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MULTIPLE-VALUED THRESHOLD LOGIC

Those of us who do not read Russian could not benefit from the pioneering work of Prof. Naum Aizenberg introducing multiple-valued threshold logic in the complex plane in the decade of the 70's. Possibly the interest in building multiple-valued threshold "gates" in hardware, motivated work mostly on ternary threshold logic. This paper summarizes some early results as well as the contributions of the author to the development of the area.

Ті з нас, хто не може читати на російській мові, не мали можливості насолоджуватися фундаментальними роботами професора Наума Айзенберга, в яких в 1970-ті роки було розвинуто теорію багатозначних порогових функцій над полем комплексних чисел. Через мовні перепони, які завадили багатьом вченим, що працювали в цій галузі, ознайомитись з підходом Н.Н. Айзенберга до багатозначної порогової логіки над полем комплексних чисел, вони продовжували розвивати багатозначну порогову логіку над множиною Z_p^n вбудовану в R^n, або, інакше кажучи, над $[0, p]^n$. До появи робіт Н.Н. Айзенберга практичні рішення в галузі багатозначної порогової логіки були обмежені переважно бажанням будувати багатозначні логічні схеми і не простягалися далі тризначної порогової логіки. Ця стаття присвячена цим більш раннім підходам до багатозначної порогової логіки і результатам, що були отримані в різні часи на основі цих підходів, і базується зокрема на власному внеску автора в розвиток цієї галузі.

1. Introduction. Prof. Naum Aizenberg introduced the complex-valued threshold logic in [1,2], in Russian. Due to the prevailing language difficulties this line was not followed by other researchers, who chose to work in Z_p^n embedded in R^n or, simply, in $[0, p]^n$. After some work on ternary threshold logic in the early 60s [3], [4], the first important result may be traced back to [5], proving that multiple-valued threshold logic is functionally complete, thus providing a theoretical support to efforts to build hardware threshold gates and assemble circuits. On the other hand, as in the binary case, the number of threshold functions relative to the total number of functions for a given p and n is very small. This motivated studies in two directions. On the one side it was important to determine the areas of application where threshold gates might be used with advantage, like in the case of arithmetic circuits. At the same time it was important to consider the increasing demands on speed and minimal size, which has drawn attention to the use of resonant tunnel diodes [6] and consider the possibilities of optical computing [7]. On the other hand, extensions on threshold logic were introduced to alleviate the constraints imposed by linear separability, by considering polynomial separability [8], as well as non-necessarily parallel hyperplanes [9] and non-necessarily parallel hypersurfaces [10] for the required separation of subdomains.

2. Formalisms.

Definition 1. *Let Z_p be the domain of p-valued variables. An n-place p-valued function is a mapping $f : (Z_p)^n \to Z_p$. Let $x = x_1x_2 \ldots x_n$ denote an n-tuple in $(Z_p)^n$. If needed, n-tuples will be processed as vectors, this becoming clear from the context. Then $\forall k \in Z_p f^{-1}(k) = def \{x|x \in (Z_p)^n \text{ and } f(x) = k\}$ will be called a subdomain. (Notice that a subdomain may be empty). A p-valued function is a threshold (or linear separable) function iff there are parallel hyperplanes in $(Z_p)^n$ embedded in R^n separating $f^{-1}(0)$ from f^{-1},..., from $f^{-1}(p-1)$ in a monotonic way.*

Let $w \in R^n$ be a vector of weights and $T = (t_1, \ldots, t_{p1})$ be an ordered superset of real valued thresholds. (A superset is a collection of objects allowing repetitions.) Finally, let the notation $a \cdot b$ denote the inner product of vectors a and b.

Definition 2. *A function $f : (Z_p)^n \to Z_p$ is a linear separable or threshold function if there exist a weight vector $w \in R^n$ and a threshold superset T with real-valued components such that*

$f(x) = 0 \Leftrightarrow t_1 > w \cdot x$
$f(x) = i \Leftrightarrow t_{i+1} \geq w \cdot x > t_i \qquad 1 \leq i \leq p-2$
$f(x) = p-1 \Leftrightarrow w \cdot x > t_{p-1}$

(See example in Fig. 1). If $f : (Z_p)^n \to Z_p$ is a threshold function, then $(w; T)$ will be called its structure. In what follows this will be written $f : (w; T)$.

Symbols			
0	1	2	3
$\diamond$	$\times$	$\bullet$	$\square$

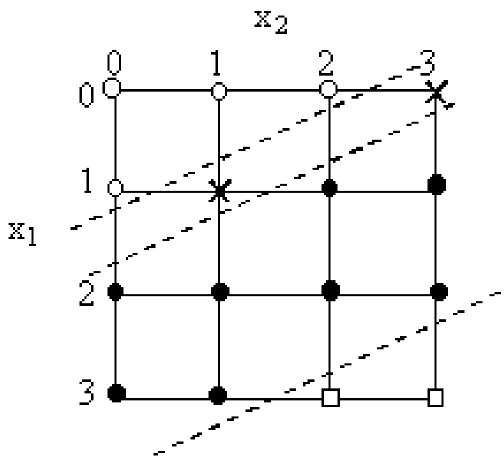

Figure 1. Example of a 2-place 4-valued threshold function w = [2, 1], T = {2.5, 3.5, 7.5}

Definition 3. [8] *A function $f : (Z_p)^n \to Z_p$ is polynomial separable if there exists a set of parallel hypersurfaces separating the subdomains of Z_p^n in a monotonic way. Fig. 2 illustrates a 2-place 4-valued function which is not linear separable (and therefore is not a threshold function), but exhibits polynomial separability, since parallel parabolas separate the subdomains in a monotonic way.*

Definition 4. [7] *A function $f : (Z_p)^n \to Z_p$ is multilineal separable if there exists a set of non necessarily parallel hyperplanes –(each one representing a binary threshold function)– separating the subdomains of $(Z_p)^n$. Notice that in this case more than one hyperplane may be needed to separate two neighbour subdomains. The summation of the separating binary threshold functions realizes the p-valued function. $(See Fig. 3)$.*

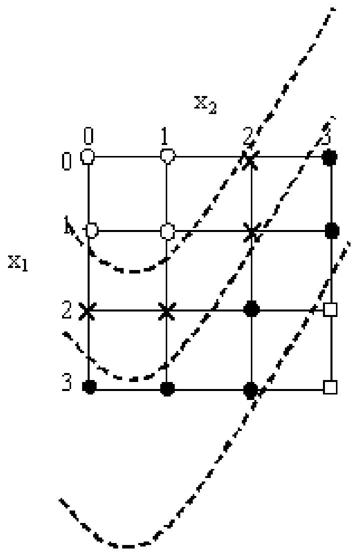

Figure 2. Example of a 2-place 4-valued non-threshold function, which has polynomial separability

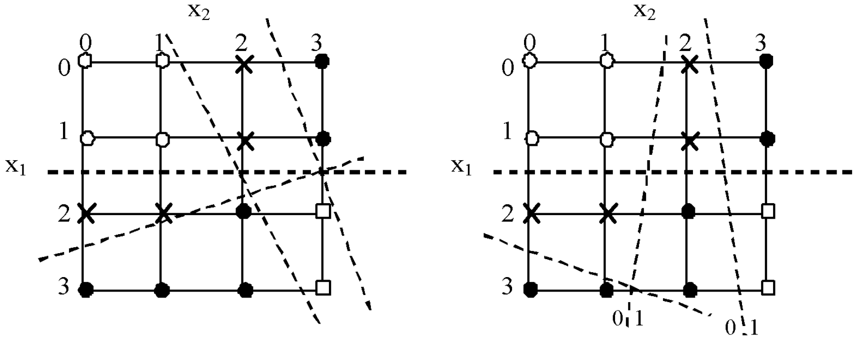

Figure 3. Different possible multilineal separations of the function shown in Fig. 2

The combination of the above extensions leads to Adaptive Separable functions [10]. Fig. 4 illustrates the adaptive separation of the same function as in Fig. 2, where two of the polynomials are replaced by overlapping window literals (which are 1-place binary threshold functions on each argument) and the third by a straight line, leading to the following realization:

if $(1/3)x_1 + x_2 \geq 3.5$ then $f_2(x) = 1$ else 0
if $(x_1)^{(2)} + x_2 \geq 2.5$ then $f_{11}(x) = 1$ else 0
if $(x_1)^{(3)} + x_2 \geq 1.5$ then $f_{12}(x) = 1$ else 0
$\forall x \in (Z_p)^n \quad f(x) = f_{11}(x) + f_{12}(x) + f_2(x)$
where $(x_i)^{(j)} = 1$ if $0 \leq j \leq x_i$ else 0.

Definition 5. *Let $x = p - 1 - x$ denote the complement of $x \in Z_p$. Notice that this is equivalent to a symmetric permutation of the domain of the variable. In the case of $x \in (Z_p)^n$, the complementation of the i-th component will be denoted by $\chi_i(x)$ and the complementation of all components, by $\chi_1(\chi_2(\ldots(\chi_n(x)\ldots)))$ or*

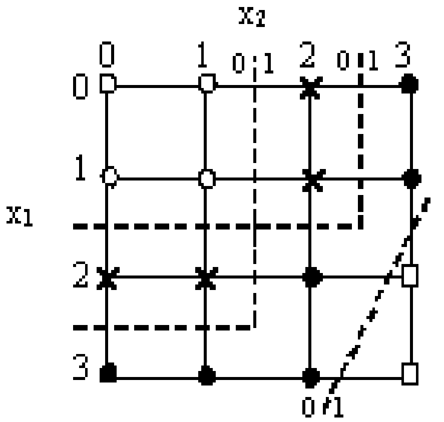

Figure 4. Adaptive separation of the function shown in Fig. 2

simply, x.

Definition 6. [11] *Let $f : (Z_p)^n \to Z_p$ be an n-place p-valued function. Moreover define $f(x) = f(\chi_1(\chi_2(\ldots(\chi_n(x)\ldots)))$; let $sym = \{x | f(x) = f(x)\}$ and $n_{sym} = |sym|$. Then $\sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_{p-1}\}$ is the set of semidual functions of f, where $\forall i, 1 \le i \le p-1$ and $\forall x \in (Z_p)^n$*

$$\sigma_i(f(x)) = \begin{cases} i-1 \quad \text{if } x \in sym \text{ and } f(x) = i, \\ i \quad \text{if } x \in sym \text{ and } f(x) = i-1, \\ \qquad f(x) \quad \text{otherwise.} \end{cases}$$

See examples in Fig. 5, where the shaded entries identify the set sym of $f(x)$. Notice that $\sigma_3(f(x)) = f(x)$ and therefore it constitutes a trivial semidual function.

Remark 1. *Given some $x \in (Z_p)^n$, x represents a point which is symmetric x with respect to the center point of $(Z_p)^n$ extended to R^n. The center point of $(Z_p)^n$ has all its coordinates equal to $(p-1)/2$.*

Remark 2. *If p is even, it is possible that $n_{sym} = 0$, meanwhile if p is odd, the smallest possible value of n_{sym} is 1, since in this case the center point is in $(Z_p)^n$ and it is its own complement.*

Remark 3. *It is easy to see from the definition, that σ_i is an involutive function: $\forall x \in (Z_p)^n$ holds that $\sigma_i(\sigma_i(f(x))) = f(x)$.*

	0	1	2	3
0	0	0	2	2
1	0	0	1	2
2	1	1	2	3
3	2	2	2	3

	0	1	2	3
0	0	0	2	2
1	0	0	0	2
2	1	0	2	3
3	2	2	2	3

	0	1	2	3
0	0	0	1	1
1	0	0	2	2
2	1	2	2	3
3	1	1	2	3

Figure 5. From left to right: $f(x)$ (with sym shaded), $\sigma_1(f(x))$ and $\sigma_2(f(x))$

3. Theorems

Theorem 1. [12]. *If $f : (Z_p)^n \to Z_p$ is a threshold function, then it is uniquely characterized by the ordered superset $(c_1, c_2, \ldots, c_n, n_0, n_1, \ldots, n_{p-1})$, where $\forall i, 1 \leq i \leq n$*

$$c_i = \sum_{x \in Z_p^n} x_i f(x)$$

and $\forall j, 0 \leq j \leq p-1,\ \ n_j = |f^{-1}(j)|$.

Theorem 2. (*Seee.g. [7]*). *Let $\Pi = \{\pi_1, \ldots, \pi_{n!}\}$ be the set of all permutations of n objects and let $f : (Z_p)^n \to Z_p$ be a threshold function. Then the following functions based on f are also threshold functions:*

i) $f(\pi_j(x)) \quad 1 \leq j \leq n!$
ii) $f(\chi_i(x)) \quad 1 \leq i \leq n$
iii) $f(x)$
iv) Any combination thereof

Furthermore if $f : (w; T)$ then
$f(\pi_j(x)) : (\pi_j(w); T)$
$f(\chi_i(x)) : (w^{(i)}; T^{(i)})$
$f(x) : (-w; -\mu(T))$
where $w^{(i)}$ denotes a modification of w such that its i-th component is multiplied by -1 meanwhile the others are preserved; μ is a mirroring function, reversing the order of the elements of T, and $T^{(i)}$ is obtained from T as follows: $\forall j, 1 \leq j < p$
$t_j^{(i)} = t_j - (p-1)w_i$

Theorem 3. [11] *Let $f : (Z_p)^n \to Z_p$ be a threshold function with a characterizing Nomura ordered superset as defined in Theorem 1. The following notation will be used:*

$$f(x) \to (c_1, c_2, \ldots, c_n, n_0, n_1, \ldots, n_{p-1})$$

furthermore let here Σ denote the summation over all x in $(Z_p)^n$. The following holds:

i) $f(\pi_j(x)) \to ((\pi_j(c_1, c_2, \ldots, c_n), n_0, n_1, \ldots, n_{p-1}) \quad 1 \leq j \leq n!$
ii) $f(\chi_i(x)) \to (c'_1, c'_2, \ldots, c'_n, n_0, n_1, \ldots, n_{p-1}) \qquad 1 \leq i \leq n$
where $c'_k = c_k,\ \ 1 \leq k \leq n,\ \ k \neq i$
$c'_i = (p-1)\Sigma f(x) - c_i$
iii) $f(x) \to (c''_1, c''_2, \ldots, c''_n, n''_0, n''_1, \ldots, n''_{p-1})$
where $c''_k = p^n(p-1)^2/2 - c_k$
and $n''_i = n_{p-1-i},\ \ 0 \leq i \leq p-1$

Theorem 4. [11] *Let $f : (Z_p)^n \to Z_p$ be an n-place p-valued threshold function with $n_{sym} > 0$. There exists at least one non-trivial semidual function $\sigma_i(f)$ that is also a threshold function with structure $(w; T^{(i)})$, where $\forall j$, $1 \leq j \leq p-1$ and $j \neq i$, $t_j^{(i)} = t_j$ meanwhile $t_i^{(i)}$ will take such a value, that the original i-th hyperplane will be parallel displaced to a new position, symmetric with respect to the center point of $(Z_p)^n$. If the i-th hyperplane contains the point $(i_1, i_2, \ldots, i_n)$ then the displaced hyperplane contains the point $(p-1-i_1, p-1-i_2, \ldots, p-1-i_n)$. Particularly, the i-th hyperplane contains the point $(0, 0, \ldots, t_i, 0, \ldots, 0)$; therefore after shifting, the hyperplane must contain the point $(p-1, p-1, \ldots, p-1-t_i, p-1, \ldots, p-1)$. See example in Fig. 6.*

In the example shown in Fig. 6, since the function has only two arguments, the

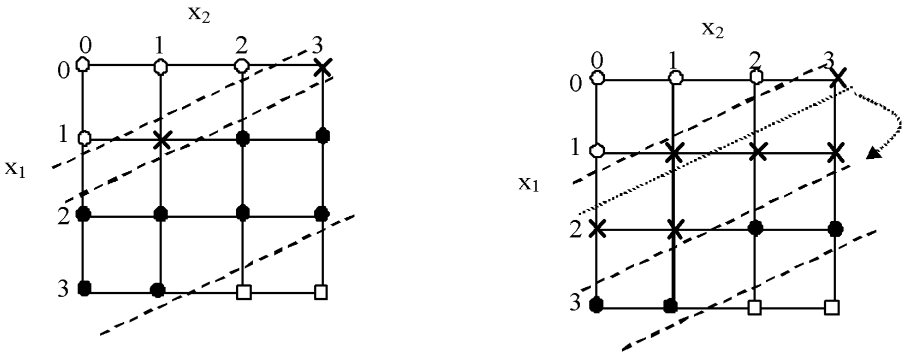

Figure 6. (Left) A 2-place quaternary threshold function. (Right) The semidual threshold function $\sigma_2(f(x))$ showing for reference, fine dotted, the position of the hyperplane before shifting.

hyperplanes are straight lines. The line separating the subdomain of 1's from the subdomain of 2's in $f(x)$ has the equation

$$x_1 = -{}^1\!/_2\, x_2 + t_2 = -{}^1\!/_2\, x_2 + 1{}^3\!/_4,$$

which obviously contains the point $(x_2, x_1) = (0, 1{}^3\!/_4)$. According to the symmetry condition stated in the theorem, the parallel-shifted separating straight line must contain the point $((p-1), ((p-1) - 1{}^3\!/_4))$, and its equation will be

$$x_1 = -{}^1\!/_2\, x_2 + t'_2,$$

from where $t'_2 = x_1 + {}^1\!/_2\, x_2 = ((p-1) - 1{}^3\!/_4) + {}^1\!/_2\, (p-1) = 1{}^1\!/_4 + {}^1\!/_2(3) = 2{}^3\!/_4$.

From Fig. 6 (Right) it is easy to see that the threshold used by the semidual function $\sigma_2(f(x))$ for the shifted separating line is indeed $2{}^3\!/_4$.

Remark 4. *Theorem 4 indicates that given a threshold function, some of its semidual functions may be non-threshold functions. An example is shown in Fig. 7. This fact may also be seen from the opposite point of view: there are non-linear separable functions such that some of their semidual functions are threshold. This is based on the involutive character of semidual functions, as mentioned above in Remark 3.*

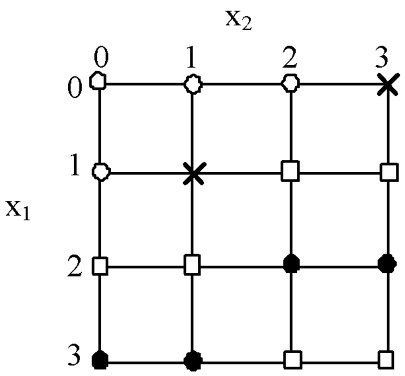

Figure 7. A non-threshold function obtained as $\sigma_3(f(x))$ based on the threshold function shown in Fig. 6 (Left)

Notice that if the function of Fig. 7 is called $g(x)$, then $\sigma_3(g(x)) = f(x)$, the threshold function of Fig. 6, i.e., in this case, a semidual function of a non-threshold function, is a threshold function.

Remark 5. *Let $f(x)$ be a threshold function such that for some $z \in Z_p$, the set sym equals the subdomain for z, i.e, there is no x outside sym such that $f(x) = z$ and there is no x inside sym such that $f(x) \quad z$. Then $\sigma_z(f(x))$ and $\sigma_{z+1}(f(x))$ are also threshold functions, since they eliminate the z subdomain by merging it with the preceding or the succeeding domains, respectively. $(SeeexampleinFig.8)$. All other semidual functions of $f(x)$ are also threshold, but trivial, since they do not change $f(x)$.*

	0	1	2	3	4
0	0	0	1	2	3
1	0	1	2	2	3
2	0	1	2	3	3
3	1	2	2	3	4
4	1	2	3	3	4

	0	1	2	3	4
0	0	0	1	1	3
1	0	1	1	1	3
2	0	1	1	3	3
3	1	1	1	3	4
4	1	1	3	3	4

	0	1	2	3	4
0	0	0	1	3	3
1	0	1	3	3	3
2	0	1	3	3	3
3	1	3	3	3	4
4	1	3	3	3	4

Figure 8. Example of a 2-place 5-valued threshold function $f(x)$ such that $f(x) = 2$ iff $x \in sym$. Both $\sigma_2(f(x))$ and $\sigma_3(f(x))$ are threshold (meanwhile $\sigma_1(f(x))$ and $\sigma_4(f(x))$ are trivial)

Remark 6. *Nomura's characterization of threshold functions is very strong. If two functions have the same characterizing superset, then they are not threshold functions. If two functions have permuted c-supersets and the same n-superset, then these functions are related by the same permutation of their arguments. If one of them is a threshold function, the other will also be a threshold function $(Theorem3)$.*

Theorems 2 and 3 have been used in the past to induce a partition on the set of ternary functions of 2 and 3 variables and to generate tables with characterization and realization supersets of canonical representative functions [13], [9]. Theorem 4 would allow to reduce the length of those ternary tables by slightly over 50% since in the original table, a threshold function and its non-trivial semidual function(s) are in different blocks [11].

4. New roads to be explored

With the advent of new technologies, other possibilities may appear to apply or extend multiple-valued logic. As an example, in what follows the case of optical computing using dual channel modulators (see *e.g.* [14]) will be considered.

It is well known that the function maximum is not a threshold function. In the case of quaternary logic, however, if the domain for the variables and the function is defined as $\{0,\ 1\}$ and a function $g(x_1^2, x_2^2) = \max(x_1, x_2)$ is considered, then, as can be seen in Fig. 9, the function g is linear separable.

If $1/9 > (x_1^2 + x_2^2)$ then $f(x) = 0$
If $4/9 > (x_1^2 + x_2^2) \geq 1/9$ then $f(x) = 1/3$
If $1 > (x_1^2 + x_2^2) \geq 4/9$ then $f(x) = 2/3$

If $(x_1^2 + x_2^2) \geq 1$ then $f(x) = 1$

As usual in kernel methods, it is possible to project the linear separation of g back to the original function maximum to obtain a set of concentric circles, centered at $(0, 0)$ and with increasing radius. Furthermore, as discussed in an earlier example, it is possible to obtain equivalent results by replacing the circles with 1-place literals, which are two-valued threshold functions. Such a realization would be more expensive (in terms of area and energy consumption) than a classical analogue one, but would exhibit a high S/N ratio.

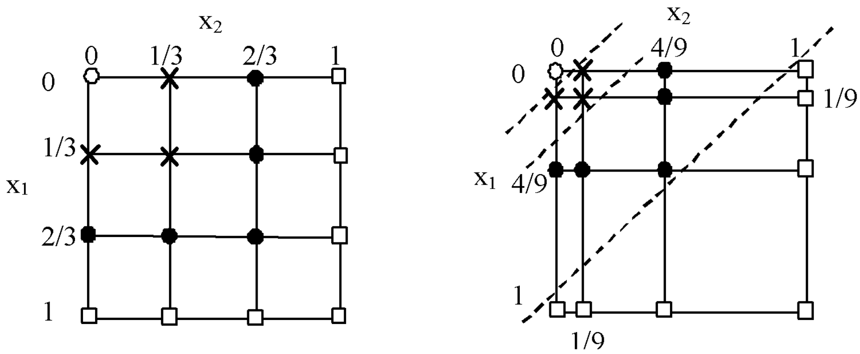

Figure 9. (Left) The quaternary non-threshold maximum function. (Right) The transformed function $g(x)$, which is linear separable.

A direct realization of g in the frame of optical computing and using the idea of multilineal separability – (a linear separable function is trivially multi-lineal separable)– is however possible, as illustrated in Fig. 10. Double channel modulators provide for the squaring of the arguments. Lenses provide for summation of light-signals, meanwhile non-linear optical devices [15] are appropriate for thresholding purposes.

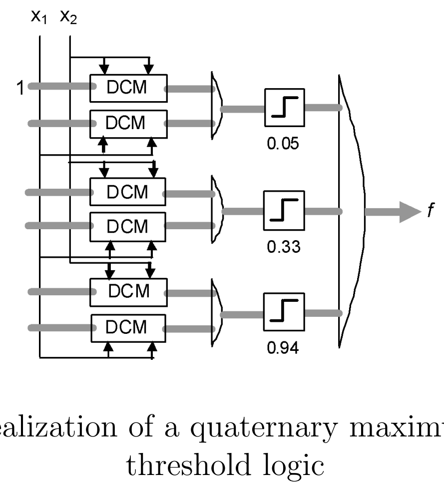

Figure 10. Optical realization of a quaternary maximum function based on threshold logic

Last but not least, it should be pointed out that multiple-valued threshold logic is closely related to neural networks. What gates are in one context, becomes neurons in the other. A combination of methods and experiences from both areas of research can only lead to richer positive results [16]. This approach was also initiated by Prof. Naum Aizenberg and is documented in [17], considering multiple-valued neural networks in the complex plane. A recent contribution in the same direction may be found in [18].

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