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CONDITIONS OF EXISTENCE WITH PROBABILITY ONE GENERALIZED SOLUTION OF THE BOUNDARY-VALUE PROBLEMS OF HYPERBOLIC EQUATIONS WITH RANDOM INITIAL CONDITIONS

Conditions of existence with probability one generalized solution of hyperbolic equations type partial differential equation of mathematical physics with random strongly $Sub_{\varphi}(\Omega)$ initial conditions are found in the multidimensional case.

В роботі знайдено умови існування з імовірністю одиниця узагальненого розв'язку гіперболічного рівняння в частинних похідних математичної фізики з строго $Sub_{\varphi}(\Omega)$ випадковими початковими умовами у баготовимірному випадку.

Introduction. Boundary value problem for a homogeneous hyperbolic partial differential equations of mathematical physics with random strongly $Sub_{\varphi}(\Omega)$ initial conditions is considered in the work. For such problem conditions of existence with probability one generalized solution are found.

Similar problems are considered in [4], [5], [7], [8]. Further references can be found in [1].

1. Stochastic processes of the space $Sub_{\varphi}(\Omega)$.

Definition 1 ([1]). *Let T be a nonempty set. A function $\rho: \quad T\times T\to[0,\infty)$ is called a pseudometric if*

1) $\rho(t,s)=\rho(s,t), \quad t,s\in T,$

2) $\rho(t,s)\leqslant\rho(t,v)+\rho(v,s), \quad t,s,v\in T,$

3) $\rho(t,s)=0,$ *if* $t=s$.

The pair (T,ρ) is called a pseudometric space.

Definition 2 ([1]). *Let (T,ρ) be a nonempty metric space and let $\varepsilon>0$. Denote by $N_{\rho}(t,\varepsilon)$ the minimum number of points of an ε-net of the set T with respect to the pseudometric ρ. The function $N_{\rho}(t,\ \varepsilon),\ \varepsilon>0$ is called the massiveness of the set T with respect to the pseudometric ρ.*

Definition 3 ([2]). *A continuous even function $u(x),\ x\in R^1$, such that $u(0)=0$, $u(x)>0$ for $x\neq 0$ and $\lim\limits_{x\to0}\frac{u(x)}{x}=0$, $\lim\limits_{x\to\infty}\frac{u(x)}{x}=\infty$ is called an N-function.*

Lemma 1 ([1]). *Let $u(x)$ be an N-function. Then*

1) $u(\alpha\, x)\leqslant\alpha\, u(x)$ *for* $0\leqslant\alpha\leqslant1$ *and* $x\in R$;

2) $u(\alpha\, x)\geqslant\alpha\, u(x)$ *for* $\alpha>1$ *and* $x\in R$;

3) $u(|x|+|y|)\leqslant u(x)+u(y)$ *for* $x,y\in R$;

4) *the function* $\frac{u(x)}{x}$ *is nondecreasing for* $x>0$.

Lemma 2 ([1]). *Let $u^{(-1)}(x)$ be the inverse to an N-function $u(x)$ for $x > 0$. Then $u^{(-1)}(x)$ is a convex increasing function such that*

1) $u^{(-1)}(\alpha\, x) \leqslant \alpha\, u^{(-1)}(x)$ *for* $\alpha > 1$ *and* $x \in R$;

2) $u^{(-1)}(\alpha\, x) \geqslant \alpha\, u^{(-1)}(x)$ *for* $0 \leqslant \alpha \leqslant 1$ *and* $x \in R$;

3) $u^{(-1)}(|x| + |y|) \geq u^{(-1)}(x) + u(y)$ *for* $x, y \in R$;

4) *the function* $\frac{u^{(-1)}(x)}{x}$ *is nonincreasing for* $x > 0$.

Definition 4 ([2]). *Let $u(x)$ be an N-function. The function*

$$u^{*}(x) = \sup_{y \in R} (xy - u(y))$$

is called the Young-Fenchel transform of the function $u(x)$.

The function $u^{*}(x)$ also is N-function.

Definition 5 ([1]). *Let $\varphi(x)$ be an N-function for which there exist constants $x_0 > 0$ and $c > 0$ such that $\varphi(x) = cx^2$ for $|x| < x_0$. The set of random variables $\xi(\omega)$, $\omega \in \Omega$ is called the space $Sub_{\varphi}(\Omega)$ generated by the N-function $\varphi(x)$ if $E\xi = 0$ and there exists a constants a_{ξ} such that*

$$E \exp\{\lambda \xi\} \leqslant \exp\{\varphi(\lambda a_{\xi})\}$$

for all $\lambda \in R^1$.

The space $Sub_{\varphi}(\Omega)$ is a Banach space with respect to the norm

$$\tau_{\varphi}(\xi) = \sup_{\lambda \neq 0} \frac{\varphi^{(-1)}(\ln E \exp\{\lambda \xi\})}{|\lambda|}.$$

Definition 6 ([1]). *A stochastic process $X = \{X(t),\ t \in T\}$ belongs to the space $Sub_{\varphi}(\Omega)$ $(X \in Sub_{\varphi}(\Omega))$ if $X(t) \in Sub_{\varphi}(\Omega)$ for all $t \in T$.*

Lemma 3 ([1]). *If $\xi \in Sub_{\varphi}(\Omega)$, then there exists a constants $C > 0$ such that*

$$(E(\xi)^2)^{1/2} \leq C \tau_{\varphi}(\xi).$$

Definition 7 ([1]). *A random variable $\xi \in Sub_{\varphi}(\Omega)$ is called strongly $Sub_{\varphi}(\Omega)$ random variable, if $\tau_{\varphi}(\xi) = (E\xi^2)^{1/2}$. The space of strongly $Sub_{\varphi}(\Omega)$ random variables is denoted by $SSub_{\varphi}(\Omega)$.*

Properties and applications of $SSub_{\varphi}(\Omega)$ random variables and stochastic processes can be found in [1].

Definition 8 ([3]). *A family Δ of random variables ξ of the space $Sub_{\varphi}(\Omega)$ is called $SSub_{\varphi}(\Omega)$ family*

$$\tau_{\varphi}\left(\sum_{i \in I} \lambda_i \xi_i\right) = \left(E\left(\sum_{i \in I} \lambda_i \xi_i\right)^2\right)^{1/2},$$

for all $\lambda_i \in R^1$, where I is at most countable and $\xi_i \in \Delta_i$, $i \in I$.

Theorem 1 ([3]). *Let Δ be a strongly $Sub_{\varphi}(\Omega)$ family of random variables. Then the linear closure $\bar{\Delta}$ of the family Δ in the $L_2(\Omega)$ and in the mean square sense is a strongly $Sub_{\varphi}(\Omega)$ family.*

Definition 9 ([1]). *A stochastic process $X_i = \{X_i(t), \ t \in T, \ i \in I\}$ is called an $SSub_{\varphi}(\Omega)$ process if the family of random variables $X_i = \{X_i(t), \ t \in T, \ i \in I\}$ is a $SSub_{\varphi}(\Omega)$ family.*

Theorem 2 ([3]). *Let $X_i = \{X_i(t), \ t \in T, \ i \in I\}$ be a family of jointly $SSub_{\varphi}(\Omega)$ stochastic processes. Then (T, O, μ) is a measurable space. If*

$$\left\{\varphi_{k_i}(t), \ i \in I, \ k = \overline{1, \infty}\right\}$$

is a family of measurable functions in (T, O, μ) and the integral

$$\xi_{k_i} = \int\limits_T \varphi_k(t) X_j(t) \, d\mu(t),$$

is well defined in the mean square sense, then the family of random variables

$$\Delta_{\xi} = \left\{\xi_{k_i}, \ i \in I, \ k = \overline{1, \infty}\right\}$$

is an $SSub_{\varphi}(\Omega)$ family.

Remark 1. *A Gaussian stochastic process with zero mean is an $SSub_{\varphi}(\Omega)$ process for*

$$u(x) = \frac{x^2}{2}.$$

2. The justification of the Fourier method for a partial differential equation with random initial conditions.

Consider the equation

$$\frac{\partial^2 u}{\partial t^2} = L(u), \tag{1}$$

for

$$L(u) = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial}{\partial x_i} \left(a_{ij}(X) \frac{\partial u}{dx_j} \right) - a(X)u.$$

The coefficients of the operator L are defined in a finite connected domain G of dimension n, let

$$X = (x_1, \ x_2, \ \ldots, x_n)$$

be are arbitrary point of G. Assume that

$$a(X) = 0, \quad a_{ij} = a_{ji}, \quad \sum_{i,j=1}^{n} a_{ij} \gamma_i \gamma_j \geqslant \alpha \sum_{i=1}^{n} \gamma_i^2, \quad \alpha > 0$$

in the domain G.

Consider the following problem for equations (1): solve equation (1) in the cylinder $Q_T = G[0 < t < T]$ for the initial conditions

$$u|_{t=0} = \xi(X), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = \eta(X) \tag{2}$$

and the boundary condition

$$u|_S = 0, \quad t \in [0, T], \tag{3}$$

where S is the boundary of the domain G. Assume that the initial conditions

$$(\xi(X), X \in G), (\eta(X) X \in G)$$

are jointly $SSub_\varphi(\Omega)$ stochastic processes.

When solving similar problems by using the Fourier method, regardless of whether initial conditions are random or nonrandom we look for a solution of the form [6]

$$u(X,\ t) = \sum_{k=1}^{\infty} \left(A_k \cos\sqrt{\lambda_k} t + B_k \sin\sqrt{\lambda_k} t \right) v_k(X), \tag{4}$$

where

$$A_k = \int_G \xi(X) v_k(X) dX, \quad B_k = \frac{1}{\sqrt{\lambda_k}} \int_G \eta(X) v_k(X) dX,$$

and the λ_k and $v_k(X)$ are eigenvalues and eigenfunctions of the Sturm-Liouville problem

$$L(v) + \lambda v = 0.$$

Definition 10. *The solution* (4) *is called generalized solution of problem* (1)–(3) *in the domain* $0 \le x_i \le S_i$, $0 \le t \le T$ *if series* (4) *converge uniformly in probability.*

Lemma 4 ([4]). *Let initial conditions*

$$(\xi(X),\ X \in G) \text{ and } (\eta(X),\ X \in G)$$

be jointly $SSub_\varphi(\Omega)$ *stochastic processes and assume that the series* (4) *converge uniformly in probability. Then the random series* (4) *also are jointly* $SSub_\varphi(\Omega)$ *stochastic process.*

For $n \geqslant 0$ put

$$S_n = \sum_{k=1}^{n} \left(A_k \cos\sqrt{\lambda_k} t + B_k \sin\sqrt{\lambda_k} t \right) v_k(X).$$

Theorem 3. *Let* $\xi(X)$, $X \in G$, *and* $\eta(X)$, $X \in G$, *be a jointly* $SSub_\varphi(\Omega)$ *stochastic processes. In order that a generalized solution of problem* (1)–(3) *exist in the domain of variables* $(t,\ x_1,\ x_2,\ \ldots,\ x_n)$ *such that* $0 \le t \le T$, $G = \{0 \le x_i \le$ $\le S_i,\ i = 1, \ldots, n\}$ *(*T *is a positive constants), and be represented in the form of series* (4) *it is sufficient that:*

1) *for all* $X \in G$ *and* $t \in [0, T]$, *the series*

$$\sum_{k=1}^{\infty} \sum_{l=1}^{\infty} v_k(X) v_l(X) \Big[EA_kA_l \cos\sqrt{\lambda_k} t \cos\sqrt{\lambda_l} t + EB_kB_l \sin\sqrt{\lambda_k} t \sin\sqrt{\lambda_l} t +$$

$$+ 2EA_kB_l \cos\sqrt{\lambda_k} t \sin\sqrt{\lambda_l} t \,\Big] < \infty;$$

2) for $n \geqslant 1$

$$\sup_{\substack{|x_i-y_i|\le h\\ |t-s|\le h}} \left(E\left|S_n(X,t)-S_n(Y,s)\right|^2\right)^{\frac{1}{2}} \le \sigma(h),$$

where $\sigma(h)$ *is a monotone increasing continuous function such that* $\sigma(h) \to 0$ *as* $h \to 0$*, moreover*

$$\int_{0+} \Psi\left(\ln\frac{1}{\sigma^{(-1)}(\varepsilon)}\right) d\varepsilon < \infty, \tag{5}$$

where $\Psi(u) = \frac{u}{\varphi^{(-1)}(u)}$, $\sigma^{(-1)}(\varepsilon)$ *is the inverse function to* $\sigma(\varepsilon)$.

Proof. Condition 2) implies that series (4) converge in the mean square sense. According to theorem 3.6 in the work [4] and Lemma 4, series (4) converge in probability in the space $C(G\times[0,T])$.

Example 1. *Assume that* $\xi(X)$ *and* $\eta(X)$ *are jointly* $SSub_\varphi(\Omega)$ *stochastic processes. Then theorem 2 and 1 (also see lemma 4) imply that* $S_n(t,X)$ *are jointly* $Sub_\varphi(\Omega)$ *stochastic processes. Let* $\varphi(x)$ *be a function such that* $\varphi(x)=|x|^p$ *for some* $p>1$ *and all* $|x|>1$*. Then* $\Psi(x)=x^{1-\frac{1}{p}}$ *for* $x>1$ *and condition* (5) *holds for all* $\varepsilon>0$*:*

$$\int_{0+}\left(\ln\frac{1}{\sigma_k^{(-1)}(u)}\right)^{1-\frac{1}{p}} du < \infty. \tag{6}$$

Conditions (6) *holds if* $\sigma(h) = \frac{C}{|\ln|h||^\delta}$ *for* $\delta > 1-\frac{1}{p}$ *and* $C>0$*. In this case, assumption* 2) *of theorem 3 is satisfied there exist constants* $C>0$ *such*

$$\left(E\left|S_n(t)-S_n(s)\right|^2\right)^{1/2} \leqslant \frac{C}{|\ln|h||^\delta}, \tag{7}$$

for $\delta > 1-\frac{1}{p}$ *and sufficiently small* $|h|$.

Lemma 5 ([4])**.** *Let*

$$G_n(X,t)=\sum_{l=1}^{n}\left(\xi_l\cos\sqrt{\lambda_l}t+\eta_l\sin\sqrt{\lambda_l}t\right)Z_l(X),\quad X\in G\ \ t\in[0,T],$$

let $Z_l(X)$ *be a continuous function, and let* ξ_l *and* η_l *be random variables such that* $E\eta_l^2<\infty$ *and* $E\xi_l^2<\infty$*. If*

$$\sup_{X\in G}|Z_l(X)|\le\delta_l,$$

$$\sup_{\substack{|x_i-y_i|\le h\\ i=1,\dots,m}}|Z_l(X)-Z_l(Y)|\le z_l\frac{1}{|\ln|h||^\delta},\quad \delta>0, |h|<1,$$

$$\sum_{l=1}^{\infty}\left(\left(E\xi_l^2\right)^{\frac{1}{2}}+\left(E\eta_l^2\right)^{\frac{1}{2}}\right)\left(z_l+\delta_l(\ln\lambda_l)^\delta\right)<\infty,$$

then

$$\sup_{\substack{|x_i-y_i|\le h\\ |t-s|\le h\\ i=1,\dots,m}} \left(E|S_n(X,t)-S_n(Y,s)|^2\right)^{\frac{1}{2}} \le \frac{C}{|ln|h||^{\delta}},$$

for $|h|<1$ *where*

$$C=\sum_{l=1}^{\infty}\left((E\xi_l^2)^{\frac{1}{2}}+(E\eta_l^2)^{\frac{1}{2}}\left(z_l+\delta_l\left(\ln\left(\frac{\sqrt{\lambda_l}}{2}+e^{\delta}\right)\right)^{\delta}\right)\right).$$

Theorem 4. *Let* $\xi(X)$, $X\in G$, *and* $\eta(X)$, $X\in G$, *be* $SSub_{\varphi}(\Omega)$ *stochastic proceses, where* $\varphi(x)$ *is a function such that* $\varphi(x)=|x|^p$ *for some* $p>1$ *and all* $|x|>1$. *Set*

$$B(X,Y)=E\xi(X)\xi(Y),$$

$$R(X,Y)=E\eta(X)\eta(Y).$$

In order that a generalized solution of problem (1)–(3) *exist with probability one in the domain* $0\le t\le T$, $G=\{0\le x_i\le S_i, i=1,\dots,m\}$, *and be represented in the form of series* (4), *uniformly convergent in probability, it is sufficient that:*

1) *for sufficiently small* h

$$\sup_{\substack{|x_i-y_i|\le h\\ i=1,\dots,m}} (B(X,X)+B(Y,Y)-2B(X,Y))^{\frac{1}{2}} \le \frac{C}{|ln\,h|^{\delta}},$$

$$\sup_{\substack{|x_i-y_i|\le h\\ i=1,\dots,m}} (R(X,X)+R(Y,Y)-2R(X,Y))^{\frac{1}{2}} \le \frac{C_{z_1}}{|ln\,h|^{\delta}},$$

where $\delta>1-\frac{1}{p}$; $i=1,\ \dots,\ n$;

2) *the series*

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} r_k r_l\left[|EA_kA_l|+|EB_kB_l|+2|EA_kB_k|\right]<\infty$$

converges where $r_k=\max\limits_{i,j=1,\dots,n}(\lambda_k v_k)$, $v_k=\sup\limits_{X\in G}|v_k(X)|$;

3) $\sup\limits_{X\in G}|v_l(X)|\le\mu_l$, *and*

$$\sup_{\substack{|x_k-y_k|<h\\ k=1,\dots,m}} |v_l(X)-v_l(Y)|\le\gamma_l\frac{1}{|ln\,h|^{\delta}},$$

$$\sum_{l=1}^{\infty}\left((EA_l^2)^{\frac{1}{2}}+(EB_l^2)^{\frac{1}{2}}\right)\lambda_l(\mu_l+(\ln\,\lambda_l)^{\delta}\gamma_l)<\infty,\ i=1,\ \dots,\ n,$$

for arbitrary $\delta>1-\frac{1}{p}$ *and* $|h|<1$.

Proof. According to example 1, conditions of theorem 3.9 [4] hold for the processes $\xi(X)$ and $\eta(X)$ if

$$\sigma(h) = \frac{C}{|\ln |h||^{\delta}}, \quad \delta > 1 - \frac{1}{p}.$$

It is clear that the series in condition 2) of theorem 3 converge if so do the series in condition 2) theorem 4. Example 1 and Lemma 5 imply that condition 3) of theorem 3 follows from condition 3) of theorem 4.

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