WKB METHOD FOR THE DIRAC EQUATION WITH A SCALAR–VECTOR COUPLING

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We outline a recursive method for obtaining WKB expansions of solutions of the Dirac equation in an external centrally symmetric field with a scalar–vector Lorentz structure of the interaction potentials. We obtain semiclassical formulas for radial functions in the classically allowed and forbidden regions and find conditions for matching them in passing through the turning points. We generalize the Bohr–Sommerfeld quantization rule to the relativistic case where a spin-1/2 particle interacts simultaneously with a scalar and an electrostatic external field. We obtain a general expression in the semiclassical approximation for the width of quasistationary levels, which was earlier known only for barrier-type electrostatic potentials (the Gamow formula). We show that the obtained quantization rule exactly produces the energy spectrum for Coulomb- and oscillatory-type potentials. We use an example of the funnel potential to demonstrate that the proposed version of the WKB method not only extends the possibilities for studying the spectrum of energies and wave functions analytically but also ensures an appropriate accuracy of calculations even for states with $n_r \backsim 1$.

1. Introduction

The relativistic hydrogen-like (HL) atoms and heavy–light quark–antiquark ($Q\bar{q}$) systems, which are their QCD analogues, are ideal objects for investigations and permit verifying quantum theory results very precisely experimentally. The Dirac equation plays a distinctive role in the development of the relativistic theory of bound states. It follows from this equation that electrons (positrons) and quarks (antiquarks) have spin and spin angular momentum. This equation also permits taking the spin–orbit and spin–spin interactions in HL atoms and $Q\bar{q}$ systems into account and thus predicting the fine and hyperfine structure of energy equations. The Dirac equation with radial corrections [1], [2] proposed by Schwinger permits taking the multiple interactions between a particle and its own and external fields into account.

The effective Dirac equation method plays an important role in the modern relativistic theory of bound states. In this method, it is possible to pass consistently from the two-particle theory to the external field approximation [3]. As follows from the results in [3], [4], this possibility is realized and has practical advantages in the case of HL atoms and $Q\bar{q}$ systems. But in the majority of problems, where the external field concept [4] is physically justified, attempts to find exact solutions of the Dirac equation with more or less realistic interaction potentials still encounter insurmountable difficulties. Either numerical or asymptotic methods are most often used to calculate the solutions. Precisely the possibility of obtaining an asymptotic solution in many theoretical and applied problems permits analyzing the problem most completely. Therefore, it hardly needs saying how important it is to create and develop asymptotic methods for solving the Dirac equation.

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The semiclassical Wentzel–Kramers–Brillouin (WKB) approximation is one of the basic and most universal asymptotic methods for solving problems in quantum mechanics and mathematical physics for which the exact solutions are unknown or too cumbersome (see, e.g., [5]–[9]). In contrast to perturbation theory, the interaction need not be very small in this approximation, and its applicability domain is hence wider, which permits analyzing qualitative laws for the behavior and properties of quantum mechanical systems. In particular, the ordinary WKB method [5]–[9] (for the Schrödinger equation) was successfully used in the case of the hydrogen atom in constant external electric and magnetic fields (see, e.g., [5], [10] and the references therein), the case of several model potentials [11], the quantum mechanical problem of two Coulomb centers [12], etc. Discussions of the modern state of this method, including its different versions and applications in the nonrelativistic theory of atoms and molecules, quantum chemistry, problems in collision theory, etc., can be found in [13], [14].

The WKB approximations were successfully used in different problems in nonrelativistic physics, which also stimulated generalizations of this method to relativistic systems described by the Dirac equation. Historically, the procedure for passing from the Dirac equation in an external field to the limit of the Hamiltonian–Jacobi equation for a classical relativistic particle was first considered by Pauli [15] and was then studied in more detail by several authors [16]. Maslov solved the problem of constructing semiclassical solutions of the Dirac equation in the general multidimensional case in the framework of the canonical operator method [17].

The first systematic studies of the theory of the semiclassical approximation for the Dirac equation in a strong external field ($E_0 > 2m_ec^2$, where E_0 is the electron binding energy) were closely related to the problem of deep levels [18]–[21], which plays a fundamental role in quantum electrodynamics (the critical nuclear charge Z_c and the spontaneous creation of positrons in a vacuum for $Z > Z_c$; see [22]–[25]). In early papers [18]–[21], applying the WKB method to the relativistic Coulomb problem with a charge $Z > 137$ was based on squaring the Dirac equation (the effective potential (EP) method [22], [23]). This approach is very close to the usual scheme for obtaining the WKB expansions [5]–[9] and works well in the subcritical region $Z \leq Z_c$, $E_0 \geq -m_ec^2$ (i.e., in the region where the spontaneous creation of positrons is still impossible). But the substitution $\chi(r) = (mc^2 + E - V)^{-1/2}F(r)$ used in this method for states with energies $E < -m_ec^2$ becomes singular at the point $r = r_g$ such that $V(r_g) = mc^2 + E$ (the attraction potential $V(r) < 0$, $0 < r < \infty$, is considered). The usual semiclassical formulas [5]–[9] thus become meaningless near the point $r = r_g$ because the phase integrals diverge. Different authors [18]–[21] overcame this difficulty differently, sometimes using rather witty approaches [19], [21], but the final complete solution was given only in [26], [27]. It turned out that this difficulty has a purely formal character because the original Dirac equation is not singular at the point $r = r_g$. Moreover, no singularity appears at all at $r = r_g$ if the WKB approximation is applied not to the second-order equation for $\chi(r)$ but directly to the original Dirac system for the radial wave functions F and G. The semiclassical formulas thus obtained for the solutions of the Dirac equation in a strong external field find numerous applications in the theory of heavy and superheavy atoms [28].

But in recent years, the behavior of quantum systems of fermions present simultaneously in an electromagnetic (vector) and a scalar external field attracted considerable interest. Such systems have several unusual features, which differ significantly from those exhibited by fermions in the presence of only an electromagnetic field. For example, in contrast to the electromagnetic field, the action of a scalar field is the same on particles as on antiparticles. The picture of energy levels of fermions interacting simultaneously with a scalar and a vector (for example, Coulomb) field can hence significantly differ from the usual picture of the spectrum in the relativistic Coulomb problem.

Additional stimuli for studying similar problems currently appear in elementary particle physics. We mean the construction models of composite mesons (QCD analogues of HL atoms) consisting, for example,

of a single light antiquark $\bar{q}$ and a single heavy quark Q ($Q\bar{q}$-mesons; see [29], [30]). If the Dirac equation is considered in the infinitely heavy quark Q limit as an equation for a single light antiquark $\bar{q}$, then several important aspects of the theory of $Q\bar{q}$ systems (like in the picture of HL atoms) such as the relativistic dynamics of a light antiquark $\bar{q}$ in the external field of a heavy quark Q, the Lorentz structure of the long-range (confining) part of the $Q\bar{q}$ interaction, the fine structure of the spectrum of composite mesons, the influence of a spontaneous violation of the chiral symmetry on the spectrum, etc., can be studied. As is known in QCD [29], [30], the asymptotic freedom at small distances implies that the main contribution to the $Q\bar{q}$ interaction is given by the usual Coulomb potential of the single-gluon exchange, $V(r) = -4\alpha_V/3r$, where α_V is the strong interaction constant. As the distance increases, the main contribution is given by the scalar confining interaction (confinement) whose "exact" form has not yet been found. The lattice calculations [31] based on the first QCD principles distinguish the linear (scalar) confinement at large distances, $S(r) = \sigma r$, where σ is the string tension. Of course, all other interactions are also important if a finer description of the meson properties is required, but they are small interactions compared with the scalar potential ensuring the confinement of quarks inside mesons (see [31] for details).

Finally, we note that the Dirac equation with a mixed (scalar–vector) Lorentz structure of interaction potentials is also interesting from the standpoint of its possible applications in the theory of hadron atoms [32]. It is not improbable in principle that the same equation can also be used to describe effects in solid state physics (for example, in two-band semiconductors [33]).

Taking into account that interest in such studies will undoubtedly increase in the future, it seems useful to generalize the techniques proposed by Maslov [17] and developed in [26], [27] for constructing the WKB solutions of the Dirac equation in an external electromagnetic field to the case of the spinor equation with a mixed scalar–vector coupling. It can be expected that this approach will also be fruitful in describing the behavior of quantum systems of fermions in external fields with a mixed Lorentz structure of the interaction potentials, for which the exact solutions of the Dirac equation either do not exist or are too cumbersome.

This paper is organized as follows. In Sec. 2, we use the well-known techniques of left and right eigenvectors of the corresponding homogeneous system to construct a recursive scheme for obtaining the WKB expansions of the solutions of the Dirac equation in an external centrally symmetric field with a mixed scalar–vector Lorentz structure of the interaction potentials. In Sec. 3, we use this scheme to obtain semiclassical formulas for the radial functions F and G in the classically allowed and forbidden regions; we also find conditions for matching them in passing through the turning points. In the same section, we generalize the Bohr–Sommerfeld quantization rule to the relativistic case where a spin-1/2 particle interacts with scalar and electrostatic external fields simultaneously. We obtain a general expression for the width of quasistationary levels in the semiclassical approximation, which was earlier known only for barrier-type electrostatic potentials. In Sec. 4, we show that the obtained quantization rule exactly reproduces the energy spectrum for some model (Coulomb-type and oscillatory) potentials with a scalar–vector structure. We use an example of the funnel potential to show that this version of the WKB method not only extends the capabilities of analytic investigations of the spectrum of energies and wave functions but also ensures a fully reasonable accuracy of calculations even for states with the radial quantum number $n_r \backsim 1$.

2. Recursive scheme for constructing the WKB expansions

The problem of describing the motion of a relativistic spin-1/2 particle in an effective field consisting of a static scalar and an external electric field can be reduced in our statement to solving the Dirac equation with a mixed scalar–vector coupling ($c = \hbar = 1$)

$$\big[\alpha\widehat{\mathbf{p}} + \beta\big(m + S(\mathbf{r})\big) + V(\mathbf{r})\big]\Psi = E\Psi, \tag{1}$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and β are the standard Dirac matrices, $\widehat{\mathbf{p}} = -i\boldsymbol{\nabla}_r$ is the momentum operator, E

and m are the total energy and the rest mass of a particle, $S(r)$ is the scalar Lorentz potential, and the potential $V(r)$ is the zero (time) component of the four-vector A_μ: $\mathbf{A} = 0$, $V(r) = -eA_0(r)$, $e > 0$.

We obtain semiclassical approximation formulas for Dirac equation (1) only in the class of potentials with a central symmetry: $S(\mathbf{r}) = S(r)$, $V(\mathbf{r}) = V(r)$. We therefore seek the wave function Ψ of stationary states (in the standard representation) as

$$\Psi = r^{-1} \begin{pmatrix} F(r)\Omega_{jlM}(\mathbf{n}) \\ iG(r)\Omega_{jl'M}(\mathbf{n}) \end{pmatrix}, \tag{2}$$

where Ω_{jlM} is the spherical spinor [2], j and M are the respective total angular momentum and its projection ($j = l \pm 1/2$), and l is the orbital angular momentum ($l + l' = 2j$), $\mathbf{n} = \mathbf{r}/r$. Separating the angular components of the vector $\mathbf{n}$ in Eq. (1), we obtain a system of first-order ordinary differential equations for the radial wave functions F and G:

$$\frac{dF}{dr} + \frac{\tilde{k}}{r}F - \frac{1}{\hbar}\big[\big(E - V(r)\big) + \big(m + S(r)\big)\big]G = 0,$$
$$\frac{dG}{dr} - \frac{\tilde{k}}{r}G + \frac{1}{\hbar}\big[\big(E - V(r)\big) - \big(m + S(r)\big)\big]F = 0. \tag{3}$$

Here, we explicitly reconstruct the dependence on the Planck constant $\hbar$ and use the new notation

$$\tilde{k} = \hbar k, \qquad k = \begin{cases} -(l+1) & \text{for } j = l + 1/2 \ (l = 0, 1, \dots), \\ l & \text{for } j = l - 1/2 \ (l = 1, 2, \dots) \end{cases} \tag{4}$$

such that $|k| = j + 1/2 = 1, 2, \dots$. The symbol k, along with $\hbar j$, $\hbar M$, and the energy E, is an integral of motion for a Dirac particle in an arbitrary central field.

Eliminating one of the unknown radial functions, we can reduce system (3) to a second-order equation, but it is more convenient to deal with the system itself in the construction of formal asymptotic solutions. We write it in the matrix form (the prime denotes the derivative with respect to r)

$$\chi' = \frac{1}{\hbar}D\chi, \qquad \chi = \begin{pmatrix} F \\ G \end{pmatrix}, \qquad D = \begin{pmatrix} -\tilde{k}/r & m + S(r) + E - V(r) \\ m + S(r) - E + V(r) & \tilde{k}/r \end{pmatrix}. \tag{5}$$

This system contains a natural small parameter $\hbar$, and the problem is to integrate this system asymptotically as $\hbar \to 0$.

We present an algorithm for constructing formal asymptotic solutions. Following the standard scheme in the asymptotic theory of systems of linear differential equations [34], we seek the solution of system (5) as an asymptotic series in powers of $\hbar$:

$$\chi(r) = \exp\bigg\{\int^r y(r')\,dr'\bigg\}\sum_{n=0}^{\infty}\hbar^n\varphi^{(n)}(r), \tag{6}$$

$$y(r) = \hbar^{-1}y_{-1}(r) + y_0(r) + \hbar y_1(r) + \dots, \qquad \varphi^{(n)}(r) = \begin{pmatrix} \varphi_F^{(n)}(r) \\ \varphi_G^{(n)}(r) \end{pmatrix}, \tag{7}$$

where the upper and lower components $\varphi_F^{(n)}$ and $\varphi_G^{(n)}$ of the vector $\varphi^{(n)}$ correspond to the radial functions F and G. Substituting expansion (6) in (5) and equating the coefficients of equal powers of $\hbar$ to zero, we

obtain an infinite system of recursive equations for the unknown scalar $y_n(r)$ and vector $\varphi^{(n)}(r)$ functions:

$$(D - y_{-1}I)\varphi^{(0)} = 0, \tag{8}$$

$$(D - y_{-1}I)\varphi^{(n+1)} = \varphi^{(n)\prime} + \sum_{k=0}^{n} y_{n-k}\varphi^{(k)}, \quad n = 0, 1, 2, \ldots, \tag{9}$$

where 0 and I are the zero and unit 2×2 matrices. It follows from Eq. (8) that $y_{-1}(r)$ must be an eigenvalue and $\varphi^{(0)} \equiv \varphi_i$ must be one of the (right) eigenvectors of the matrix $D(r)$. The eigenvalues $y_{-1} \equiv \lambda_i$ are the roots of the characteristic equation $\det(D - y_{-1}I) = 0$:

$$y_{-1} \equiv \lambda_i = \pm q, \qquad q = \sqrt{\left(m + S(r)\right)^2 - \left(E - V(r)\right)^2 + \left(\frac{k}{r}\right)^2}. \tag{10}$$

Then the corresponding right eigenvectors φ_i written in component form are equal to

$$\varphi_i = A_1 \begin{pmatrix} m + S + E - V \\ \lambda_i + kr^{-1} \end{pmatrix} = A_2 \begin{pmatrix} \lambda_i - kr^{-1} \\ m + S - E + V \end{pmatrix}. \tag{11}$$

Hereafter, $\hbar = c = 1$, the subscript i takes one of the two values $\pm$ corresponding to the two values of the function $\lambda_i(r) = \pm q(r)$, and $A_1(r)$ and $A_2(r)$ are the normalization factors, which we fix by agreement.

Because $D(r)$ is not a symmetric matrix, in addition to the right eigenvectors, we must also introduce the left eigenvectors $\check{\varphi}_i$, which are determined by the conditions

$$\check{\varphi}_i(D - \lambda_i I) = 0, \tag{12}$$

$$\check{\varphi}_i = B_1(m + S - E + V, \lambda_i + kr^{-1}) = B_2(\lambda_i - kr^{-1}, m + S + E - V). \tag{13}$$

We must also note that $\check{\varphi}_i$ does not coincide with φ_i^{T}; the left and right eigenvectors of the matrix $D(r)$ are orthogonal:

$$(\check{\varphi}_i, \varphi_j) = \sum_{\alpha=1}^{2} (\check{\varphi}_i)_\alpha (\varphi_j)_\alpha = \mathrm{const}\, \delta_{ij}. \tag{14}$$

As usual, δ_{ij} is the Kronecker symbol.

We use Eq. (9) with $n = 0$ to find the function $y_0(r)$. We set $\varphi^{(0)} = \varphi_i$ in this equation and multiply both sides from the left by the row vector $\check{\varphi}_i$. Then, in view of (12), the left-hand side of the final relation is zero, and we obtain an equation for $y_0(r)$, which implies

$$y_0(r) = -\frac{(\check{\varphi}_i, \varphi_i')}{(\check{\varphi}_i, \varphi_i)}. \tag{15}$$

As we saw above (see (11), (13)), the eigenvectors φ_i and $\check{\varphi}_i$ are determined up to arbitrary factors $A_{1,2}$ and $B_{1,2}$. This arbitrariness can be removed by imposing the natural condition

$$(\check{\varphi}_i, \varphi_i') = (\check{\varphi}_i', \varphi_i) \tag{16}$$

on φ_i and $\check{\varphi}_i$. In this case, the integral in (6) is calculated in closed form as

$$\int^r y_0(r')\, dr' = \log\left[(\check{\varphi}_i, \varphi_i)^{-1/2}\right]. \tag{17}$$

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Then the desired semiclassical solution of system (5) becomes

$$\chi_i = (\check{\varphi}_i, \varphi_i)^{-1/2} \exp\left\{\int^r \lambda_i(r')\, dr'\right\} \varphi_i. \tag{18}$$

Continuing these constructions, we can successively find all the other terms $y_1, y_2, \ldots$ and $\varphi^{(1)}, \varphi^{(2)}, \ldots$ in expansion (6). But the formulas for them turn out to be very cumbersome. Hence, only the leading term (18) in expansion (6), corresponding to the well-known expression $\psi \sim p^{-1/2} \exp\left\{i \int^r p(r')\, dr'\right\}$ in the usual nonrelativistic semiclassical theory [5], is customarily used in applications. Indeed, an essential point is that formal expansion (6) in powers of $\hbar$ generally does not converge but is the so-called asymptotic series whose first several terms give a good approximation to the exact solution only if the expansion parameter $\hbar$ is sufficiently small.

Finally, it remains to show that condition (16) can always be satisfied by appropriately choosing the normalization factors $A_{1,2}$ and $B_{1,2}$ in formulas (11) and (13). Substituting (11) and (13) in (16) leads to the equation

$$\frac{A_1 B_1' - A_1' B_1}{A_1 B_1} = -\frac{(m+S)V' + (E-V)S'}{q(q \pm kr^{-1})}. \tag{19}$$

This and representation (18) give the expression for the leading term of the asymptotic approximation of the radial functions χ_i in the subbarrier region:

$$\chi_\pm = \left[2q\left(q \pm \frac{k}{r}\right)\right]^{-1/2} \times$$

$$\times \exp\left\{\pm \int^r q\, dr + \frac{1}{2}\int^r \frac{(m+S)V' + (E-V)S'}{q(q \pm kr^{-1})}\, dr\right\} \begin{pmatrix} m+S+E-V \\ kr^{-1} \pm q \end{pmatrix}. \tag{20}$$

If we start from the second expressions for the eigenvectors φ_i and $\check{\varphi}_i$ (with the normalization factors A_2 and B_2 in formulas (11) and (13)), then similar constructions lead to the WKB representation

$$\chi_\pm = \left[2q\left(q \mp \frac{k}{r}\right)\right]^{-1/2} \times$$

$$\times \exp\left\{\pm \int^r q\, dr - \frac{1}{2}\int^r \frac{(m+S)V' + (E-V)S'}{q(q \mp kr^{-1})}\, dr\right\} \begin{pmatrix} \pm q - kr^{-1} \\ m+S-E+V \end{pmatrix}. \tag{21}$$

Before we discuss the meaning of the obtained formulas, we note that $y_{-1}(r)$ and $y_0(r)$ are real-valued functions in the classically forbidden regions $q(r) > 0$. The plus and minus signs in (20) and (21) correspond to solutions exponentially increasing and decreasing as r increases. We must use formula (20) for $k < 0$ and formula (21) for $k > 0$ for the decreasing solution (the minus sign), and the situation is opposite for the increasing solution. A simple and often efficient method for choosing a convenient expression of the solutions is based on the requirement that the quantity $Q_\pm = q \pm k/r$ be positive in the classically forbidden region. Under the opposite choice of the solution, an indeterminacy of the form $0/0$ can appear in expressions for F and G at the point $r = r_g$, where $Q_\pm = 0$. This fictitious singularity makes the direct use of formulas (20) and (21) impossible in a neighborhood of r_g. Therefore, the procedure for obtaining semiclassical formulas for F and G free from singularities requires evaluating these indeterminacies at the point r_g, which in turn requires additional, rather cumbersome calculations.

We now consider applications of this version of the WKB method to specific physical problems. The semiclassical approximation is typically used in the case of a discrete spectrum. Somewhat less often, it is

used to calculate the wave functions of the continuous spectrum and in the scattering theory [5]–[9], [13]. But in many areas of physics, we encounter potentials with a barrier for which the quasistationary states (resonances) with the complex energy $E = E_r - i\Gamma/2$ appear instead of discrete levels. In the next section, we consider this problem in the semiclassical approximation, which permits finding general expressions for the resonance position E_r and its width Γ, and these expressions hold for arbitrary barrier-type potentials $V(r)$ and $S(r)$.

3. The WKB method for subbarrier resonances

In this section, we briefly show how the WKB method can be used to solve one important problem related to the study of decaying (quasistationary) states of quantum objects [35]. As is known, such states are introduced by analogy with the usual stationary states of the discrete part of the spectrum of eigenvalues of the Dirac Hamiltonian $\widehat{H}$. They are associated with the poles of the stationary Green's function $G(E) = (E - \widehat{H} + i\eta)^{-1}$ on the complex plane of the energy E [35].

But the introduction of complex energy levels violated one of the main postulates in quantum theory according to which the spectrum of eigenvalues of any Hermitian operator must be real and the corresponding eigenfunctions must be normalized. The analytic continuations of the stationary solutions to the complex plane of energies hence have a principally new meaning. They give the simplest and most convenient approximation of nonstationary solutions in the main domain of the variable range, i.e., in the domain where the exact wave functions $\widetilde{\Psi}(\mathbf{r}, t)$ come closest to the functions of stationary bound states, $\widetilde{\Psi}(\mathbf{r}, t) \approx \Psi_n(\mathbf{r}) \exp(-iE_n t)$, $\operatorname{Im} E_n \ll \operatorname{Re} E_n$.

The problem of obtaining a more precise determination of the applicability domain of the original representations in the theory of quasistationary states and improving the calculation methods has been the main problem in these studies for a long time. This problem also remains relevant now, especially in applications to specific systems. As examples, we can here mention the fundamental problems of atomic and nuclear physics such as the ionization of atomic systems under the action of an external electromagnetic field [10], [36], decay of radioactive nuclei or unstable particles [37], reaction of electron separation in low-energy collisions of atomic particles [38], etc. At the same time, the logic for developing the theory of decaying states obviously dictates that several qualitatively new problems must be stated similarly to those previously solved only in the nonrelativistic approximation. The natural question arises of how different physical factors (such as relativistic effects and spin–orbit interaction, external fields with a mixed Lorentz structure of the interaction potentials, adiabatically slow variations in the parameters, etc.) influence the properties of decaying quasistationary states. The results obtained by solving similar problems are interesting for the quantum mechanics of a non-Abelian particle in an external Yang–Mills gauge field [39], for the study of the vacuum shell of supercritical atoms [18]–[24], for the description of effects of the spontaneous creation of positrons under slow collisions of heavy nuclei [23]–[28], and also from the standpoint of the study of solutions of the Dirac equation in strong external fields.

3.1. Structure of WKB solutions. We return to WKB asymptotic approximations (20) and (21) and see that the function $q(r)$ involved there coincides with the radial momentum of a classical relativistic particle (up to the factor $i = \sqrt{-1}$). If we write this function as $q = \sqrt{2m(U - \overline{E})}$, then expression (10) is associated with the effective energy $\overline{E} = (E^2 - m^2)/(2m)$ and the EP for the radial motion,

$$U(r, E) = \frac{E}{m}V + S + \frac{S^2 - V^2}{2m} + \frac{k^2}{2mr^2}. \tag{22}$$

Further analysis significantly depends on the form of the EP $U(r, E)$ and the position of the turning points (the zeros of $q(r)$ on the semiaxis $0 < r < \infty$). To describe the phenomena related to the creation
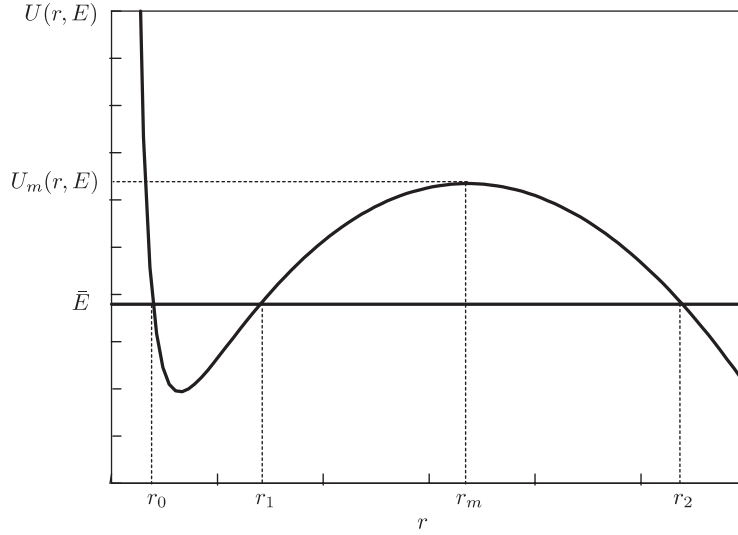
Fig. 1. Form of the barrier-type effective potential $U(r, E)$; r_0, r_1, and r_2 are roots of quasimomentum (10).

and decay of quasistationary states, we distinguish the class of potentials $V(r)$ and $S(r)$ for which the EP $U(r, E)$ has the form of a well, separated from the exterior region by a potential barrier. In what follows, we assume that $V(r)$ and $S(r)$ do not have singularities for $0 < r < \infty$ and satisfy the conditions (at the limit point $r = 0$)

$$\lim r^{1+\delta} V(r) = 0, \qquad \lim r^{1+\delta} S(r) = 0 \quad \text{as } r \to 0 \text{ with } \delta > 0. \tag{23}$$

For potentials exhibiting a power-law behavior at small distances,

$$V(r) \sim r^{-\beta_1}, \qquad S(r) \sim r^{-\beta_2}, \quad \beta_i \leq 1, \quad r \to 0. \tag{24}$$

These conditions eliminate the "fall to the center" [2], [23]. For scalar potentials $S(r)$ more singular than r^{-1}, the term with the centrifugal potential in expression (22) does not play any essential role, and the asymptotic approximation of the solutions F and G as $r \to 0$ is independent of k. If $V(r)$ and $S(r)$ satisfy conditions (23), then the EP $U(r, E)$ has a second-order pole at $r = 0$. The qualitative behavior of the barrier-type EP $U(r, E)$ and the position of the turning points r_j for $\overline{E} < U_m$ (U_m is the top of the potential barrier) are shown in Fig. 1.

In the semiclassical approximation, the wave function of a quasistationary state has different asymptotic forms in the following regions: (I) the potential well $r_0 < r < r_1$, (II) the subbarrier region $r_1 < r < r_2$, and (III) the classically allowed region $r > r_2$ with a quasidiscrete energy spectrum [5]. The boundaries between the different regions (i.e., the turning points r_j, $j = 0, 1, 2$) are here the roots of the equation $\overline{E} = U(r, E)$. We omit the calculation details and write the smooth terms of the asymptotic approximation of the solutions of Dirac system (3) in each of these three regions.

I. In the classically allowed region $r_0 < r < r_1$, the asymptotic approximation of the WKB radial functions F and G are oscillatory:

$$F(r) = C_1^{\pm} \left[\frac{E - V + m + S}{p(r)} \right]^{1/2} \cos \Theta_1,$$

$$G(r) = C_1^{\pm} \operatorname{sgn} k \left[\frac{E - V - m - S}{p(r)} \right]^{1/2} \cos \Theta_2. \tag{25}$$

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In this case, according to (10),

$$p(r) = \left[\left(E - V(r) \right)^2 - \left(m + S(r) \right)^2 - \left(\frac{k}{r} \right)^2 \right]^{1/2} \tag{26}$$

is the semiclassical momentum for the radial motion of a particle, $\operatorname{sgn} k$ is the sign of k, and the phases Θ_1 and Θ_2 are determined by the relations

$$\Theta_1(r) = \int_{r_1}^{r} \left(p + \frac{kw}{pr} \right) dr + \frac{\pi}{4}, \qquad \Theta_2(r) = \int_{r_1}^{r} \left(p + \frac{k\widetilde{w}}{pr} \right) dr + \frac{\pi}{4}, \tag{27}$$

$$w = \frac{1}{2} \left(\frac{V' - S'}{m + S + E - V} - \frac{1}{r} \right), \qquad \widetilde{w} = \frac{1}{2} \left(\frac{V' + S'}{m + S - E + V} + \frac{1}{r} \right). \tag{28}$$

In formulas (25) and in all the WKB representations of solutions given below, the normalization constants C_j related to the states with $k > 0$ and $k < 0$ are denoted by the superscripts $+$ and $-$.

Under the natural assumption that the level width Γ is sufficiently small, the wave function of a quasistationary state can be normalized to a single particle localized in the region of potential well I (see [5]–[7] for details) as

$$\int_{r_0}^{r_1} (F^2 + G^2) \, dr = 1.$$

In this case, the fast oscillating functions $\cos^2 \Theta_1(r)$ and $\cos^2 \Theta_2(r)$ must be replaced with their average value $1/2$; we then obtain the relation for the normalization constants $C_1^{\pm}$ in (25):

$$|C_1^{\pm}| = \left\{ \int_{r_0}^{r_1} \frac{E - V(r)}{p(r)} \, dr \right\}^{-1/2} = \left(\frac{2}{T} \right)^{1/2}. \tag{29}$$

The quantity T in this expression coincides with the period of radial oscillations of a classical relativistic particle with the energy E in potential well I. We note that the relation $E - V(r_1) = \left[\left(m + S(r_1) \right)^2 + k^2/r_1^2 \right]^{1/2}$ holds at the turning point $r = r_1$; therefore, the quantity $E - V$ is positive in region I.

II. In the subbarrier region $r_1 < r < r_2$, the quantity p takes purely imaginary values, $p = iq$, $p^2(r) < 0$, and $q(r)$ and $y_0(r)$ are real functions. Oscillating-type WKB solution (25) is then continued here by a solution exponentially decreasing as the distance increases in the classically forbidden region II. For states with $k > 0$ (sufficiently far from the turning points r_1 and r_2), we have

$$\chi = \begin{pmatrix} F \\ G \end{pmatrix} = \frac{C_2^+}{\sqrt{qQ}} \exp \left\{ - \int_{r_2}^{r} \left[q + \frac{(m+S)V' + (E-V)S'}{2qQ} \right] dr \right\} \begin{pmatrix} -Q \\ m + S - E + V \end{pmatrix}. \tag{30}$$

The same solution for the states with $k < 0$ can be written as

$$\chi = \frac{C_2^-}{\sqrt{qQ}} \exp \left\{ - \int_{r_2}^{r} \left[q - \frac{(m+S)V' + (E-V)S'}{2qQ} \right] dr \right\} \begin{pmatrix} m + S + E - V \\ -Q \end{pmatrix}. \tag{31}$$

We have $Q = q + |k| r^{-1}$ in the last two formulas, and the function $q(r)$ is given by formula (10).

III. In the "exterior" classically allowed region $r > r_2$, a quasistationary state is associated with the divergent wave

$$\chi = \frac{C_3^+}{\sqrt{pP}} \exp\left\{ \int_{r_2}^r \left[ip + \frac{(m+S)V' + (E-V)S'}{2pP} \right] dr \right\} \begin{pmatrix} iP \\ m+S-E+V \end{pmatrix}. \tag{32}$$

This expression must be used to study the states with $k > 0$; in this case, $P = p + i|k|r^{-1}$, and the radial momentum $p(r)$ is again positive. A similar expression for the states with $k < 0$ has the form

$$\chi = \frac{C_3^-}{\sqrt{p\overline{P}}} \exp\left\{ \int_{r_2}^r \left[ip - \frac{(m+S)V' + (E-V)S'}{2pP} \right] dr \right\} \begin{pmatrix} m+S+E-V \\ iP \end{pmatrix}. \tag{33}$$

3.2. Formulas relating the WKB solutions. The semiclassical formulas obtained in Sec. 3.1 approximate the desired solution of Dirac system (3) for all r except in small neighborhoods of the turning points r_j, $j = 0, 1, 2$. The unified solution of system (3) on the entire interval of r can be obtained by passing around the turning points r_j (in the plane of the complex variable r) and establishing relations between the coefficients $C_j^\pm$ of the semiclassical solutions (25)–(33) constructed in different regions. A rigorous mathematical consideration of the relations between the WKB solutions generally requires taking into account that the second-order pole at the point $r = 0$ and at the turning point r_0, as well as at the turning points r_0 and r_1 (a narrow well and deep subbarrier resonances with an energy near the bottom of the potential well), can approach each other and the effective potential energy $U(r, E)$ can have points of inflection. In the problems of quasistationary states with large quantum numbers, additional difficulties always arise when the approach (towards each other) of the turning points r_1 and r_2 (a narrow barrier and the Rydberg states of atoms with energies close to the top of the barrier, $E_r \to U_m$) and the drift of the turning points r_1 and r_2 into the complex plane (complex potentials and overbarrier resonances situated over the classical ionization threshold, $E_r > U_m$) must be taken into account. But we do not dwell on this class of more complicated, but simultaneously more interesting, problems; the study of these problems is beyond the scope of this paper and needs special investigations. We here consider only the most typical case where all three turning points r_j, $j = 0, 1, 2$, are strongly separated and can be analyzed independently of each other. More precisely, we assume that the conditions $\left| \int_{r_j}^{r_{j+1}} p(r) \, dr \right| \gg 1$, $j = 0, 1$, are satisfied. (As a rule, precisely this situation occurs in the physical problems considered in the next section.) We can then use the standard methods [5]–[8] to bypass these points and match the WKB solutions. One of these methods, proposed by Zwaan and Stueckelberg and later developed in many papers, is based on the Stokes phenomenon: the coefficients $C_j^\pm$ of the asymptotic representation of the solution in a neighborhood of the turning point r_j jump in passing through the Stokes line $\mathrm{Re} \int_{r_j}^r p(r') \, dr' = 0$. It is also based on the requirement that the exact solution be unique while going around the turning point along a closed contour.

Another possible method is to reduce Dirac system (3) near the turning point r_j, $j = 1, 2$, to a second-order equation (a nonrelativistic Schrödinger-type equation) with a potential linearly depending on $r - r_j$, whose solution is described by the so-called Airy function. This exact solution is then matched with the semiclassical solution on both sides of the turning point r_j. Because the details in both methods are described sufficiently fully in numerous sources in the literature, we here present only the scheme for solving the problem of matching the WKB expansions and, as the concluding result, write the final formulas relating the coefficients $C_j^\pm$ in the semiclassical solutions of the Dirac equation for the problem with three isolated turning points considered in this paper:

    1. The usual formulas [5]–[8] relating the WKB solution to the left and to the right of the turning point r_j remain valid if in the asymptotic formulas of form (30), (31) and (32), (33), we

consider only the behavior of the leading factors $p^{-1/2} \exp\left\{\pm \int_{r_j}^r p \, dr\right\}$, singular as $r \to r_j$, in a neighborhood of this turning point.

2. The formulas thus constructed, relating the coefficients $C_j^{\pm}$ of semiclassical solutions (25)–(33) in regions I–III, have the form

$$
C_2^{\pm} = -i C_3^{\pm} = \mp \frac{C_1^{\pm}}{2} \left[\frac{E - V(r_1) + m + S(r_1)}{|k| r_1^{-1}} \right]^{\pm 1/2} \times
$$

$$
\times \exp\left\{ -\int_{r_1}^{r_2} \left[q \pm \frac{(m + S)V' + (E - V)S'}{2qQ} \right] dr \right\}.
\tag{34}
$$

Although obtained semiclassical formulas (25)–(34) have a very complicated analytic structure, using them in specific problems does not create any principal difficulties, because all the quantities contained in F and G are determined in terms of one-dimensional quadratures.

3.3. Equations for the energy of quasistationary levels. As already noted in Sec. 3.1, to find the quasistationary states, it is usually required that the solution of the Dirac equation at infinity be a divergent wave given by (2), (32), (33); this corresponds to a particle eventually escaping from a decaying system [5]. The condition that there is no converging component in the asymptotic expression for the wave function of the quasistationary states permits choosing complex energy eigenvalues $E = E_r - i\Gamma/2$, where E_r is the resonance position and Γ is the resonance width corresponding to the quasistationary state. The quantity Γ is positive and determines the probability of decay per unit time: $W = \Gamma/\hbar$.

We derive the condition determining the position of the quasistationary levels in the semiclassical case. Neglecting the barrier penetrability in the region $r_1 < r < r_2$ and assuming that only exponentially decreasing WKB solutions remain to the left and to the right of the potential well $r_0 < r < r_1$, we obtain the quantization condition

$$
\int_{r_0}^{r_1} \left(p + \frac{kw}{pr} \right) dr = \left(n_r + \frac{1}{2} \right) \pi, \quad n_r = 0, 1, 2, \ldots,
\tag{35}
$$

from (25), where n_r is the radial quantum number. This equation determines the real part of the energy level E_r. It differs from the usual semiclassical Bohr–Sommerfeld quantization condition [5] in the relativistic expression for the momentum $p(r)$ and by including a correction $\sim w(r)$ taking the spin–orbit interaction into account and resulting in splitting the levels with different signs of the quantum number k.

As follows from expression (28) for $w(r)$, the difference $V' - S'$, where the additive contributions due to the scalar $(-S')$ and vector (V') versions of the interaction have opposite signs, is completely responsible for the fine splitting of the levels. Of course, this effect is based on dependence of the sign of the spin–orbit interaction on the Lorentz structure of the interaction potentials. At the same time, this can be treated as a reflection of the spins of particles in the vector field being oriented in the direction $[\mathbf{F}_{\text{vec}} \mathbf{p}]$, where $\mathbf{F}_{\text{vec}} = -\mathbf{n} \, dV/dr$ is the force acting on the particle, $\mathbf{p}$ is its momentum, and $\mathbf{n}$ is the unit vector aligned with $\mathbf{r}$. In the scalar field, the spins of particles are oriented in the direction $-[\mathbf{F}_{\text{sc}} \mathbf{p}]$, where $\mathbf{F}_{\text{sc}} = -\mathbf{n} \, dS/dr$. This reasoning clearly explains that the level $j = 3/2$, $l = 1$ lies below the level $j = 1/2$, $l = 1$ in the scalar field and the situation is opposite in the vector field.

In what follows (Sec. 4), we see that in the more general case where a particle interacts with a scalar and a vector field simultaneously, the value and the character of the spin–orbit splitting of the levels depends significantly on the relative role of these interactions. Thus, the combined information about the position and the fine splitting of the levels can already reveal the roles of $S(r)$ and $V(r)$ separately.

Because quantization condition (35) takes only the spin–orbit correction into account and neglects the correction due to the spin–spin coupling, it is necessary to assume the following. Of course, if we acted

formally, then we could use the recursive technique developed in the preceding section to obtain a more precise equation for the level energy E_r, which together with the spin–orbit correction also takes the spin–spin interaction into account. We recall that to include this correction in (35), we needed to calculate the functions $y_1(r)$ and $\varphi^{(1)}(r)$, i.e., terms of the order of $\hbar$ in semiclassical expansion (6). But this refinement of quantization condition (35) is hardly meaningful because taking the spin–spin interaction into account would exceed the accuracy of semiclassical calculations (in contrast to the spin–orbit interaction). In what follows (Sec. 4), we illustrate this with an example of the Coulomb and oscillatory potentials; for these potentials, quantization formula (35) gives the energy spectrum exactly.

We now calculate the level width $\Gamma = 2\,\mathrm{Im}\,E$. For this, we multiply the first equation in system (3) by G^* and the second equation by F (after passing to the complex conjugates G^*, F^*, and $E^* = E_r + i\Gamma/2$), sum the obtained relations, and integrate them over the variable r from 0 to ∞. The integral of $G^*F' + G^{*'}F = (G^*F)'$ over r can be easily calculated in the general form; we must remember in this case that the product G^*F is zero at the lower integration limit (at $r = 0$) and only the upper integration limit ($r = \infty$) contributes nontrivially. The majority of the obtained terms are real in the final relation; taking the imaginary parts everywhere, we readily obtain the desired result

$$\Gamma = -2\,\mathrm{Im}\big[G^*(r)F(r)\big]\big|_{r\to\infty}. \tag{36}$$

This result for the resonance width Γ can also be obtained by directly calculating the particle flux to infinity normalized to a single particle localized in the region $r_0 < r < r_1$. We now use asymptotic representations (32) and (33) of the radial functions F and G to the right of the turning point r_2 and formulas (34) relating the normalization constants $C_j^{\pm}$ in regions I–III and thus obtain the expression for the leading term of the WKB approximation of the level width:

$$\Gamma = \frac{\Gamma_0}{T}\exp\left\{-2\int_{r_1}^{r_2} q(r)\,dr\right\}, \tag{37}$$

where

$$\Gamma_0 = \exp\left\{2k\int_{r_1}^{r_2}\frac{w}{rq}\,dr\right\} \tag{38}$$

and the period T of radial oscillations is given in (29).

We note that obtained semiclassical formula (37) is a relativistic generalization of the well-known Gamow formula for the width of a quasistationary level. A nontrivial point in this generalization is that the expression for the oscillation period T is different and an additional factor Γ_0 appears in expression (37) before the exponential (this factor depends only on the sign of the quantum number k and originates from the spin–orbit coupling in the mixture of the scalar $S(r)$ and vector $V(r)$ interaction potentials). In the nonrelativistic case, this factor is practically equal to unity, and formula (37) has a clear interpretation. Namely, $1/T$ is the number of impacts per unit time of a particle (localized inside region I) on the interior wall ($r = r_1$) of the potential barrier $r_1 < r < r_2$, and the exponential factor $\exp\left\{-2\int_{r_1}^{r_2} q\,dr\right\}$ corresponds to the probability that the particle penetrates this barrier at each impact.

Formulas (35) and (37) are the main result in our study: they determine the spectrum of quasistationary states of a spinor particle ($s = 1/2$) in a barrier-type EP $U(r, E)$.

To avoid misunderstanding, we stress the following here: when we speak about a barrier-type potential (or, equivalently, a potential with a barrier), we mean not the original potentials $V(r)$ and $S(r)$ directly contained in system (3) of Dirac equations but the energy-dependent EP $U(r, E)$ generated by the original potentials (according to expression (22)). Somewhat more generally (than in definition (22)), the EP appears in squaring the Dirac equation, i.e., in reducing system of equations (3) to a (single)

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mathematically equivalent (Schrödinger-type) second-order equation; moreover, small spin corrections containing the function $w(r)$ are added to expression (22) for $U(r, E)$. In the nonrelativistic case, we have $U(r, E) \approx S(r) + V(r)$; but if the level binding energy $E_b = m - E_r$ is comparable to the rest energy m, then the difference between the potentials U and $S + V$ becomes very essential.

We conclude this section by remarking that if the scalar field is switched off ($S = 0$), then the semiclassical formulas obtained here for E_r and Γ agree completely with the well-known results [26], [27] for the same characteristics of resonances in the purely vector field $V(r) = -eA_0(r)$.

Before we pass to specific applications of the method considered above, we first test quantization rule (35) in the case of model potentials where all calculations can be performed analytically.

4. Comparison with exact solutions

We consider several examples of the potentials $V(r)$ and $S(r)$ for which semiclassical quantization condition (35) gives the exact values of all energy levels including the ground state.

Example 1. We begin with one of the simplest problems: to calculate the spectrum of relativistic bound states in the mixture of a scalar $S(r)$ and a vector $V(r)$ attractive Coulomb-type potential,

$$V(r) = -\frac{\xi}{r}, \qquad S(r) = -\frac{\xi'}{r}, \tag{39}$$

where ξ and ξ' are the electrostatic and scalar coupling constants. This problem is often used as a model approximation in the relativistic description of the spectra of "exotic" HL systems, for example, lepton atoms for which the interaction between their components is realized as an exchange of quanta of fields of two types [40], [41]. Namely, if the Coulomb interaction $V(r) = -\xi/r$ is caused by an exchange of a virtual photon (an electromagnetic field quantum), then the lepton–nucleus interaction $S(r) = -\xi'/r$ responsible for the scalar coupling can be caused by an exchange of a virtual neutral spin-0 particle. The main candidate for this role is the scalar σ-meson, whose existence is confirmed by serious theoretical arguments (see, e.g., [42] and the references therein). Moreover, it was reported quite recently [43] that an anomalously wide scalar resonance in cascades of nonlepton decays of heavy mesons (D, B, and J/ψ) was observed by two groups of experimenters. We also note that the discovered scalar meson has a comparatively large mass ($M_\sigma = 390\,\text{MeV}$ [43], [44]). Hence, the scalar potential $S(r)$ corresponding to the exchange of such a particle is indeed a short-range (Yukawa-type) potential. Nevertheless, as noted in [40] and [41], in the framework of a comparatively simple model with a scalar–vector version of interaction such as model (39) considered here, many interesting characteristics of the energy spectrum of lepton atoms can already be found, and these characteristics are preserved when a more realistic model is considered (see [40], [41] for further physical details of model (39)).

We use the notation

$$\lambda = (m^2 - E^2)^{1/2}, \qquad \gamma = (k^2 - \xi^2 + \xi'^2)^{1/2} \tag{40}$$

to rewrite expressions (26) and (28) for $p(r)$ and $w(r)$ as

$$p(r) = \frac{\lambda}{r}\sqrt{(r - r_0)(r_1 - r)}, \qquad w(r) = -\frac{m + E}{2\big[\xi - \xi' + (m + E)r\big]}, \tag{41}$$

where the turning points r_0 and r_1 are determined by the formulas

$$r_{0,1} = \frac{1}{\lambda^2}\left[\xi E + \xi'm \mp \sqrt{(\xi E + \xi'm)^2 - \lambda^2\gamma^2}\right]. \tag{42}$$

Quantization condition (35) in this notation becomes

$$\lambda \int_{r_0}^{r_1} \sqrt{(r - r_0)(r_1 - r)}\, \frac{dr}{r} - \frac{k(m + E)}{2\lambda} \int_{r_0}^{r_1} \frac{dr}{\left[\xi - \xi' + (m + E)r\right] \sqrt{(r - r_0)(r_1 - r)}} =$$

$$= \left(n_r + \frac{1}{2}\right)\pi, \quad n_r = 0, 1, 2, \ldots. \quad (43)$$

It is convenient to let I_1 and I_2 denote the respective first and second integrals in this formula. Each of these integrals can be calculated in three steps. First, we change the integration variable as $r = \left[(r_1 - r_0)x + (r_1 + r_0)\right]/2$ to reduce the quadratic forms under the radical signs in I_1 and I_2 to diagonal form. Second, we substitute $y = \left[(1 - x)/(1 + x)\right]^{1/2}$ to reduce the integrands in I_1 and I_2 to rational form:

$$I_1 = \lambda \frac{(r_1 - r_0)^2}{r_0} \int_{-\infty}^{+\infty} \frac{y^2}{(1 + y^2)^2(y^2 + a^2)}\, dy, \quad (44)$$

$$I_2 = -\frac{k}{2}\sqrt{\frac{m + E}{m - E}} \cdot \frac{1}{\sqrt{\left[\xi - \xi' + r_1(m + E)\right]\left[\xi - \xi' + r_0(m + E)\right]}} \int_{-\infty}^{+\infty} \frac{dy}{y^2 + 1}, \quad (45)$$

where $a^2 = r_1/r_0$. These integrals can be calculated using the residue theorem. For this, we must close the integration path in both integrals (44) and (45) by a semicircle of infinitely large radius in the upper half-plane of the complex variable y and take into account that the integrand in (44) has poles at the points $y = ia$ and $y = i$; moreover, the last pole is a second-order pole. In turn, the integrand in (45) has a unique simple pole at the point $y = i$ in the upper half-plane. Calculating the integrals in (44) and (45) over the residues of the integrands, we finally obtain

$$\frac{\pi}{2}\left\{\lambda r_0 \left(\sqrt{\frac{r_1}{r_0}} - 1\right)^2 - \sqrt{\frac{m + E}{m - E}} \frac{k}{\sqrt{(\xi - \xi')^2 + (\xi - \xi')(m + E)(r_0 + r_1) + r_0 r_1(m + E)^2}}\right\} =$$

$$= \left(n_r + \frac{1}{2}\right)\pi.$$

Simplifying this expression, we obtain the equation for the energy eigenvalues:

$$\frac{\xi E + \xi' m}{\sqrt{m^2 - E^2}} - \gamma - \frac{1}{2}\operatorname{sgn} k = n_r + \frac{1}{2}. \quad (46)$$

We solve this equation for E and obtain

$$E = m\left\{\frac{-\xi\xi'}{\xi^2 + (N + \gamma)^2} \mp \left[\left(\frac{\xi\xi'}{\xi^2 + (N + \gamma)^2}\right)^2 - \frac{\xi'^2 - (N + \gamma)^2}{\xi^2 + (N + \gamma)^2}\right]^{1/2}\right\}, \quad (47)$$

where $N = n_r + (1 + \operatorname{sgn} k)/2 = n - |k|$, $k = \mp(j + 1/2)$ for the states with $j = l \pm 1/2$, and $n = 1, 2, \ldots$ is the principal quantum number. This formula coincides (for all values of n_r and k) with the well-known exact expression (see p. 186 in [40]) for the spectrum of spinor equation (1) with scalar and electrostatic Coulomb-type potentials (39), although according to the semiclassical method used to derive this formula, it can formally be used only for $n_r \gg 1$.

We briefly analyze the results following from (47) in some most important cases.

A. We first consider the situation where the external electrostatic field is switched off ($\xi = 0$) and expression (47) for the discrete energy levels then becomes

$$E_\pm = \pm m\sqrt{1 - \frac{\xi'^2}{(n - |k| + \gamma)^2}}, \quad (48)$$

where we now have $\gamma = \sqrt{k^2 + \xi'^2}$. This formula shows that the Dirac equation with a scalar coupling for $S(r) = -\xi'/r$ has two branches of the massive fermion energy spectrum symmetrically situated (with respect to the zero level $E = 0$) corresponding to the two possible values of the square root in (48). More precisely, the positive sign of the root in formula (48) corresponds to the energy spectrum of a particle, and the negative sign corresponds to the energy spectrum of an antiparticle (with the minus sign). We can thus see that considered spinor equation (1) with a scalar coupling simultaneously describes the behavior of particles and antiparticles. In this case, the energy levels in each branch of the spectrum are ordered in succession, starting from the ground state: $1S_{1/2}$, $2P_{3/2}$, $2S_{1/2}$ and $2P_{1/2}$, $3S_{1/2}$, and so on.

We also note that the energy gap between the spectra of particles and antiparticles decreases as the scalar coupling constant ξ' increases, and in the limit as $\xi' \to \infty$, the energy levels tend asymptotically to zero ($E_\pm \to \pm 0$) but never reach it. Therefore, neither the creation of pairs nor the collision of levels of particles and antiparticles occur here, i.e., the Dirac vacuum is stable in the case of the scalar coupling.

B. We now consider what happens when the external scalar field is switched off. We set $\xi' = 0$ in (47) and obtain the well-known Sommerfeld formula [2] for the fine structure of the levels of an HL atom,

$$E_{nj} = m \left[1 + \frac{\xi^2}{\left(n - |k| + \gamma \right)^2} \right]^{-1/2}, \tag{49}$$

where $\gamma = \sqrt{k^2 - \xi^2}$, $\xi = Z\alpha = Z/137$, Z is the nuclear charge, and α is the fine structure constant. The second branch of the energy spectrum corresponding to the negative sign of the root in (47) is neglected because it gives extra solutions of original equation (46) in the special case $\xi' = 0$ under study: for $E < 0$ and $\xi' = 0$, the left-hand side of (46) is negative, and the right-hand side is positive. This mathematically explains the well-known fact that the discrete spectrum of the Dirac equation with a vector coupling does not contain bound states for antiparticles under the given condition $V(r) < 0$, although there is the effective attraction $U(r, E) \sim -V^2/(2m)$ at small distances, which holds for both particles and antiparticles. As follows from (49), each level of the discrete spectrum in the Coulomb field of the nuclear charge Z arises only from the upper continuum and decreases monotonically to zero as Z increases. For $\xi = |k|$, the parameter γ in (49) has a root singularity, which gives complex values of the energy E_{nj} when (49) is further continued to the domain $\xi > |k| = j + 1/2$ (a detailed discussion of problems related to this singularity is contained in [18], [23], [24], [25], [40], [41]). As is known, if the finite dimensions of the nucleus are taken into account, then this anomaly disappears, and the bound states of an electron in the strong field of the Coulomb potential (cut off at small distances, $r \le 10^{-12}$ cm) exist up to Z_c, where Z_c is the value of the charge Z at which the energy of the considered state attains the boundary m of the lower (positron) continuum. For the first four levels $1S_{1/2}$, $2P_{1/2}$, $2S_{1/2}$, and $2P_{3/2}$, calculations give the respective values of the critical charge Z_c 172, 185, 245, and 255 [24]. It is essential that the electron and positron states are not mixed for $Z < Z_c$ ($1S_{1/2}$); the Dirac equation with a vector coupling does not have "positron levels" that would arise from the lower continuum. Therefore, the spectrum of the Dirac equation in the field of the nucleus is completely determined for $Z < Z_c$. Obviously, the phenomena arising for $Z > Z_c$ have an essentially multiparticle character and must be described in the framework of the field theory formalism [25], [45].

Based on the above discussion, we conclude that because the Coulomb field of the nucleus considered here acts on particles and antiparticles differently (attracts electrons and repels positrons), the energy spectra of particles (electrons) and antiparticles (positrons) are nonsymmetric. This is precisely the principal difference between this case and the preceding case of interactions of massive fermions with external Coulomb-type scalar field (39); the influence of the latter leads only to variations in the mass (which is the same for both types of Fermi particles, i.e., electrons and positrons). This means that in contrast to the electrostatic field, the scalar field acts equally on both particles and their antiparticles (and this is the

reason that the discrete spectra of electrons and positrons are symmetric for the Dirac equation with a scalar coupling).

C. We consider another important particular case realized for $\xi = \xi'$. In this situation, $\gamma = |k|$, $n - |k| + \gamma = n$, and expression (47) becomes noticeably simpler:

$$E_\pm = m \left[\frac{-\xi^2}{\xi^2 + n^2} \pm \frac{n^2}{\xi^2 + n^2} \right]. \tag{50}$$

In this expression, the spectrum branch $E_- = -m$ corresponding to the lower (minus) sign must be omitted because this branch leads to a violation of relation (46) in our particular case $\xi = \xi'$. The branch of the spectrum remaining in (50) then gives the particle energy

$$E_+ = m \left[1 - \frac{2\xi^2}{\xi^2 + n^2} \right]. \tag{51}$$

This implies that the energy gap $\Delta = m + E_+$ between the bound state and the lower continuum decreases as the coupling constant ξ increases. Nevertheless, in the limit as $\xi \to \infty$, the energy levels do not enter the lower continuum but only asymptotically approach its boundary and never attain the value $E = -m$. The energy of the lower level $n = 1$ becomes zero for the particular value $\xi = 1$.

We now consider the problem of whether the semiclassical approximation can be applied to compound Coulomb-type field (39) studied here. To obtain a qualitative estimate, we transform expression (41) for the semiclassical momentum $p(r)$ as

$$p(r) = \frac{\gamma}{r} \sqrt{\left(\frac{r}{r_0} - 1 \right) \left(1 - \frac{r}{r_1} \right)}, \tag{52}$$

where γ is again determined by expression (40). The applicability condition for the semiclassical approach then becomes

$$\frac{d}{dr} \left(\frac{1}{p} \right) = \gamma^{-1} \left(\frac{r}{r_{\min}} - 1 \right) \left[\left(\frac{r}{r_0} - 1 \right) \left(1 - \frac{r}{r_1} \right) \right]^{-3/2} \ll 1 \tag{53}$$

and is better satisfied for larger γ. Here, $r_{\min} = 2/(r_0^{-1} + r_1^{-1})$ is the point at which effective potential (22) attains its minimum. Under the condition $\xi' > \xi$ (which corresponds to the case of a strong scalar field), the solution of Dirac system (3) for potentials (39) under study has a semiclassical form in the entire domain of r except in narrow intervals immediately adjacent to the turning points r_0 and r_1. But if $\xi \gg \xi'$, then the situation is similar to the case of a strong Coulomb field in the theory of supercritical atoms [26], [27].

For $\gamma \gg 1$, the centrifugal potential $\gamma^2/(2mr^2)$ plays an important role (dominating as $r \to 0$) in the effective potential (22) in the region $r < r_1$. In this region, the momentum $p(r)$ is similar to γ/r, and the ratio of two terms in the integrand in (35) has the order of $k\gamma^{-2} r w(r) \sim (\xi' - \xi)^{-1}$. Because the semiclassical approximation for the wave function has the order of magnitude of $(\xi' - \xi)^{-2}$ for $\xi' \gg \xi \sim 1$ (see [18]), the spin–orbit terms must be preserved (as corrections of the order of $(\xi' - \xi)^{-1}$).

Example 2. We find the energy spectrum of a relativistic particle of mass m and spin $1/2$ in the mixture of scalar and vector potentials of the oscillatory form,

$$S(r) = V(r) = \omega \frac{r^2}{4}, \quad \omega > 0. \tag{54}$$

The solution of the spectral problem for the Dirac equation with such potentials is particularly interesting from the theoretical standpoint in the spectroscopy of hadrons [31], [46].

It is clear that the model of the interquark $Q\bar{q}$ interaction composed only of oscillatory-type potentials (54) does not take the Coulomb attraction at small distances corresponding to the interaction of free quarks into account. But for large r, the effective string tension responsible for the confinement of quarks inside hadrons is taken into account in this model. Moreover, the Dirac equation can be solved exactly for this simple model of the $Q\bar{q}$ interaction, which is particularly interesting in itself.

For potentials (54), we obtain

$$p(r) = \frac{\beta}{\sqrt{2}\,r}\left[(r^2 - r_0^2)(r_1^2 - r^2)\right]^{1/2}, \qquad w(r) = -\frac{1}{2r}, \tag{55}$$

$\beta = \sqrt{\omega(E + m)}$, and the turning points r_0 and r_1 are determined by the formulas

$$r_{0,1} = \frac{1}{\sqrt{\omega}}\left[E - m \mp \sqrt{(E - m)^2 - 2k^2\omega^2\beta^{-2}}\right]^{1/2}. \tag{56}$$

We replace the momentum $p(r)$ and the function $w(r)$ in (35) with their explicit expressions (55) and integrate over the new variable $x = r^2$. As a result of the above transformations, quantization rule (35) becomes

$$\frac{\beta}{2^{3/2}}\int_{x_0}^{x_1}\sqrt{(x - x_0)(x_1 - x)}\,\frac{dx}{x} - \frac{k}{2^{3/2}\beta}\int_{x_0}^{x_1}\frac{dx}{x\sqrt{(x - x_0)(x_1 - x)}} = \left(n_r + \frac{1}{2}\right)\pi, \tag{57}$$

where $n_r = 0, 1, 2, \ldots$ and the new boundaries of the integration domain are given by the formulas $x_0 = r_0^2$ and $x_1 = r_1^2$.

The quantization integrals can be calculated using the residue theory as in the case of the integrals I_1 and I_2 in formula (43). Because all technical details necessary for these calculations have already been described in detail, we here present only the final result

$$\frac{\beta(E - m)}{2^{3/2}\omega} - \frac{|k|}{2} - \frac{1}{4}\,\mathrm{sgn}\,k = n_r + \frac{1}{2}. \tag{58}$$

If we introduce the notation $\mathcal{K} = |k| + (1 + \mathrm{sgn}\,k)/2$ and replace the parameter β with its explicit expression, then the last relation can be rewritten as

$$(E - m)\sqrt{2(E + m)} - (4n_r + 2\mathcal{K} + 1)\sqrt{\omega} = 0. \tag{59}$$

We solve this equation for E and obtain the energy eigenvalues as functions of the quantum numbers n_r and $\mathcal{K}$:

$$E_{n_r,\mathcal{K}} = \frac{2m + 8 \cdot 2^{2/3}m^2 A^{-1/3} + 2^{1/3}A^{1/3}}{6}, \tag{60}$$

where we introduce the notation

$$A = -B + \sqrt{B^2 - 1024m^6}, \qquad B = 32m^3 - 27\omega(1 + 2\mathcal{K} + 4n_r)^2.$$

The applicability condition for semiclassical expression (60) is $n_r \gg 1$. As we show in the appendix, in the exact solution of the Dirac equation with scalar $S(r)$ and vector $V(r)$ oscillatory-type potentials (54), the energy of stationary states is given by formula (60) for all values of n_r.

As follows from (60), each of the states is characterized by two quantum numbers n_r and $\mathcal{K}$. The energy depends only on the combination $2n_r + \mathcal{K} = \Lambda$ of these quantum numbers, and $\Lambda = 1, 2, 3, \ldots$ can hence be called the principal quantum number. Each value of $\Lambda \geq 3$ can be realized in several combinations of the values of n_r and $\mathcal{K}$, and energy levels (60) with a value of $\Lambda \geq 3$ are hence degenerate.

Example 3. We derive the energy spectrum of the Dirac equation for a massless fermion in an external scalar field with a combined "funnel"-type potential

$$S(r) = -\frac{\xi'}{r} + \sigma r, \quad \sigma > 0, \qquad V(r) = 0. \tag{61}$$

The specific character of the considered model with such a scalar interaction is manifested, in particular, in the spontaneously violated chiral symmetry of an originally symmetric system. The point is that for a massless particle ($m = 0$), Dirac equation (1) with a purely vector coupling (for $S(\mathbf{r}) = 0$) is invariant under the global transformation of the wave function $\Psi \to \exp(i\alpha'\gamma_5)\Psi$, $\widetilde{\Psi} \to \widetilde{\Psi}\exp(i\alpha'\gamma_5)$. The chiral symmetry of the spectrum is manifested in the degeneration of all the states with respect to their parity; more precisely, the masses of the states 0^+ and 0^- (or 1^+ and 1^-) are the same.

The invariance under global transformations does not exhaust all the symmetry properties of the massless Dirac equation. It is easy to verify that system of equations (3) in the chiral limit ($m = 0$) is invariant under more general transformations of the form [29]

$$E \to E, \quad k \to -k, \quad S \to -S, \quad V \to V, \quad G(r) \to -F(r), \quad F(r) \to G(r), \tag{62}$$

which are not related to a geometric space–time symmetry. This implies that the spectrum is degenerate with respect to the sign of the Dirac quantum number k in the absence of an external scalar field ($S(r) = 0$), i.e., the spectrum depends only on the total momentum j and not on its components (the orbital l and spin $s = 1/2$ momenta) separately (the chiral degeneration).

A straightforward verification easily shows that the massless Dirac equation is also invariant under charge conjugation transformations

$$E \to -E, \quad k \to -k, \quad S \to S, \quad V \to -V, \quad G(r) \leftrightarrow F(r). \tag{63}$$

We see that these symmetry transformations contrast with preceding transformations (62) because they couple states with positive and negative values of the energy. In particular, in the absence of an external electrostatic field (the purely scalar interaction), there are pairs of states with a given value of $|E_n|$ but with opposite signs of the energy $E_n = \pm|E_n|$ itself. This fact permits restricting our consideration to studying only one (positive) branch of the spectrum $E_n > 0$ in detail, which simplifies the problem of finding the energy eigenvalues E_n for Dirac system (3) with scalar interaction version (61). This last problem can be solved either numerically or in the semiclassical approximation, which can formally be applied to higher excited states but (as we show below) gives results with a good accuracy even for the ground and first excited states.

For potential (61) and a particle with zero mass, expressions (26) and (28) for $p(r)$ and $w(r)$ become

$$p(r) = \frac{\sigma}{r}\left[(r^2 - r_0^2)(r_1^2 - r^2)\right]^{1/2}, \qquad w(r) = -\frac{1}{2}\left(\frac{1}{r - P_+} + \frac{1}{r - P_-}\right), \tag{64}$$

where

$$P_\pm = \frac{1}{2\sigma}\left(-E \pm \sqrt{E^2 + 4\xi'\sigma}\right), \qquad \gamma = \sqrt{k^2 + \xi'^2}, \tag{65}$$

and the position of the turning points r_0 and r_1 is determined by the relation

$$r_{0,1} = \frac{1}{\sqrt{2}\,\sigma}\sqrt{E^2 + 2\xi'\sigma \mp \sqrt{(E^2 + 2\xi'\sigma)^2 - 4\sigma^2\gamma^2}}\,. \tag{66}$$

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In this notation, the semiclassical quantization condition can be written as

$$\sigma \int_{r_0}^{r_1} \sqrt{(r^2 - r_0^2)(r_1^2 - r^2)}\, \frac{dr}{r} -$$

$$- \frac{k}{2\sigma} \sum_{i=\pm} \int_{r_0}^{r_1} \frac{dr}{(r - P_i)\sqrt{(r^2 - r_0^2)(r_1^2 - r^2)}} = \left(n_r + \frac{1}{2}\right)\pi, \quad n_r = 0, 1, 2, \ldots. \tag{67}$$

We can use appropriate changes of integration variables to calculate the first integral in the left hand-side of (67) over the residues of the corresponding path integral. The second integral in (67) (under the summation sign) can be expressed in terms of the complete elliptic integrals of the first $K(\nu)$ and the third $\Pi(\alpha^2, \nu)$ kind [47],

$$K(\nu) = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - \nu^2 \sin^2 \varphi}}, \qquad \Pi(\alpha^2, \nu) = \int_0^{\pi/2} \frac{d\varphi}{(1 - \alpha^2 \sin^2 \varphi)\sqrt{1 - \nu^2 \sin^2 \varphi}}. \tag{68}$$

As a result of these calculations, semiclassical quantization condition (67) becomes the transcendental equation

$$\frac{E^2 + 2\sigma(\xi' - \gamma)}{4\sigma} - \frac{k}{\sigma(r_0 + r_1)\pi}\left[2r_0\left(\frac{\Pi(\alpha_+^2, \nu)}{r_0^2 - P_+^2} + \frac{\Pi(\alpha_-^2, \nu)}{r_0^2 - P_-^2}\right) - \right.$$

$$\left. - \left(\frac{1}{r_0 + P_+} + \frac{1}{r_0 + P_-}\right)K(\nu)\right] = n_r + \frac{1}{2}, \tag{69}$$

where we use the new notation

$$\nu = \sqrt{\frac{E^2 + 2\sigma(\xi' - \gamma)}{E^2 + 2\sigma(\xi' + \gamma)}}, \qquad \alpha_\pm^2 = \nu\frac{P_\pm + r_0}{P_\pm - r_0}.$$

Equation (69) can be solved explicitly in the two limit cases $\sigma \to 0$ and $\sigma \to \infty$. In what follows, we only consider the practically important case of weak coupling, where for small values of the parameter σ (namely, for $\sigma \lesssim 0.2\,\mathrm{GeV}^2$), the condition $E_{n_r,k}^2 \gg 2\sigma\gamma$ is exactly satisfied for all possible values of the energy levels $E_{n_r,k}$. In this case, the above formulas become noticeably simpler, and Eq. (69) for the semiclassical spectrum takes the sufficiently simple final form

$$E_{n_r,k}^2 = 4\sigma\left[n_r + \frac{1}{2} + \frac{\gamma - \xi'}{2} + \frac{k}{4\gamma} + \frac{\sigma k}{2E_{n_r,k}^2}R(E_{n_r,k})\right] + O\left(\left(\frac{\sigma\gamma}{E_{n_r,k}^2}\right)^2\right), \tag{70}$$

where $\sigma > 0$ and the notation

$$R(E_{n_r,k}) = \frac{1}{\pi}\left(0.38 + \log\frac{E_{n_r,k}^2}{\sigma\gamma}\right)$$

is used. In the case where potential (61) dose not contain the Coulomb-type term (i.e., for $\xi' = 0$), Eq. (70) exactly coincides with the semiclassical quantization condition proposed in [30] for the energy levels in the scalar well $U(r, E)$ generated by the linear confining interaction $S(r) = \sigma r$, $V(r) = 0$.

Equation (70) for $E_{n_r,k}$ can be easily solved numerically. Comparing the results of such calculations for $E_{n_r,k}$ with the exact values [29] obtained by numerically integrating Dirac system (3), we see that semiclassical equation (70) ensures a fully appropriate accuracy for the energy spectrum: even for lower states with $n_r \sim 1$, the error in calculating $E_{n_r,k}$ does not exceed 5 % and rapidly decreases as n_r increases.

Along with directly solving transcendental equations (69) and (70) numerically, it is expedient, after several simplifications or approximations, to construct approximate analytic expressions for the energy levels, which would rather easily permit tracing the dependence of $E_{n_r,k}$ on the quantum numbers n_r and k and the parameters of model interaction (61). For this, we note that $R(E_{n_r,k}) \cong 0.6$ to 0.8 for lower states with the radial quantum number $n_r \sim 1$. But as n_r increases, this quantity rapidly attains unity, and, as numerical calculations show, there exists a sufficiently large region of the energy spectrum (see Table 1) where we can set $R(E_{n_r,k}) = 1$. In this approximation, Eq. (70) has the analytic solution

$$\varepsilon_{n_r,k} = \frac{E_{n_r,k}}{\sqrt{\sigma}} = \pm\sqrt{N' - \xi' + \left[(N' - \xi')^2 + 2k\right]^{1/2}}, \tag{71}$$

where $N' = 2n_r + 1 + \gamma + k/(2\gamma)$. The positive sign of the root corresponds to the energy of a particle, and the negative sign corresponds to the energy of the antiparticle taken with the minus sign. We make several remarks concerning formula (70) and some singularities of the relativistic spectrum of model (61) under study.

Table 1

States		$\xi' = 0$			$\xi' = 0.4$	
n_r	k	$\varepsilon_{n_r,k}$[29]	$\varepsilon_{n_r,k}^{\mathrm{WKB}}$	$\varepsilon_{n_r,k}^{(\mathrm{as})}$	$E_{n_r,k}^{\mathrm{WKB}}$, GeV	$E_{n_r,k}^{(\mathrm{as})}$, GeV
0	−1	1.61944	1.62292	1.4142	0.55809	
1	−1	2.60263	2.60381	2.5887	1.02972	1.04765
2	−1	3.29118	3.29182	3.2886	1.33819	1.35699
3	−1	3.85541	3.85581	3.8555	1.58633	1.60356
0	−2	2.14652	2.14721	2.0009	0.82198	0.72702
1	−2	2.95197	2.95230	2.9208	1.18994	1.18375
2	−2	3.57353	3.57371	3.5616	1.46502	1.46751
3	−2	4.09947	4.09961	4.0941	1.69492	1.69967
0	−3	2.56927	2.56951	2.4495	1.01854	0.95639
1	−3	3.26852	3.26871	3.2287	1.33151	1.31723
0	−4	2.93218	2.93231	2.8284	1.18253	1.15227
0	1	2.29403	2.29251	2.3178	0.93348	0.92051
1	1	3.03103	3.03038	3.0359	1.25659	1.23879
2	1	3.62598	3.62557	3.6265	1.51411	1.49669
0	2	2.70440	2.70391	2.7443	1.09944	1.10954
1	2	3.35376	3.35350	3.3693	1.38451	1.38364
0	3	3.05967	3.05589	3.1021	1.24973	1.26591
0	4	3.40866	3.36945	3.4183	1.38513	1.40086

Eigenvalues $E_{n_r,k}$ and $\varepsilon_{n_r,k}$ of the massless Dirac equation with scalar interaction (61) for two values of the Coulomb parameter ξ': $\varepsilon_{n_r,k}$ is the result of numerical calculations [29], $\varepsilon_{n_r,k}^{\mathrm{WKB}}$ and $E_{n_r,k}^{\mathrm{WKB}}$ are the results of solving transcendental equation (69) numerically, $\varepsilon_{n_r,k}^{(\mathrm{as})}$ and $E_{n_r,k}^{(\mathrm{as})}$ are asymptotic approximations of (70), and $|\sigma| = 0.18\,\mathrm{GeV}^2$.

1. The dependence $E_{n_r,k}(\sigma) \propto \sigma^{1/2}$ already follows from the scaling argument; the change $r \to \mu r$ in system of equations (3) with the mass $m = 0$ and combined potential (61) gives $E_{n_r,k} = \sqrt{\sigma}\,\varepsilon_{n_r,k}$ under an appropriate choice of the scale factor μ $(r \to r/\sqrt{\sigma})$.

2. The massless Dirac equation in external scalar field (61) with arbitrary ξ' and $\sigma \neq 0$ has only a discrete spectrum of energy levels, which is easily explained. The EP $U(r, E)$ at small distances in model (61) contains the prevailing term γ^2/r^2 corresponding to the repulsion and eliminating the "fall to

the center" for each value of the parameter ξ'. On the other hand, at large distances (where the energy spectrum is just formed), the EP $U(r, E)$ contains the dominating relativistic term $S^2/(2m)$ leading to the quadratic confinement $(\sigma r)^2/(2m)$ (independently of the sign of the parameter σ). Therefore, the EP $U(r, E)$ in model (20) always (for both positive and negative values of σ and ξ') has the form of an oscillatory potential well. This is the essential distinction from the nonrelativistic potential model in which the effective potential in the radial Schrödinger equation has a barrier for negative values of σ, and the quasistationary states with complex energy hence appear instead of discrete levels.

3. The spectrum of the eigenvalues $E_{n_r,k}$ of massless equation (1) with a scalar interaction of the linear form $S(r) = \sigma r$ ($V(r) \equiv 0$) was calculated with a high accuracy in [29] by directly integrating this equation numerically. Semiclassical expression (70) is compared with the exact values of $\varepsilon_{n_r,k}$ obtained in [29] and with the results of solving transcendental equation (69) by a computer implementation of the minimization method. This comparison is shown in Table 1 for two values of the Coulomb parameter ξ': $\xi' = 0$ and $\xi' = 0.4$. We see that the applicability domain of semiclassical asymptotic approximation (70), which formally holds under the quantization condition $E_{n_r,k}^2 \gg 2\sigma\gamma$ (for excited states with $n_r \gg 1$), can be "extended" up to the conventional string tension $\sigma \cong 0.18\,\text{GeV}^2$ even for the ground state ($n_r = 0$). This shows that asymptotic approximations of the semiclassical type can be useful in the qualitative analysis of the spectrum of original equation (1).

4. Numerical calculations of $E_{n_r,k}$ using transcendental equation (69) show that the general characteristics of the spectrum depend weakly on the value and even on the sign of the parameter σ if $|\sigma| \lesssim 0.18\,\text{GeV}^2$. But this assertion does not hold for the fine structure of P levels. For example, the $P_{1/2}$ level lies above the $P_{3/2}$ level for $\sigma > 0$ and below the $P_{3/2}$ level for $\sigma < 0$.

Appendix

We obtain the exact solutions and the energy spectrum of Dirac equation (1) in external centrally symmetric fields determined by scalar $S(r)$ and vector $V(r)$ oscillatory-type potentials (54). We eliminate the function $G(r)$ from system (3) and obtain the second-order equation for $F(r)$

$$\frac{d^2 F(r)}{dr^2} + \left[(E + m)\left(E - m - \frac{\omega r^2}{2} \right) - \frac{k(k+1)}{r^2} \right] F(r) = 0. \tag{A.1}$$

According to the character of the asymptotic behavior of the radial functions $F(r)$ and $G(r)$ for large and small r, we seek the solution of Eq. (A.1) in the form

$$F(r) = e^{-\beta r^2/2^{3/2}} r^{\mathcal{K}} f(\rho), \tag{A.2}$$

where $\rho = \beta r^2/\sqrt{2}$ and the parameters β and $\mathcal{K}$ are respectively determined in (55) and (59). Substituting this expression in (A.1) gives the equation for the function $f(\rho)$:

$$\rho f''(\rho) + (4\alpha - \rho)f'(\rho) - \left[2\alpha - \frac{\beta(E - m)}{2^{3/2}\omega} \right] f(\rho) = 0, \tag{A.3}$$

where $\alpha = (2\mathcal{K} + 1)/8$. The solution of this equation, which is finite for $\rho = 0$, can be expressed (up to a constant factor C) in terms of the degenerate hypergeometric function $F(a, b; z)$ as

$$f(\rho) = CF\left(2\alpha - \frac{\beta(E - m)}{2^{3/2}\omega}, 4\alpha; \rho \right). \tag{A.4}$$

For the hypergeometric function $F(a, b; \rho)$ in the right-hand side of (A.4) to reduce to a polynomial, the parameter a must be a nonpositive integer, which leads to Eq. (59) for determining the discrete energy levels.

The solution for $G(r)$ is given by the formula

$$G(r) = \frac{1}{m + E}\left(\frac{dF(r)}{dr} + \frac{k}{r}F(r)\right), \qquad (A.5)$$

where the expression for $F(r)$ obtained above and the recursive relations for the degenerate hypergeometric functions [47] are used. The overall normalization coefficient C, which remains undetermined in $F(r)$ and $G(r)$, can be obtained from the condition $\int_0^\infty (F^2 + G^2)\, dr = 1$.

In conclusion, the final expressions for the radial wave functions of the discrete spectrum are

$$F(r) = Ce^{-\beta r^2/2^{3/2}} r^{\mathcal{K}} F\left(-n_r, \frac{2\mathcal{K}+1}{2}; \frac{\beta}{\sqrt{2}}r^2\right), \qquad (A.6)$$

$$G(r) = C\frac{2^{3/2}\beta}{(2\mathcal{K}+1)(m + E_{n_r\mathcal{K}})}e^{-\beta r^2/2^{3/2}} r^{\mathcal{K}+1} \times$$

$$\times \left[n_r F\left(1 - n_r, \frac{2\mathcal{K}+3}{2}; \frac{\beta}{\sqrt{3}}r^2\right) + \frac{2\mathcal{K}+1}{4}F\left(-n_r, \frac{2\mathcal{K}+1}{2}; \frac{\beta}{\sqrt{2}}r^2\right)\right], \qquad (A.7)$$

where the normalization coefficient is

$$C = \frac{\beta^{(2\mathcal{K}+1)/4}}{2^{-1+(2\mathcal{K}+1)/8}\Gamma((2\mathcal{K}+1)/2)}\sqrt{\frac{\Gamma((2\mathcal{K}+1)/2 + n_r)(E_{n_r,\mathcal{K}} + m)}{n_r!\,(3E_{n_r,\mathcal{K}} + m)}}.$$

The exact solutions and the energy spectrum of the Dirac equation with oscillatory potential (54) were recently studied in [48], where only the states with $k < 0$, $\mathcal{K} = |k|$, were considered. In this particular case, our expressions (60), (A.6), and (A.7) give the results obtained in [48].

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