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Research Article

Construction of a Model of Gaussian Stationary Random Process in Some Orlicz Spaces with Given Accuracy and Reliability

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Abstract

In the paper we propose estimates for sub-Gaussian standard models of Gaussian stationary random processes. A model of Gaussian stationary random process in some Orlicz spaces is constructed with given accuracy and reliability.

Keywords: Stochastic process; Sub-Gaussian random variables; Orlicz space; Spectral function; Accuracy and Reliability of model

1. Introduction

The problem of modelling of stochastic process has been a matter of active research during the last decades. Descriptions of methods of modelling of random processes and fields can be found, for example, in books by Ogorodnikov and Prigarin (1996), Kozachenko and Pashko (1999), Kozachenko et al. (2016). In this article wedeal with the method of modelling of Gaussian stationary stochastic processes which is based on randomization of the spectrum. This method isdescribed in the paper byVojtyshek (1983). In this article we investigate the accuracy and reliability of models of Gaussian stationary random process in some Orlicz spaces. We used properties of random processes in Orlicz space described in articles byKozachenko and Pashko (1988), Tegza (2002),Antoniniet al (2002).

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2. Basic Definitions and Statements

Let $\{\Omega, B, P\}$ be a standart probability space.

Definition 2.1 A random variable ξ is called sub-Gaussian, if there exists a number $a \geq 0$, such that the inequality

$$E\exp\{\lambda\xi\} \leq \exp\left\{\frac{a^2\lambda^2}{2}\right\}$$

holds true for all $\lambda \in R$.

The class of all sub-Gaussian random variables defined on a standart probability space $\{\Omega, B, P\}$ is denoted by $Sub(\Omega)$. Consider the following numerical characteristic of a sub-Gaussian random variable ξ:

$$\tau(\xi) = \inf\left\{a \geq 0 : E\exp\{\lambda\xi\} \leq \exp\left\{\frac{a^2\lambda^2}{2}\right\}, \lambda \in R\right\}. \tag{1}$$

We will call $\tau(\xi)$ the sub-Gaussian standard of the random variable ξ. By definition, $\xi \in Sub(\Omega)$ if and only if $\tau(\xi) < \infty$. In the book by Buldygin and Kozachenko (2000) it is shown that the space $Sub(\Omega)$ is a Banach space with the norm $\tau(\xi)$.

Let $(T, \Im, \mu)$, $\mu(T) < \infty$, be a measurable space.

Definition 2.2 A stochastic process $X = \{X(t), t \in T\}$ is called sub-Gaussian if for any $t \in T$ $X(t) \in Sub(\Omega)$ and $\sup_{t \in T} \tau(X(t)) < \infty$.

Let $L_U(T)$ be the Orlicz space generated by an C-function $U = \{U(x), x \in R\}$. Remind that a continuous even convex function $U(\cdot)$ is called C- function if it is monotonically increasing, $U(0) = 0$, and $U(x) > 0$ as $x \neq 0$. For example, $U(x) = \exp\{|x|^\alpha\} - 1$, $\alpha \geq 1$, is an C-function The Orlicz space generated by an C-function function $U(x)$ is defined as a family of functions $\{f(t), t \in T\}$ where for each function $f(t)$ there exists a constant r such that

$$\int_T U\left(\frac{f(t)}{r}\right) d\mu(t) < \infty.$$

The space $L_U(T)$ is a Banach space with the norm

$$\|f\|_{L_U} = \inf\left\{r > 0 : \int_T U\left(\frac{f(t)}{r}\right) d\mu(t) \leq 1\right\}. \tag{2}$$

The norm $\|f\|_{L_U}$ is calledthe Luxemburg norm .

Random processes in Orlicz spaces were introduced and studied in the paper by Kozachenko (1985).

Let $X = \{X(t), t \in T\}$ be a sub-Gaussian random process and let $\tau = \sup_t \tau(X(t)) < \infty$.

The following theorem is proved in the paper by Kozachenkoand Pashko (1988) .

Theorem 2.1 Let $U=\{U(x), x\in R\}$ be an C -function such that the function $G_U(t)=\exp\{(U^{(-1)}(t-1))^2\}$ is convex for $t\geq 1$. Then with probability one the process $X\in L_U(\mathrm{T})$ and for all ε such that

$$\varepsilon\geq \hat{\mu}(\mathrm{T})\cdot\tau\cdot\left(2+\left(U^{(-1)}(1)\right)^{-2}\right)^{\frac{1}{2}}$$

the following inequality holds true

$$P\left\{\|X\|_{L_U}>\varepsilon\right\}\leq\sqrt{e}\,\frac{\varepsilon U^{(-1)}(1)}{\hat{\mu}(\mathrm{T})\cdot\tau}\cdot\exp\left\{-\frac{\varepsilon^2(U^{(-1)}(1))^2}{2(\hat{\mu}(\mathrm{T}))^2\cdot\tau^2}\right\}, \quad (3)$$

where $\hat{\mu}(\mathrm{T})=max(\mu(\mathrm{T}),1)$

The proof of the theorem can be found in the book by Kozachenko et al. (2016).

3. Problem Statement

Let $X=\{X(t), t\in\mathrm{T}\}$ be a Gaussian stationary real valued centered mean square continuous random process with the covariance function

$$EX(t+\tau)X(t)=r(\tau)=\int_0^\infty\cos(\lambda\tau)dF(\lambda),$$

where $F(\lambda)$ is a continuous spectral function of the process. It is known that the process $X(t)$ can be represented as in the form

$$X(t)=\int_0^\infty\cos(\lambda t)d\eta_1(\lambda)+\int_0^\infty\sin(\lambda t)d\eta_2(\lambda),$$

where $\eta_1(\lambda)$, $\eta_2(\lambda)$ are independent centered Gaussian processes with independent increments.
Let us represent the process $X(t)$ in the form

$$X(t)=X_\Lambda(t)+X^\Lambda(t),$$

where

$$X_\Lambda(t)=\int_0^\Lambda\cos(\lambda t)d\eta_1(\lambda)+\int_0^\Lambda\sin(\lambda t)d\eta_2(\lambda),$$

$$X^\Lambda(t)=\int_\Lambda^\infty\cos(\lambda t)d\eta_1(\lambda)+\int_\Lambda^\infty\sin(\lambda t)d\eta_2(\lambda). \quad (4)$$

As a model of the process $X_\Lambda(t)$ we will take the process

$$X_\Lambda^M(t)=\sum_{k=0}^{M-1}(\eta_{k1}\cos(\zeta_k t)+\eta_{k2}\sin(\zeta_k t)), \quad (5)$$

Here $\Lambda=\{\lambda_0,\ldots,\lambda_M\}$ is a partition of the set $[0,\Lambda]$, such that $\lambda_0=0$, $\lambda_k<\lambda_{k+1}$, $\lambda_M=\Lambda$;
η_{k1}, η_{k2}, ζ_k are independent random variables; $\eta_{k1}=\eta_1(\lambda_{k+1})-\eta_1(\lambda_k)$, $\eta_{k2}=\eta_2(\lambda_{k+1})-\eta_2(\lambda_k)$ are Gaussian random variables such that $E\eta_{k1}=E\eta_{k2}=0$,

$$E\eta_{k1}^2 = E\eta_{k2}^2 = F(\lambda_{k+1}) - F(\lambda_k) = b_k^2, \quad k = 0, \dots, M-1,$$

ζ_k -- are independent for any k, and are defined on $[\lambda_k, \lambda_{k+1}]$ with cumulative distribution function

$$P\{\zeta_k < \lambda\} = F_k(\lambda) = \frac{F(\lambda) - F(\lambda_k)}{F(\lambda_{k+1}) - F(\lambda_k)}.$$

Let $\eta_\Lambda(t) = X_\Lambda(t) - X_\Lambda^M(t)$. Then

$$\eta_\Lambda(t) = \sum_{k=0}^{M-1} \left[\int_{\lambda_k}^{\lambda_{k+1}} (\cos(\lambda t) - \cos(\zeta_k t)) d\eta_1(\lambda) + \int_{\lambda_k}^{\lambda_{k+1}} (\sin(\lambda t) - \sin(\zeta_k t)) d\eta_2(\lambda) \right] \tag{6}$$

In the book by Kozachenko et al. (2016) it is proved that $\eta_\Lambda(t)$ is a sub-Gaussian random process.

4. Construction of a Model with Given Accuracy and Reliability

Definition 4.1 A random process $X_\Lambda^M(t)$ approximates the process $X(t)$ with reliability $(1-\beta)$, $0 < \beta < 1$ and accuracy $\delta > 0$ in the Orlich space L_U, if there exists a partition $\Lambda = \{\lambda_0, \dots, \lambda_M\}$ of the set $[0, \Lambda]$ such that

$$P\left\{ \left\| X(t) - X_\Lambda^M(t) \right\|_{L_U} > \delta \right\} \le \beta.$$

We will describe conditions under which it is possibleto select a partition Λ, such that the model X_Λ^M approximates the centered Gaussian process $X(t)$ in the Orlicz space with given accuracy and reliability

Lemma 4.1 For $a > 1.322$ the inequality $\sin x \le \ln(1+ax)$ holds true.

*Proof:*In the case $x > \frac{\pi}{2}$ the inequality $\sin x \le \ln(1+ax)$ is obvious. This inequality it is necessary to provein the case $x \in [0, \frac{\pi}{2}]$ .But in this case the inequality $\sin x \le \ln(1+ax)$ is equivalent to

$$a \ge \max_{x \in [0, \frac{\pi}{2}]} \frac{e^{\sin x} - 1}{x} \approx 1.322$$

Lemma 4.2 For the sub-Gaussian process $\eta_\Lambda(t)$ determined by (6) the following inequality is satisfied

$$\tau(\eta_\Lambda(t)) \le 4 \ln\left(1 + \frac{3t\Lambda}{4M}\right) \sqrt{F(\Lambda)}.$$

Proof: Using the corresponding results from papers by Buldygin and Kozachenko (2000) and Tegza (2002) we have that for anySub-Gaussian random variable the following inequality holds true: $\tau(\xi) \le \Theta_1(\xi)$, where

$$\Theta_1(\xi)=\sup_{k\ge1}\left[\frac{2^k k!}{(2k)!}E\xi^{2k}\right]^{\frac{1}{2k}}<\infty.$$

Since $\eta_1(\lambda)$, $\eta_2(\lambda)$ are independent centered Gaussian processes we havethe followng inequalities:

$$\tau^2\left(\sum_{k=0}^{M-1}\left[\int_{\lambda_k}^{\lambda_{k+1}}(\cos(\lambda t)-\cos(\zeta_k t))d\eta_1(\lambda)+\int_{\lambda_k}^{\lambda_{k+1}}(\sin(\lambda t)-\sin(\zeta_k t))d\eta_2(\lambda)\right]\right)\le$$

$$\le\sum_{k=0}^{M-1}\left(\tau\left(\int_{\lambda_k}^{\lambda_{k+1}}(\cos(\lambda t)-\cos(\zeta_k t))d\eta_1(\lambda)\right)+\tau\left(\int_{\lambda_k}^{\lambda_{k+1}}(\sin(\lambda t)-\sin(\zeta_k t))d\eta_2(\lambda)\right)\right)^2$$

$$\tau\left(\int_{\lambda_k}^{\lambda_{k+1}}(\cos(\lambda t)-\cos(\zeta_k t))d\eta_1(\lambda)\right)\le\Theta_1\left(\int_{\lambda_k}^{\lambda_{k+1}}(\cos(\lambda t)-\cos(\zeta_k t))d\eta_1(\lambda)\right)=$$

$$=\sup_{m\ge1}\left[\frac{2^m m!}{(2m)!}E\left(\int_{\lambda_k}^{\lambda_{k+1}}(\cos(\lambda t)-\cos(\zeta_k t))d\eta_1(\lambda)\right)^{2m}\right]^{\frac{1}{2m}}\qquad(7)$$

$$E\left(\int_{\lambda_k}^{\lambda_{k+1}}(\cos(\lambda t)-\cos(\zeta_k t))d\eta_1(\lambda)\right)^{2m}\le4^m\frac{(2m)!}{2^m m!}E\left(\int_{\lambda_k}^{\lambda_{k+1}}\sin^2\frac{t(\zeta_k-\lambda)}{2}dF(\lambda)\right)^m.$$

Substituting the last inequality to (7), and using the equality from lemma 4.1 (with $a=1.5$), we will have

$$\tau\left(\int_{\lambda_k}^{\lambda_{k+1}}(\cos(\lambda t)-\cos(\zeta_k t))d\eta_1(\lambda)\right)\le2b_k\sup_{m\ge1}\left[\int_{\lambda_k}^{\lambda_{k+1}}\left(\int_{\lambda_k}^{\lambda_{k+1}}\sin^2\frac{t(u-\lambda)}{2}dF_k(\lambda)\right)^m dF_k(u)\right]^{\frac{1}{2m}}\le$$

$$\le2b_k\left(\int_{\lambda_k}^{\lambda_{k+1}}\int_{\lambda_k}^{\lambda_{k+1}}\left(\ln^2(1+\frac{3t|u-\lambda|}{4})\right)^m dF_k(\lambda)dF_k(u)\right)^{\frac{1}{2m}}\le2b_k\ln\left(1+\frac{3t(\lambda_{k+1}-\lambda_k)}{4}\right)$$

Similarly

$$\tau\left(\int_{\lambda_k}^{\lambda_{k+1}}(\sin(\lambda t)-\sin(\zeta_k t))d\eta_2(\lambda)\right)\le2b_k\ln\left(1+\frac{3t(\lambda_{k+1}-\lambda_k)}{4}\right).$$

Finaly we have

$$\tau^2(\eta_\Lambda(t)) \le 16\sum_{k=0}^{M-1} b_k^2 \ln^2\left(1+\frac{3t(\lambda_{k+1}-\lambda_k)}{4}\right) =$$
$$=16\ln^2\left(1+\frac{3t\Lambda}{4M}\right)\sum_{k=0}^{M-1}\left(F(\lambda_{k+1})-F(\lambda_k)\right) = 16\ln^2\left(1+\frac{3t\Lambda}{4M}\right)F(\Lambda).$$

Lemma 4.3 For thesub-Gaussian process $X^\Lambda(t)$ *defined by* (4), the next inequality is satisfied

$$\tau(X^\Lambda(t)) \le 2\sqrt{F(+\infty)-F(\Lambda)}.$$

*Proof:*As in the previous lemma we estimate each term of the sub-Gaussian standard separately.

$$\tau\left(\int_\Lambda^\infty \cos(\lambda t)d\eta_1(\lambda)\right) \le \Theta_1\left(\int_\Lambda^\infty \cos(\lambda t)d\eta_1(\lambda)\right) = \sup_{m\ge 1}\left[\frac{2^m m!}{(2m)!}E\left(\int_\Lambda^\infty \cos(\lambda t)d\eta_1(\lambda)\right)^{2m}\right]^{\frac{1}{2m}} \le$$
$$\le \sup_{m\ge 1}\left[E\left(\int_\Lambda^\infty \cos^2(\lambda t)dF(\lambda)\right)^m\right]^{\frac{1}{2m}} \le \left(\int_\Lambda^\infty \cos^2(\lambda t)dF(\lambda)\right)^{1/2} \le \sqrt{F(+\infty)-F(\Lambda)}.$$

Similarly

$$\tau\left(\int_\Lambda^\infty \sin(\lambda t)d\eta_2(\lambda)\right) \le \Theta_1\left(\int_\Lambda^\infty \sin(\lambda t)d\eta_2(\lambda)\right) \le \sqrt{F(+\infty)-F(\Lambda)}.$$

Finaly we have

$$\tau\left(X^\Lambda(t)\right) = \tau\left(\int_\Lambda^\infty \cos(\lambda t)d\eta_1(\lambda) + \int_\Lambda^\infty \sin(\lambda t)d\eta_2(\lambda)\right) \le 2\sqrt{F(+\infty)-F(\Lambda)}$$

Denote by $\tau_1 = \sup_{t\in[0,T]} \tau(X_\Lambda(t) - X_\Lambda^M(t))$; $\tau_2 = \sup_{t\in[0,T]} \tau(X^\Lambda(t))$.

Theorem 4.4 Let in the model $X_\Lambda^M(t)$ the partition Λ is such that for

$$\delta \ge max\left\{\frac{\tau_1}{\alpha} max(T,1)\sqrt{2+(U^{(-1)}(1))^{-2}}; \frac{\tau_2}{1-\alpha} max(T,1)\sqrt{2+(U^{(-1)}(1))^{-2}}\right\}$$

the system of inequalities

$$\begin{cases} \tau_1 \le \dfrac{\alpha\delta U^{(-1)}(1)}{max(T,1)\cdot x_\beta}, \\ \tau_2 \le \dfrac{(1-\alpha)\delta U^{(-1)}(1)}{max(T,1)\cdot x_\beta}, \end{cases}$$

holds true, where $x_\beta > 1$ is a root of the equation $xe^{(x^2-1)/2} = \beta$; $0<\alpha<1$.Then the model $X_\Lambda^M(t)$ approximates the process $X(t)$ with reliability $(1-\beta)$, $0<\beta<1$ and accuracy $\delta>0$ in the Orlich space $L_U([0,T])$, where C-function $U(x)$ satisfies conditions of Theorem 2.1.

Proof: This statement follows from Theorem 2.1 and lemms 4.2 and 4.3. Indeed

$$P\left\{\left\|X(t)-X_\Lambda^M(t)\right\|_{L_U}\geq\delta\right\}=P\left\{\left\|X_\Lambda(t)-X_\Lambda^M(t)+X^\Lambda(t)\right\|_{L_U}\geq\delta\right\}\leq$$
$$\leq P\left\{\left\|X_\Lambda(t)-X_\Lambda^M(t)\right\|_{L_U}>\alpha\delta\right\}+P\left\{\left\|X^\Lambda(t)\right\|_{L_U}>(1-\alpha)\delta\right\}.$$

Let $\mathrm{T}=[0,T]$. Consider each term of the last inequality individually.

a) Under the condition $\alpha\delta\geq max(T,1)\cdot\tau_1\cdot\sqrt{2+\left(U^{(-1)}(1)\right)^{-2}}$ we have

$$P\left\{\left\|X_\Lambda(t)-X_\Lambda^M(t)\right\|_{L_U}>\alpha\delta\right\}\leq\sqrt{e}\,\frac{\alpha\delta U^{(-1)}(1)}{max(T,1)\cdot\tau_1}e^{-\frac{\alpha^2\delta^2(U^{(-1)}(1))^2}{2(max(T,1))^2\cdot\tau_1^2}},$$

where, according to Lemma 4.2: $\tau_1=4\ln\left(1+\frac{3T\Lambda}{4M}\right)\sqrt{F(\Lambda)}.$

b) Under the condition $(1-\alpha)\delta\geq max(T,1)\cdot\tau_2\cdot\sqrt{2+\left(U^{(-1)}(1)\right)^{-2}}$ we have

$$P\left\{\left\|X^\Lambda(t)\right\|_{L_U}>(1-\alpha)\delta\right\}\leq\sqrt{e}\,\frac{(1-\alpha)\delta U^{(-1)}(1)}{max(T,1)\cdot\tau_2}e^{-\frac{(1-\alpha)^2\delta^2(U^{(-1)}(1))^2}{2(max(T,1))^2\cdot\tau_2^2}},$$

where, according to Lemma 4.3: $\tau_2=2\sqrt{F(+\infty)-F(\Lambda)}.$
Thus, by Theorem 2.1, under the condition

$$\delta\geq max\left\{\frac{\tau_1}{\alpha}max(T,1)\sqrt{2+(U^{(-1)}(1))^{-2}};\frac{\tau_2}{1-\alpha}max(T,1)\sqrt{2+(U^{(-1)}(1))^{-2}}\right\}$$

we have

$$P\left\{\left\|X_\Lambda(t)-X_\Lambda^M(t)+X^\Lambda(t)\right\|_{L_U}\geq\delta\right\}\leq$$
$$\leq\sqrt{e}\,\frac{\alpha\delta U^{(-1)}(1)}{max(T,1)\cdot\tau_1}e^{-\frac{\alpha^2\delta^2(U^{(-1)}(1))^2}{2(max(T,1))^2\cdot\tau_1^2}}+\sqrt{e}\,\frac{(1-\alpha)\delta U^{(-1)}(1)}{max(T,1)\cdot\tau_2}e^{-\frac{(1-\alpha)^2\delta^2(U^{(-1)}(1))^2}{2(max(T,1))^2\cdot\tau_2^2}}\leq\beta \quad (8)$$

if inequalities $\frac{\alpha\delta U^{(-1)}(1)}{max(T,1)\cdot\tau_1}>x_\beta$ and $\frac{(1-\alpha)\delta U^{(-1)}(1)}{max(T,1)\cdot\tau_2}>x_\beta$ are satisfied.

The proof is completed.

Example 4.1 Consider a particular case of the Orlicz space of random variables which is generated by C-function $U(x)=e^{x^2}-1$. In this case we have $U^{(-1)}(1)=\sqrt{\ln 2}.$
Let $T=1,\alpha=0.5, F(\Lambda)=1-e^{-\Lambda},\ \delta=0.1, \beta=0.1$

It follows from Theorem 4.4 that in this case $x_\beta = 2.768$, $\tau_1 \le 0.015$ and $\tau_2 \le 0.015$. Substitutingthese values to the corresponding formulas, we get that $M > \frac{3\Lambda}{4}\left(exp(\frac{0.015}{4\sqrt{1-e^{-\Lambda}}}) - 1 \right)^{-1}$ for $\Lambda > 9.8$. In particularly, we get $M = 197$ for $\Lambda = 10$.

Substituting to the model (5) this number M and simulating random variables η_{k1}, η_{k2}, ζ_k for $k = \overline{0, M-1}$, it is possible to get a graphical representation of the considered process.

5. Conclusion

In this article we propose a method of construction of models of Gaussian stationary random processes in some Orlicz spaces with given accuracy and reliability.

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