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AN ESTIMATE OF THE PARAMETER OF THE RANDOM FIELD COVARIANCE FUNCTION

The problem of the parameter estimation of the non-Gaussian random field covariance function is considered. We obtained a consistent estimation of the unknown parameter by using Baxter statistics. Also the non-asymptotic confidence regions are constructed.

Розглядається задача оцінювання параметра коваріаційної функції негауссового випадкового поля. За допомогою бакстерівських статистик отримано консистентну оцінку невідомого параметра. Також побудовано неасимптотичні довірчі області.

Introduction. The Baxter type theorems for random processes and fields have been investigated by P. Levi [1], G. Baxter [2], E.G. Gladyshev [3], S.M. Berman [4], T.V. Arak [5], T. Kawada [6], S.M. Krasnitskii [7], X. Guyon [8], O.O. Kurchenko [9] and others. V. V. Buldygin and Y. V. Kozachenko [10] obtained the conditions of the convergence of Baxter sums for jointly strictly sub–Gaussian random processes and jointly pseudo–Gaussian random processes. O. O. Kurchenko [11] proved the Baxter type theorem for jointly strictly sub–Gaussian random fields. Y. V. Kozachenko and O. O. Kurchenko [12] obtained the Levy–Baxter limit theorems for certain class of non–Gaussian random processes.

The Baxter type theorems we used for parametric estimation in statistics of random processes and fields in papers by R. E. Maiboroda [13], Y. V. Kozachenko and O. O. Kurchenko [14], O. O. Kurchenko [15], J-C. Breton, I. Nourdin, G. Peccati [16] and others. O. O. Synyavska in paper [17] obtained the strong consistent estimate of the unknown parameter of the covariance function for fractional anisotropic Wiener field.

In this paper we study the problem of a estimation of the parameter of the covariance function of one non-Gaussian random field. This consistent estimate is based on the Levy–Baxter theorems. We also construct the confidence regions of this estimate.

1. Preliminaries. Let (Ω, F, P) be a probabilistic space.

Definition 1. *[12] A random vector $(\xi,\eta) \in L_4(\Omega)\times L_4(\Omega)$ has the property K if*

1) $E\xi = E\eta = 0$,

2) $E(\xi \pm \eta)^4 \le 3\left(E\left(\xi\pm\eta\right)^2\right)^2$.

The class of all two-dimensional vectors with property K is denoted by K. Let us define the subclass K_1 of the class K as the set of all vectors of class K for which $E(\xi \pm \eta)^4 = 3\left(E\left(\xi\pm\eta\right)^2\right)^2$.

Lemma 1. *Let $\xi^{(1)},\dots,\xi^{(n)} \in K_1$ be a independent random vectors. Then for any real numbers $\alpha_1,\dots,\alpha_n$ the random vector $\sum_{i=1}^n \alpha_i\xi^{(i)}$ belongs to class K_1.*

Proof. The random vector $\alpha_1\xi^{(1)}+\ldots+\alpha_n\xi^{(n)}$ has zero mean and $E\Big(\xi_1^{(i)}\pm\xi_2^{(i)}\Big)^4=3E\left(\left(\xi_1^{(i)}\pm\xi_2^{(i)}\right)^2\right)^2$, where $\xi^{(i)}=\left(\xi_1^{(i)},\xi_2^{(i)}\right)$, $\alpha_i\in\mathbb{R}$, $i=\overline{1,n}$. Taking the consideration the independence of the random variables $\alpha_i\xi_1^{(i)}$, $\overline{1,n}$ and $\alpha_i\xi_2^{(i)}$, $i=\overline{1,n}$, we obtain

$$E\left(\sum_{i=1}^{n}\alpha_i\left(\xi_1^{(i)}\pm\xi_2^{(i)}\right)\right)^4=\sum_{i=1}^{n}\alpha_i^4E\left(\xi_1^{(i)}\pm\xi_2^{(i)}\right)^4+$$

$$+6\sum_{\substack{i,j=1\\i<j}}^{n}\alpha_i^2\alpha_j^2E\left(\xi_1^{(i)}\pm\xi_2^{(i)}\right)^2E\left(\xi_1^{(j)}\pm\xi_2^{(j)}\right)^2=$$

$$3\left(\sum_{i=1}^{n}\alpha_i^4\left(E\left(\xi_1^{(i)}\pm\xi_2^{(i)}\right)^2\right)^2+2\sum_{\substack{i,j=1\\i<j}}^{n}\alpha_i^2\alpha_j^2E\left(\xi_1^{(i)}\pm\xi_2^{(i)}\right)^2E\left(\xi_1^{(j)}\pm\xi_2^{(j)}\right)^2\right)=$$

$$3\left(\sum_{i=1}^{n}\alpha_i^2E\left(\xi_1^{(i)}\pm\xi_2^{(i)}\right)^2\right)^2=3\left(E\left(\sum_{i=1}^{n}\alpha_i\left(\xi_1^{(i)}\pm\xi_2^{(i)}\right)\right)^2\right)^2.$$

So, the random vector $\sum_{i=1}^{n}\alpha_i\xi^{(i)}$ belongs to class K_1. The lemma is proved.

Lemma 2. *Let a random vector (ξ,η) belong to class K_1. Then the following inequality holds:*

$$E(\xi^2\eta^2)\le 2(E\xi\eta)^2+E\xi^2E\eta^2.$$

Proof. Thanks the Definition 1, we can write

$$E\left(\xi^4+4\xi\eta^3+6\xi^2\eta^2+4\xi^3\eta+\eta^4\right)\le 3\left(E\xi^2+2E\xi\eta+E\eta^2\right)^2,$$

$$E\left(\xi^4-4\xi\eta^3+6\xi^2\eta^2-4\xi^3\eta+\eta^4\right)\le 3\left(E\xi^2-2E\xi\eta+E\eta^2\right)^2.$$

Let us multiply this inequalities on $\frac{1}{2}$ and add them:

$$E\left(\xi^4+6\xi^2\eta^2+\eta^4\right)\le 3\left(\left(E\xi^2\right)^2+4\left(E\xi\eta\right)^2+\left(E\eta^2\right)^2+2E\xi^2E\eta^2\right).$$

Since the random vectors (ξ,η) belong to class K_1, it follows that $E(\xi^2\eta^2)\le 2(E\xi\eta)^2+E\xi^2E\eta^2$. The lemma is proved.

Definition 2. *[12] A stochastic process $\{X(t),t\in[0,1]\}$ is said to be a process with first and second-order increments of class K_1 if for any $0\le s\le t\le u\le v\le 1$ the random vectors $(\xi_1,\eta_1),(\xi_2,\eta_2)$, where $\xi_1=X(t)-X(s),\eta_1=X(v)-X(u)$, $\xi_2=X(t)-2X\left(\frac{t+s}{2}\right)+X(s),\eta_2=X(v)-2X\left(\frac{u+v}{2}\right)+X(u)$, belong to class K_1.*

2. Statement of the problem. Let the random field $B(t),t\in[0,1]^2$ with zero mean and covariance function

$$EB(t)B(s)=\frac{1}{4}\left(|t_1|^{2H_1}+|s_1|^{2H_1}-|t_1-s_1|^{2H_1}\right)\left(|t_2|^{2H_2}+|s_2|^{2H_2}-|t_2-s_2|^{2H_2}\right),$$

where $H = (H_1, H_2)$, $0 < H_i < 1, i = 1,2; t = (t_1, t_2), s = (s_1, s_2) \in [0,1]^2$, is observed on the sides $[0,1]^2$:

$$E_2 = \left\{ \left(1, \frac{k}{a_n}\right), \left(\frac{k}{a_n}, 1\right) \,\Big|\, 0 \le k \le a_n - 1, n \ge 1 \right\},$$

$(a_n) \subset \mathbb{N}, a_n \to \infty, n \to \infty$. It is necessary to construct the estimate of the unknown parameter $H = (H_1, H_2)$, $H_i \in (0, H_i^*]$, $H_i^* \in (0,1)$ by those observations. Let us assume that for any $\alpha > 0$ the series $\sum_{n=1}^{\infty} (a_n)^{-\alpha}$ is convergent.

The process $B_1(t_1) = B(t_1, 1)$, $0 \le t_1 \le 1$ is a process with zero mean and covariance function $r(t_1, s_1) = \frac{1}{2}\left(|t_1|^{2H_1} + |s_1|^{2H_1} - |t_1 - s_1|^{2H_1}\right)$, $H_1 \in (0,1)$, $t_1, s_1 \in [0,1]$. The process $B_2(t_2) = B(1, t_2)$, $0 \le t_2 \le 1$ is a process with zero mean and covariance function $r(t_2, s_2) = \frac{1}{2}\left(|t_2|^{2H_2} + |s_2|^{2H_2} - |t_2 - s_2|^{2H_2}\right)$, $H_2 \in (0,1)$, $t_2, s_2 \in [0,1]$. We demand that the processes $\{B_i(t_i),\, t_i \in [0,1],\}$, $i = 1,2$ were random processes with first and second-order increments of class K_1.

3. The results. We denote

$$\Delta_1 B_i\left(\frac{k}{a_n}\right) = B_i\left(\frac{k+1}{a_n}\right) - B_i\left(\frac{k}{a_n}\right),$$

$$\Delta_2 B_i\left(\frac{k}{a_n}\right) = B_i\left(\frac{k+1}{a_n}\right) - 2B_i\left(\frac{k+0.5}{a_n}\right) + B_i\left(\frac{k}{a_n}\right), \ 0 \le k \le a_n - 1.$$

Let

$$S_{n,i}^{(j)} = \sum_{k=0}^{a_n-1} \left(\Delta_j B_i\left(\frac{k}{a_n}\right)\right)^2, \ \widehat{S}_{n,i}^{(j)} = a_n^{2H_i-1} S_{n,i}^{(j)}, \ i = 1,2, \ n \ge 1, \ j = 1,2,$$

be a sequence of Baxter sums.

Lemma 3. *Let $\{B_i(t_i),\, t_i \in [0,1],\}$, $i = 1,2$ be a random process with zero mean and covariance function $r(t_i, s_i)$, $i = 1,2$. Then the following inequality holds:*

$$\sup_{H_i \in (0, H_i^*]} \mathrm{Var}\widehat{S}_{n,i}^{(1)} \le \begin{cases} \frac{4}{a_n}\left(3 + 2\zeta(4 - 4H_i^*)\right), & \text{if } H_i^* \in (0, \frac{3}{4}); \\ \frac{4}{a_n}\left(3 + 2\left(1 + \ln a_n\right)\right), & \text{if } H_i^* = \frac{3}{4}; \\ \frac{4}{a_n}\left(3 + 2\frac{a_n^{4H_i^*-3}}{4H_i^*-3}\right), & \text{if } H_i^* \in (\frac{3}{4}, 1), \end{cases}, \ i = 1,2,$$

where $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$, $s > 1$.

Proof. We have $\mathrm{Var}\widehat{S}_{n,i}^{(1)} = a_n^{4H_i-2}\mathrm{Var}S_{n,i}^{(1)}$;

$$\mathrm{Var}S_{n,i}^{(1)} = E\left(S_{n,i}^{(1)} - ES_{n,i}^{(1)}\right)^2 = E\left(\sum_{k=0}^{a_n-1}\left(\Delta_1 B_i\left(\frac{k}{a_n}\right)\right)^2 - \sum_{k=0}^{a_n-1} E\left(\Delta_1 B_i\left(\frac{k}{a_n}\right)\right)^2\right)^2 =$$

$$= E\left(\sum_{k,j=0}^{a_n-1}\left(\left(\Delta_1 B_i\left(\frac{k}{a_n}\right)\right)^2 - E\left(\Delta_1 B_i\left(\frac{k}{a_n}\right)\right)^2\right)\left(\left(\Delta_1 B_i\left(\frac{j}{a_n}\right)\right)^2 - \right.\right.$$

$$\left.\left. - E\left(\Delta_1 B_i\left(\frac{j}{a_n}\right)\right)^2\right)\right) = \sum_{k,j=0}^{a_n-1}\left(E\left(\Delta_1 B_i\left(\frac{k}{a_n}\right)\right)^2 \times\right.$$

$$\times\left(\Delta_1 B_i\left(\frac{j}{a_n}\right)\right)^2 - E\left(\Delta_1 B_i\left(\frac{k}{a_n}\right)\right)^2 E\left(\Delta_1 B_i\left(\frac{j}{a_n}\right)\right)^2\Bigg).$$

By using the Lemma 2, we can write

$$\mathrm{Var} S_{n,i}^{(1)} \le 2\sum_{k,j=0}^{a_n-1}\left(E\Delta_1 B_i\left(\frac{k}{a_n}\right)\Delta_1 B_i\left(\frac{j}{a_n}\right)\right)^2.$$

Further, $E\Delta_1 B_i\left(\frac{k}{a_n}\right)\Delta_1 B_i\left(\frac{j}{a_n}\right) = \frac{1}{2}\left|\frac{(k-j)+1}{a_n}\right|^{2H_i} - \left|\frac{k-j}{a_n}\right|^{2H_i} + \frac{1}{2}\left|\frac{(k-j)-1}{a_n}\right|^{2H_i}$, $0 \le k, j \le a_n - 1$, $i = 1, 2$. We put $z_l := (l+1)^{2H_i} - 2l^{2H_i} + (l-1)^{2H_i}$, $l \ge 1$. Hence,

$$\mathrm{Var} S_{n,i}^{(1)} \le 4\left(a_n^{1-4H_i} + \frac{a_n^{-4H_i}}{2}\sum_{\substack{k,j=0,\\ k<j}}^{a_n-1} z_{j-k}^2\right) \le 2a_n^{1-4H_i}\left(1 + \frac{1}{2}\sum_{l=1}^{a_n-1} z_l^2\right),\ i = 1, 2.$$

Since $z_1^2 = \left(2^{2H_i} - 2\right)^2 \le 4$ by $H_i \in (0,1)$, then $\mathrm{Var} S_{n,i}^{(1)} \le 4a_n^{1-4H_i}\left(3 + \frac{1}{2}\sum_{l=2}^{a_n-1} z_l^2\right)$, $i = 1, 2$. Moreover, z_l be a increment second-order of the function $f(x) = x^{2H_i}$, $x \ge 1$, $i = 1, 2$ in segment $[l-1, l+1]$. From the formula for the increments n–order [18, p. 244] , we have $z_l = 2H_i(2H_i - 1)\theta_l^{2H_i-2}$, $i = 1, 2$, where $\theta_l \in (l-1, l+1)$. Further, $\sup_{H_i\in(0,1)} |2H_i(2H_i - 1)| = 2$. As $l - 1 < \theta_l$, $l \ge 2$, then

$$z_l^2 \le \frac{4}{(l-1)^{4-4H_i}},\ l \ge 2,$$

Hence, $\mathrm{Var} S_{n,i}^{(1)} \le 4a_n^{1-4H_i}\left(3 + 2\sum_{l=2}^{a_n-1}\frac{1}{(l-1)^{4-4H_i}}\right)$, $i = 1, 2$. Consequently, $\mathrm{Var}\widehat{S}_{n,i}^{(1)} \le \frac{4}{a_n}\left(3 + 2\sum_{l=2}^{a_n-1}\frac{1}{(l-1)^{4-4H_i}}\right)$ and

$$\sup_{H_i\in(0,H_i^*]}\mathrm{Var}\widehat{S}_{n,i}^{(1)} \le \frac{4}{a_n}\left(3 + 2\sum_{l=2}^{a_n-1}\frac{1}{(l-1)^{4-4H_i^*}}\right),\ i = 1, 2. \tag{1}$$

Using the fact that $\sum_{l=1}^{a_n}\frac{1}{l^{4-4H_i^*}} \le \begin{cases}\zeta(4-4H_i^*), & H_i^* \in (0, \frac{3}{4});\\ 1 + \ln a_n, & H_i^* = \frac{3}{4};\\ \frac{a_n^{4H_i^*-3}}{4H_i^*-3}, & H_i^* \in \left(\frac{3}{4}, 1\right),\end{cases}$, $i = 1, 2$, where $\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}$, $s > 1$, and (1) the proof of lemma is done.

Theorem 1. $\widehat{S}_{n,i}^{(1)} \to 1$, $i = 1, 2$ *with probability one as* $n \to \infty$.

Proof. We have:

$$E\widehat{S}_{n,i}^{(1)} = a_n^{2H_i-1}a_n E\left(\Delta_1 B_i\left(\frac{k}{a_n}\right)\right)^2 = 1,\ n \ge 1,\ 0 \le k \le a_n - 1,\ i = 1, 2. \tag{2}$$

Using the Lemma 3 and the assumption for convergence of the series $\sum_{n=1}^{\infty} a_n^{-\alpha}$ for any $\alpha > 0$ we get $\sum_{n=1}^{\infty}\mathrm{Var}\widehat{S}_{n,i}^{(1)} < +\infty$ for $H_i \in (0,1)$, $i = 1, 2$. Then from [19, p. 24] $\widehat{S}_{n,i}^{(1)} - E\widehat{S}_{n,i}^{(1)} \to 0$ with probability one as $n \to \infty$. Thus from (2) it follows that $\widehat{S}_{n,i}^{(1)} \to 1$ with probability one as $n \to \infty$.

Lemma 4. *Let $\{B_i(t_i),\ t_i \in [0,1],\}$, $i=1,2$ be a random process with zero mean and covariance function $r(t_i,s_i)$, $i=1,2$. Then the following inequality holds:*

$$\sup_{H_i\in(0,1)} \mathrm{Var}\widehat{S}_{n,i}^{(2)} \le \frac{\kappa}{a_n},\ i=1,2,$$

where $\kappa = 38 + \left(\frac{9}{256}\right)^2 \frac{\pi^4}{45}$.

Proof. We found: $\mathrm{Var}\widehat{S}_{n,i}^{(2)} = a_n^{4H_i-2}\mathrm{Var}S_{n,i}^{(2)}$, $i=1,2$. From Lemma 2 we get

$$\mathrm{Var}S_{n,i}^{(2)} \le 2\sum_{k,j=0}^{a_n-1}\left(E\Delta_2 B_i\left(\frac{k}{a_n}\right)\Delta_2 B_i\left(\frac{j}{a_n}\right)\right)^2.$$

Then $E\left(\Delta_2 B_i\left(\frac{k}{a_n}\right)\Delta_2 B_i\left(\frac{j}{a_n}\right)\right) = \frac{1}{2}\Big(-\left|\frac{(k-j)-1}{a_n}\right|^{2H_i} + 4\left|\frac{(k-j)-1/2}{a_n}\right|^{2H_i} - 6\left|\frac{k-j}{a_n}\right|^{2H_i} + 4\left|\frac{(k-j)+1/2}{a_n}\right|^{2H_i} - \left|\frac{(k-j)+1}{a_n}\right|^{2H_i}\Big)$, $0 \le k,j \le a_n-1$, $i=1,2$. For $k=j$ also we have $E\left(\Delta_2 B_i\left(\frac{k}{a_n}\right)\right)^2 = a_n^{-2H_i}(2^{2-2H_i}-1)$, $i=1,2$.

Put $g_l = \left(|l-1|^{2H_i} - 4|l-\frac{1}{2}|^{2H_i} + 6l^{2H_i} - 4|l+\frac{1}{2}|^{2H_i} + |l+1|^{2H_i}\right)$, $l \ge 1, i=1,2$. Hence, $g_1 = \frac{6\cdot 2^{2H_i}-4-4\cdot 3^{2H_i}+2^{4H_i}}{2^{2H_i}}$, $i=1,2$ for $l=1$. Thus

$$\mathrm{Var}S_{n,i}^{(2)} \le 4\left(a_n^{1-4H_i}(2^{2-2H_i}-1)^2 + \frac{a_n^{-4H_i}}{2}\sum_{\substack{k,j=0,\\ k<j}}^{a_n-1} g_{j-k}^2\right) \le$$

$$\le 4\Bigg(a_n^{1-4H_i}(2^{2-2H_i}-1)^2 + \frac{a_n^{1-4H_i}}{2}\left(\frac{6\cdot 2^{2H_i}-4-4\cdot 3^{2H_i}+2^{4H_i}}{2^{2H_i}}\right)^2 +$$

$$+\frac{a_n^{1-4H_i}}{2}\sum_{l=2}^{a_n-1} g_l^2\Bigg).$$

Since $\sup_{H\in(0,1)}(2^{2-2H}-1)^2 = 9$, then

$$\mathrm{Var}S_{n,i}^{(2)} \le 4a_n^{1-4H_i}\left(9 + \frac{c}{2} + \frac{1}{2}\sum_{l=2}^{a_n-1} g_l^2\right),$$

where $c = \sup_{H\in(0,1)}\left(\frac{6\cdot 2^{2H}-4-4\cdot 3^{2H}+2^{4H}}{2^{2H}}\right)^2$. Further, g_l be a increment for the function $f(x) = x^{2H_i}$, $x \ge 1$, $i=1,2$ in segment $[l-1,l+1]$. Using a formula for the increments n-order [18, p. 244] we get that $g_l = \frac{2H_i(2H_i-1)(2H_i-2)(2H_i-3)}{2^4}\tilde{\theta}_l^{2H_i-4}$, where $\tilde{\theta}_l \in (l-1,l+1)$. Further,

$$\tilde{c} = \sup_{H\in(0,1)}\left|\frac{2H(2H-1)(2H-2)(2H-3)}{2^4}\right|.$$

Since $l-1 < \tilde{\theta}_l$, $l \ge 2$, that $g_l^2 \le \tilde{c}^2\frac{1}{(l-1)^{8-4H}}$, $l \ge 2$. Hence

$$\mathrm{Var}S_{n,i}^{(2)} \le 4a_n^{1-4H_i}\left(9 + \frac{c}{2} + \frac{\tilde{c}^2}{2}\sum_{l=2}^{a_n-1}\frac{1}{(l-1)^{8-4H_i}}\right) \le$$

$$\leq 4a_n^{1-4H_i}\left(9+\frac{1}{2}+\frac{1}{2}\left(\frac{9}{256}\right)^2\zeta(4)\right)=a_n^{1-4H_i}\kappa,$$

where $\kappa = 38 + \left(\frac{9}{256}\right)^2 \frac{\pi^4}{45}$, $i = 1, 2$ and by Wolfram Mathematica 7.0 $c = \frac{9}{256}$, $\tilde{c} = 1$, $\zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$. Consequently, $\sup_{H_i \in (0,1)} \mathrm{Var}\widehat{S}_{n,i}^{(2)} \leq \frac{\kappa}{a_n}$, $i = 1, 2$. The lemma is proved.

Theorem 2. $\widehat{S}_{n,i}^{(2)} \to 2^{2-2H_i} - 1$, $i = 1, 2$ *with probability one as* $n \to \infty$.

Proof. We have: $E\widehat{S}_{n,i}^{(2)} = 2^{2-2H_i} - 1$. Combining the Lemma 4 end $\sum_{n=1}^{\infty} \frac{1}{a_n} < +\infty$ we deduce that for any $H_i \in (0, 1)$ the series $\sum_{n=1}^{\infty} \mathrm{Var}\widehat{S}_{n,i}^{(2)}$, $i = 1, 2$ is convergent. Then from [19, p. 24] $\widehat{S}_{n,i}^{(2)} - E\widehat{S}_{n,i}^{(2)} \to 0$ with probability one as $n \to \infty$. Thus it follows that $\widehat{S}_{n,i}^{(1)} \to 2^{2-2H_i} - 1$ with probability one as $n \to \infty$. The theorem is proved.

From the Theorem 1 and 2 the next corollary is true.

Corollary 1.

$$\frac{S_{n,i}^{(2)}}{S_{n,i}^{(1)}} = \frac{\widehat{S}_{n,i}^{(2)}}{\widehat{S}_{n,i}^{(1)}} \to 2^{2-2H_i} - 1,\ H_i \in (0,1),\ i = 1,2 \tag{3}$$

with probability one as $n \to \infty$.

We put

$$\theta(H) := 2^{2-2H} - 1,\ H \in (0,1). \tag{4}$$

The function

$$\varphi(\theta) = 1 - \frac{1}{2}\log_2(\theta + 1),\ \theta \in (0,3), \tag{5}$$

be a inverse function of the $\theta(H), H_\in(0,1)$.

Theorem 3. *The statistics*

$$\widehat{H}_n^{(i)} = 1 - \frac{1}{2}\log_2(\widehat{\theta}_n^{(i)} + 1),\ n \geq 1,$$

where $\widehat{\theta}_n^{(i)} = \frac{S_{n,i}^{(2)}}{S_{n,i}^{(1)}}$, $n \geq 1$, $i = 1, 2$ *is the strong consistent estimate of the parameter* H_i, $i = 1, 2$.

Proof. Since, the function (5) be a inverse function of the (4), then from (5) and (3) we have that $\widehat{H}_n^{(i)} \to H_i$, $i = 1, 2$ with probability one as $n \to \infty$.

4. Confidence regions.

Lemma 5. *Let* $\{X_k \big| 0 \leq k \leq a_n,\ a_n \in N\}$, $\{Y_k \big| 0 \leq k \leq a_n,\ a_n \in N\}$ *be a sequences of random variables with zero mean and finite fourth moment for which:*

$$EX_k = EY_k = 0,\ EX_k^2 = EX_0^2,\ EY_k^2 = EY_0^2,\ 0 \leq k \leq a_n;$$

$$S_1 = \sum_{k=0}^{a_n} X_k^2,\ S_2 = \sum_{k=0}^{a_n} Y_k^2,\ \delta = \frac{EX_0^2}{EY_0^2}.$$

Then for any $\varepsilon > 0$ the following inequality holds:

$$P\left\{\left|\frac{S_1}{S_2} - \delta\right| \geq \varepsilon\right\} \leq \frac{\mathrm{Var}Q_1}{(EQ_1)^2} + \frac{\mathrm{Var}Q_2}{(EQ_2)^2},$$

where $Q_1 = (\delta - \varepsilon) S_2 - S_1$, $Q_2 = S_1 - (\delta + \varepsilon) S_2$.

Proof. We have:

$$P\left\{\left|\frac{S_1}{S_2} - \delta\right| \geq \varepsilon\right\} = P\{|S_1 - \delta S_2| \geq \varepsilon S_2\} = P\{S_1 - \delta S_2 \leq -\varepsilon S_2\} +$$

$$+P\{S_1 - \delta S_2 \geq \varepsilon S_2\} = P\{Q_1 \geq 0\} + P\{Q_2 \geq 0\}.$$

Moreover,

$$EQ_1 = (\delta - \varepsilon) ES_2 - ES_1 = (\delta - \varepsilon)(a_n + 1)EY_0^2 - (a_n + 1)EX_0^2 =$$

$$= (a_n + 1)(\delta - \varepsilon - \delta)EY_0^2 = -\varepsilon(a_n + 1)EY_0^2 < 0;$$

$$EQ_2 = (a_n + 1)EX_0^2 - (\delta + \varepsilon)(a_n + 1)EY_0^2 = -\varepsilon(a_n + 1)EY_0^2 < 0.$$

Using the Tchebychev inequality we obtain the upper estimate for the probability $P\{Q_1 \geq 0\}$:

$$P\{Q_1 \geq 0\} = P\{Q_1 - EQ_1 \geq -EQ_1\} \leq P\{|Q_1 - EQ_1| \geq -EQ_1\} \leq \frac{\mathrm{Var}Q_1}{(EQ_1)^2}.$$

The analogous upper estimate is true for the probability $P\{Q_2 \geq 0\}$. The lemma is proved.

Now we construct the confidence regions for estimate $\theta_i = 2^{2-2H_i} - 1$, $H_i \in (0, 1)$, $i = 1, 2$ by using the Lemma 5. Let the level of confidence $1 - p \in (0, 1)$ is given. We will seek the variable $\gamma_n^{(i)}(p)$, $i = 1, 2$ so that

$$P\left\{\left|\widehat{\theta}_n^{(i)} - \theta_i\right| \geq \gamma_n^{(i)}(p)\right\} < p.$$

Put

$$Q_1^{(i)} = \left(\theta_i - \gamma_n^{(i)}(p)\right)\widehat{S}_{n,i}^{(1)} - \widehat{S}_{n,i}^{(2)},\ Q_2^{(i)} = \widehat{S}_{n,i}^{(2)} - \left(\theta_i + \gamma_n^{(i)}(p)\right)\widehat{S}_{n,i}^{(1)},\ i = 1, 2,$$

where $\gamma_n^{(i)}(p)$ is unknown variable. It is easily to prove that $EQ_1^{(i)} = EQ_2^{(i)} = -\gamma_n^{(i)}(p)$. Then

$$\mathrm{Var}Q_1^{(i)} \leq 2\left(\theta_i - \gamma_n^{(i)}(p)\right)^2 \mathrm{Var}\widehat{S}_{n,i}^{(1)} + 2\mathrm{Var}\widehat{S}_{n,i}^{(2)},$$

$$\mathrm{Var}Q_2^{(i)} \leq 2\left(\theta_i + \gamma_n^{(i)}(p)\right)^2 \mathrm{Var}\widehat{S}_{n,i}^{(1)} + 2\mathrm{Var}\widehat{S}_{n,i}^{(2)},\ i = 1, 2.$$

Recall that $\theta \in (0, 3)$.

Select $\gamma_n^{(i)}(p)$ such that

$$\frac{\mathrm{Var}Q_1^{(i)}}{\left(EQ_1^{(i)}\right)^2} \leq \frac{2\left(\theta_i - \gamma_n^{(i)}(p)\right)^2 \mathrm{Var}\widehat{S}_{n,i}^{(1)} + 2\mathrm{Var}\widehat{S}_{n,i}^{(2)}}{\left(\gamma_n^{(i)}(p)\right)^2} \leq \frac{p}{2},$$

$$\frac{\mathrm{Var}Q_2^{(i)}}{\left(EQ_2^{(i)}\right)^2} \le \frac{2\left(\theta_i + \gamma_n^{(i)}(p)\right)^2 \mathrm{Var}\widehat{S}_{n,i}^{(1)} + 2\mathrm{Var}\widehat{S}_{n,i}^{(2)}}{\left(\gamma_n^{(i)}(p)\right)^2} \le \frac{p}{2},\ i = 1,2. \qquad (6)$$

We put for $H_i^* \in (0,1)$

$$\lambda_n^{(i)} = \begin{cases} \frac{4}{a_n}\left(3 + 2\zeta(4 - 4H_i^*)\right), & H_i^* \in (0, \frac{3}{4}); \\ \frac{4}{a_n}\left(3 + 2\left(1 + \ln a_n\right)\right), & H_i^* = \frac{3}{4}; \\ \frac{4}{a_n}\left(3 + 2\frac{a_n^{4H_i^*-3}}{4H_i^*-3}\right), & H_i^* \in \left(\frac{3}{4}, 1\right), \end{cases}, \quad \mu_n^{(i)} = \frac{\kappa}{a_n}, \qquad (7)$$

where $\zeta(\cdot)$ is a zeta function, $\kappa = 38 + \left(\frac{9}{256}\right)^2 \frac{\pi^4}{45}$, $i = 1,2$.

From Lemma 2, 4 we get

$$2 \sup_{H_i \in (0, H_i^*]} \mathrm{Var}\widehat{S}_{n,i}^{(1)} \le \lambda_n^{(i)},\ 2 \sup_{H_i \in (0,1)} \mathrm{Var}\widehat{S}_{n,i}^{(2)} \le \mu_n^{(i)}.$$

For $H_i \in (0, H_i^*]$ from the inequality (6) it follow that

$$\frac{\left(3 + \gamma_n^{(i)}(p)\right)^2 \lambda_n^{(i)} + \mu_n^{(i)}}{\left(\gamma_n^{(i)}(p)\right)^2} \le \frac{p}{2},\ i = 1,2.$$

Hence for $\frac{p}{2} - \lambda_n^{(i)} > 0$, $i = 1,2$ we get $\gamma_n^{(i)}(p) \ge \frac{3\lambda_n^{(i)} + \sqrt{\frac{D_n^{(i)}}{4}}}{\left(\frac{p}{2} - \lambda_n^{(i)}\right)}$, where $D_n^{(i)} = 36\left(\lambda_n^{(i)}\right)^2 + 4\left(\frac{p}{2} - \lambda_n^{(i)}\right)\left(9\lambda_n^{(i)} + \mu_n^{(i)}\right)$, $i = 1,2$. Put

$$\gamma_n^{(i)}(p) = \frac{3\lambda_n^{(i)} + \sqrt{\frac{D_n^{(i)}}{4}}}{\left(\frac{p}{2} - \lambda_n^{(i)}\right)},\ i = 1,2. \qquad (8)$$

Thus the next theorem is true.

Theorem 4. *Let $H_i \in (0,\ H_i^*]$, where $H_i^* \in (0,1)$ be fixed and $\frac{p}{2} - \lambda_n^{(i)} > 0$, $i = 1,2$. Then*

$$P\left\{H_i \in \left(H_{n,l}^{(i)}, H_{n,r}^{(i)}\right)\right\} \ge 1 - p,$$

where $H_{n,l}^{(i)} = \varphi_i\left(\min\left(\theta_n^{(i)} + \gamma_n^{(i)}(p), \theta(H_i^*)\right)\right)$, $H_{n,r}^{(i)} = \varphi_i\left(\max\left(0, \theta_n^{(i)} - \gamma_n^{(i)}(p)\right)\right)$, $\theta_n^{(i)} = \frac{S_{n,i}^{(2)}}{S_{n,i}^{(1)}}$, $i = 1,2$, $\varphi(H)$ is a function defined in (5), $\gamma_n^{(i)}(p)$ defined in (8), $1 - p \in (0,1)$.

From the Lemma 1 [17] it follow that $P\left\{H \in \left(H_{n,l}^{(1)}, H_{n,r}^{(1)}\right) \times \left(H_{n,l}^{(2)}, H_{n,r}^{(2)}\right)\right\} \ge 1 - 2p$, where $H = (H_1, H_2)$, $p \in [0,1]$.

Conclusions. In this work were constructed the strong consistent estimate of the parameter $H = (H_1, H_2)$ of covariance function for random field $B(t)$, $t \in [0,1]^2$. We also obtained the confidence regions.

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