the BVP (1), (2), its application faces difficulties due to the fact that the explicit form of $x_\infty^{(j)}(s, z^{(j-1)}, z^{(j)})$ and the functions

$$\Delta^{(k)}(z^{(k-1)}, z^{(k)}) : D^{k-1} \times D^k \to \mathbb{R}^n,\ k = 1,2,...,N$$

$$\Delta^{(N+1)}(z^{(0)}, z^{(1)}, ..., z^{(N)}) : D^0 \times D^1 \times ... \times D^N \to \mathbb{R}^n,$$

appearing in (19) are usually unknown. This complication can be overcome by using $x_m^{(k)}(s, z^{(k-1)}, z^{(k)})$ of form (15) for a fixed m, which will lead to the so-called m-th ap*proximate determining equations:*

$$\Delta^{(k)}(z^{(k-1)}, z^{(k)}) = z^{(k)} - z^{(k-1)} - \int_{t_{k-1}}^{t_k} f(s, x_m^{(k)}(s, z^{(k-1)}, z^{(k)}))ds = 0,\ k = 1,2,...,N, \tag{20}$$

$$\Delta^{(N+1)}(z^{(0)}, z^{(1)}, ..., z^{(l)}, z^{(l+1)}) =$$

$$g\left(z^{(0)}, z^{(N)}, \sum_{j=0}^{l} \int_{t_j}^{t_{j+1}} x_m^{(j)}(s, z^{(j-1)}, z^{(j)})ds + + \int_{t_l}^{c} x_m^{(l)}(s, z^{(l)}, z^{(l+1)})ds\right) - d = 0. \tag{21}$$

Note that, unlike system (19), the m-th approximate determining system (20), (21) contains only terms involving the functions $x_m^{(j)}(\cdot, z^{(j-1)}, z^{(j)})$ and, thus, are known explicitly.

It is natural to expect that approximations to the unknown solution of problem (1), (2) can be obtained by using the function

$$u_m(t, z^{(0)}, z^{(1)}, ..., z^{(N)}) := \begin{cases} x_m^{(1)}(t, z^{(0)}, z^{(1)}),\ \text{if } t \in [t_0, t_1], \\ x_m^{(2)}(t, z^{(1)}, z^{(2)}),\ \text{if } t \in [t_1, t_2], \\ .... \\ x_m^{(N)}(t, z^{(N-1)}, z^{(N)}),\ \text{if } t \in [t_{N-1}, t_N] \end{cases} \tag{22}$$

which is an "approximate" version of (17) well defined for all $t \in [a,b]$ and $z^{(k)} \in D^k, k = 0,1,2,...,N$.

Lemma 1. *If $z^{(0)}, z^{(1)}, ..., z^{(N)}$ satisfy equations (20), (21) for a certain m, then the function $u_m(t, z^{(0)}, z^{(1)}, ..., z^{(N)})$ determined by equality (22) is continuously differentiable on $[a,b]$.*

Proof. We recall that the functions of the sequences (15) have the property

$$x_m^{(k)}(t_k, z^{(k-1)}, z^{(k)}) = x_m^{(k+1)}(t_k, z^{(k)}, z^{(k+1)}) = z^{(k)}, k = 0,1,2,...,N-1. \tag{23}$$

It follows immediately from (15) that

$$\frac{dx_m^{(k)}(t, z^{(k-1)}, z^{(k)})}{dt} = f\left(t, x_m^{(k)}(t_k, z^{(k-1)}, z^{(k)})\right) -$$

$$\frac{1}{h_k}\int_{t_{k-1}}^{t_k} f\left(s, x_{m-1}^{(k)}\left(s, z^{(k-1)}, z^{(k)}\right)\right) ds + \frac{1}{h_k}\left[z^{(k)} - z^{(k-1)}\right] \tag{24}$$

and

$$\frac{dx_m^{(k+1)}(t, z^{(k)}, z^{(k+1)})}{dt} = f\left(t, x_m^{(k+1)}(t_k, z^{(k)}, z^{(k+1)})\right) -$$

$$\frac{1}{h_{k+1}}\int_{t_{k-1}}^{t_k} f\left(s, x_{m-1}^{(k)}\left(s, z^{(k-1)}, z^{(k)}\right)\right) ds + \frac{1}{h_{k+1}}\left[z^{(k)} - z^{(k-1)}\right]. \tag{25}$$

In view of (20) it follows from (24), (25) that

$$\frac{dx_m^{(k)}(t_k, z^{(k-1)}, z^{(k)})}{dt} = f\left(t, x_m^{(k)}(t_k, z^{(k-1)}, z^{(k)})\right) \tag{26}$$

and

$$\frac{dx_m^{(k+1)}(t_k, z^{(k)}, z^{(k+1)})}{dt} = f\left(t, x_m^{(k+1)}(t_k, z^{(k)}, z^{(k+1)})\right) \tag{27}$$

and on the base of (23) it follows from (26), (27) that

$$\frac{dx_m^{(k)}(t_k, z^{(k-1)}, z^{(k)})}{dt} = \frac{dx_m^{(k+1)}(t_k, z^{(k)}, z^{(k+1)})}{dt},$$

i.e. the derivative of the function $u_m(t, z^{(0)}, z^{(1)}, ..., z^{(N)})$ of form (22) is continuous at the points $t_k,\ k = 1,2,...,N-1$. The continuous differentiability of the function $u_m(t, z^{(0)}, z^{(1)}, ..., z^{(N)})$ at other points is obvious from its definition.

5 Example

Let us apply the approach described above to the system of differential equations

$$\begin{cases} x_1'(t) = \frac{1}{2}x_2^2(t) - \frac{t}{4}x_1(t) + \frac{1}{32}t^3 - \frac{1}{32}t^2 + \frac{9}{40}t = f_1(t, x_1, x_2), \\ x_2'(t) = \frac{t}{8}x_1(t) - t^2x_2(t) + \frac{15}{64}t^3 + \frac{1}{80}t + \frac{1}{4} = f_2(t, x_1, x_2),\ t \in [0, 1.9] \end{cases} \tag{28}$$

considered with the integral boundary conditions

$$\begin{cases} x_1(0)x_2(1.9) + \left[\int\limits_0^1 x_1(s)\ ds\right]^2 = -0.044097222, \\ x_1(1.9)x_2(0) - \int\limits_0^1 x_2(s)\ ds = -0.125. \end{cases} \tag{29}$$

	m=1	m=2	m=7	m=9
$z_1^{(0)}$	-0.4748602215	-0.1075734488	-0.1000001272	-0.09999999351
$z_2^{(0)}$	-0.00625079464	-0.1312018998	1.6548×10^{-8}	-1.3781466×10^{-8}
$z_1^{(1)}$	-0.3059609811	0.02054437808	0.02499988756	0.02500000602
$z_2^{(1)}$	0.2226033171	0.2360172089	0.250000009	0.2499999989
$z_1^{(2)}$	-0.1059129	0.1752333605	0.1812499387	0.1812500041
$z_2^{(2)}$	0.3470424761	0.3696960632	0.3749998443	0.3750000035
$z_1^{(3)}$	0.1051963874	0.3445674853	0.3512500029	0.3512500008
$z_2^{(3)}$	0.4532927092	0.4728012162	0.4749995422	0.4750000178

Table 1

Let us choose in (3) $N = 3$, and introduce the variable mesh-points $t_0 = 0 = a$, $t_1 = t_0+h_1 = 0+1 = 1$, $t_2 = t_1+h_2 = 1+0.5 = 1.5$, $t_3 = t_2+h_3 = 1.5+0.4 = 1.9 = b$.

It is easy to check that

$$x_1^*(t) = \frac{t^2}{8} - \frac{1}{10},\ x_2^*(t) = \frac{t}{4}. \tag{30}$$

is a continuously differentiable solution of the problem (28), (29).

Let us choose the sets $D_{0,1}$, $D_{1,2}$ and $D_{2,3}$ as follows:

$$\begin{aligned} D_{0,1} &= \{(x_1, x_2) : -0.11 \le x_1 \le 0.03,\ -0.01 \le x_2 \le 0.25\}, \\ D_{1,2} &= \{(x_1, x_2) : 0.026 \le x_1 \le 0.18,\ 0.24 \le x_2 \le 0.37\}, \\ D_{2,3} &= \{(x_1, x_2) : 0.18 \le x_1 \le 0.35,\ 0.37 \le x_2 \le 0.47\}. \end{aligned} \tag{31}$$

Using vectors $\rho^{(1)} = \begin{bmatrix} 0.2 \\ 0.3 \end{bmatrix}$, $\rho^{(2)} = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}$, $\rho^{(3)} = \begin{bmatrix} 0.1 \\ 0.4 \end{bmatrix}$ on the base of (31) and (10), we define the domains

$$\begin{aligned} D^{[1]} &= \{(x_1, x_2) : -0.31 \le x_1 \le 0.23,\ -0.31 \le x_2 \le 0.55\}, \\ D^{[2]} &= \{(x_1, x_2) : -0.074 \le x_1 \le 0.28,\ 0.04 \le x_2 \le 0.57\}, \\ D^{[3]} &= \{(x_1, x_2) : 0.08 \le x_1 \le 0.45,\ -0.03 \le x_2 \le 0.87\}. \end{aligned}$$

Direct computations show that the conditions of Theorem 1 for the problem (28), (29) in the domains $D^{[1]}, D^{[2]}, D^{[3]}$ hold.

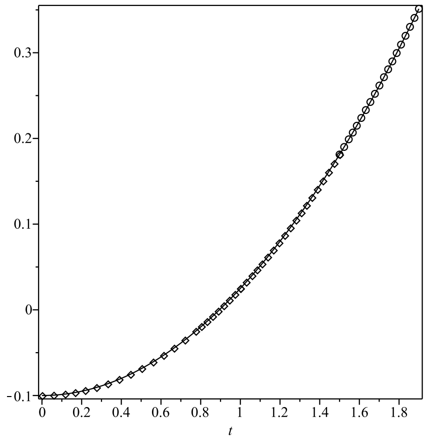
Applying Maple 14 by solving the approximate determining equations (20), (21) for $m = 1, 2, 7, 9$ we obtain the numerical results which are presented in Table 1.

The graphs of the ninth (m=9) approximation and the exact solution (30) of the BVP (28), (29) are shown on Figure 1.

The number of the solutions of the algebraic determining system (20), (21) coincides with the number of solutions of the given integral BVP.

Using vectors $\rho^{(1)} = \begin{bmatrix} 0.4 \\ 0.4 \end{bmatrix}$, $\rho^{(2)} = \begin{bmatrix} 0.2 \\ 0.3 \end{bmatrix}$, $\rho^{(3)} = \begin{bmatrix} 0.1 \\ 0.3 \end{bmatrix}$ and sets

$$D_{0,1} = \{(x_1, x_2) : -0.14 \le x_1 \le 0.57,\ 1.03 \le x_2 \le 1.08\},$$



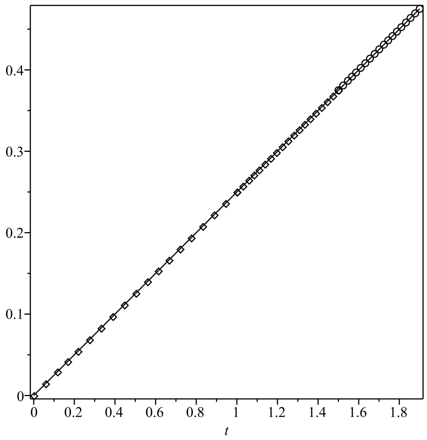


Figure 1. The components of the exact solution (solid line) and its ninth approximation (drawn with dots)

	m=1	m=2	m=7	m=9
$\hat{z}_1^{(0)}$	-0.1521237421	-0.1397657264	-0.1383259999	-0.1383258115
$\hat{z}_2^{(0)}$	1.075809281	1.074153009	1.073568831	1.073568702
$\hat{z}_1^{(1)}$	0.5310279374	0.5570093765	0.5608197314	0.5608200765
$\hat{z}_2^{(1)}$	1.046713842	1.040090793	1.038597963	1.038597766
$\hat{z}_1^{(2)}$	0.7822010976	0.8008809184	0.8042478589	0.8042481471
$\hat{z}_2^{(2)}$	0.7647114402	0.7654378361	0.7640757655	0.7640756219
$\hat{z}_1^{(3)}$	0.9097001195	0.9256427415	0.928760109	0.9287602397
$\hat{z}_2^{(3)}$	0.6279451019	0.6281507937	0.6263505218	0.6263509057

Table 2

$$D_{1,2} = \{(x_1, x_2) : 0.56 \le x_1 \le 0.8,\ 0.76 \le x_2 \le 1.04\},$$
$$D_{2,3} = \{(x_1, x_2) : 0.8 \le x_1 \le 0.93,\ 0.62 \le x_2 \le 0.77\}$$

we define domains

$$D^{[1]} = \{(x_1, x_2) : -0.54 \le x_1 \le 0.97,\ 0.63 \le x_2 \le 1.48\},$$
$$D^{[2]} = \{(x_1, x_2) : 0.36 \le x_1 \le 1,\ 0.46 \le x_2 \le 1.34\},$$
$$D^{[3]} = \{(x_1, x_2) : 0.7 \le x_1 \le 1.03,\ 0.32 \le x_2 \le 1.07\}.$$

Computations show that the approximate determining system (20), (21) side by side with the solution indicated in Table 1 for $m = 1, 2, 7, 9$ has another solutions, which are presented in Table 2.

The graphs of the first, the seventh and the ninth approximations to the second solution of the given BVP are shown on Figure 2.

The residual obtained as a result of substitution of the ninth approximation into the given differential system (28) is estimated as follows:

$$\max_{t\in[0,1.9]} \left| x'_{91}(t) - \frac{1}{2}x_{92}^2(t) + \frac{t}{4}x_{91}(t) - \frac{1}{32}t^3 + \frac{1}{32}t^2 - \frac{9}{40}t \right| = 1.5 \cdot 10^{-7},$$
$$\max_{t\in[0,1.9]} \left| x'_{92}(t) - \frac{t}{8}x_{91}(t) + t^2 x_{92}(t) - \frac{15}{64}t^3 - \frac{1}{80}t - \frac{1}{4} \right| = 6 \cdot 10^{-7}.$$

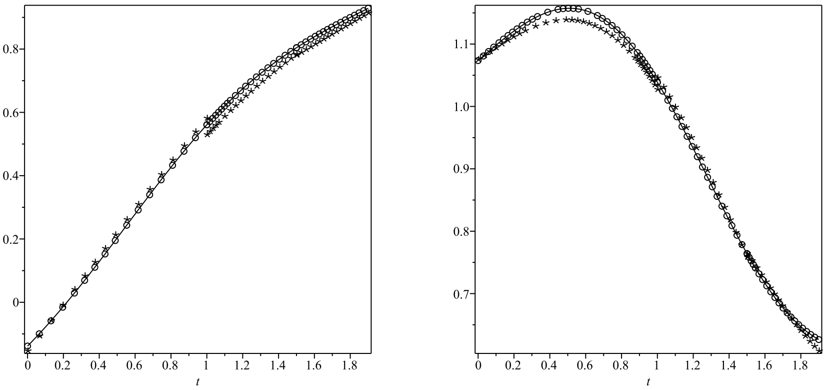


Figure 2. The components of the first ($\star$), the seventh ($\circ$) and the ninth (solid line) approximations to the second solution

References

1. Benchohra, M., Nieto, J.J. and Qahab, A. Second- order boundary value proble with integral boundary conditions, *Boundary Value Problems* 2011, 2011:260309, doi:10.1155/2011/260309, 9 pages
2. Mao Jinxiu, Zhao Zengqin and Xu Naiwei, On existence and uniqueness of positive solutions for integral boundary value problems, *Electron. J. Qual. Theory Differ. Equ.,* 2010, No.16, 1-8
3. Rontó M. and Marynets. K., On numerical-analytic method for nonlinear boundary value problem with integral conditions, *Electron. J. Qual. Theory Differ. Equ.,* No. 99 (2012, p.1-23), http.//www.math. u-szeged.hu/ejqtde/
4. Ronto, A. and Ronto, M., Periodic successive approximations and interval halving, *Miskolc Mathematical Notes,* **13,** No.2 (2012), 459-482
5. Ronto, A., Ronto, M. and Shchobak N, Constructive analysis of periodic solutions with interval halving, *Boundary Value Problems* 2013 2013:57, doi:10.1186/1687-2770-2013-57, 34 pages
6. Ronto, A., Ronto, M. and Shchobak N, Notes on interval halving procedure for periodic and two-point problems, *Boundary Value Problems* 2014 2014:164, doi:10.1186/s13661-014-0164-9, 20 pages
7. Rontó, M., Varha, Y. and Marynets K., Further results on the investigation of solutions of integral boundary value problems, *Tatra Mountains Mathematical Publications, **63** (2015), 247-267, DOI 10515/tmmp-2015-0035*
8. Ronto, A., Ronto, M. and Varha, Y., A new approach to non-local boundary value problems for ordinary differential systems, *Applied Mathematics and Computation,* (2015), **250,** 689-700
9. Rontó, M. and Varha, Y., Constructive existence analysis of solutions of non-linear integral boundary value problems, *Miskolc Mathematical Notes*, vol. 15 , No.2 (2014), 725-742
10. Ronto, A. and Ronto, M., Successive Approximation Techniques in Non- Linear Boundary Value Problems for Ordinary Differential Equations , Handbook of Differential Equations, Ordinary Differential Equations, vol. IV, 441- 592, Edited by F. Batelli and M. Feckan, (2008), Elsevier B.V.
11. Ronto A. and Ronto M., On non-separated three-point boundary value problerms for linear functional differential equations, *Abstract and Applied Analysis*, Volume 2011, ID 326052, 22 pages, doi:10.1155/2011/ 326052,
12. Rontó, M. and Varha, Y., Successive approximations and interval halving for integral boundary value problems, *Miskolc Mathematical Notes*, vol. 16 , No.2 (2015), 24 pages

Одержано 11.11.2015