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Research Article

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Reliability and accuracy in the space $L_p(T)$ for the calculation of integrals depending on a parameter by the Monte Carlo method

Abstract: This paper is devoted to the estimation of the accuracy and reliability (in $L_p(T)$ metrics) for the calculation of improper integrals depending on a parameter t using the Monte Carlo method. For this estimates we use the theory of $\mathbf{F}_\psi(\Omega)$ spaces.

Keywords: $\mathbf{F}_\psi(\Omega)$ space of random variables, condition $\mathbf{H}$, Monte Carlo method, random process

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1 Introduction

This paper is a continuation of the investigation started in the paper by Kozachenko and Mlavets [6]. In that paper we developed a theory for finding reliability and accuracy for the calculation of integrals depending on a parameter by the Monte Carlo method in the uniform metrics. Unlike the previous paper, where the theory of Orlicz space of random variables had been used, here we utilized the theory of $\mathbf{F}_\psi(\Omega)$ spaces.

The choice of the space depends on particular integrals and allows one to find better accuracy. In this paper, the accuracy is defined via the norm in the space $L_p(T)$. It is worth to note that by considering the spaces $\mathbf{F}_\psi(\Omega)$ it is also possible to find the accuracy and reliability in $C(T)$ space. But this is the subject of future work.

There are many works devoted to the usage of the Monte Carlo method for calculation of integrals. Among them are books by Yermakov [3] and Yermakov and Mikhailov [4].

But there are not so many works studying reliability and accuracy of the calculation of integrals via Monte Carlo methods, especially, when the integral depends on a parameter. In the papers by Dmitrovskii and Ostrovskii [2], Dmitrovskii [1], Voitishek and Prigarin [13], Voitishek [12], the conditions of weak convergence to the integral value had been investigated and for the large n the accuracy and reliability were determined if the estimates were Gaussian processes.

The paper by Kurbanmuradov and Sabelfeld [9] contains the estimate for the accuracy in the space $C(T)$ and the reliability of the calculation of integrals depending on a parameter if the set of integration is bounded. To obtain these results the theory of sub-Gaussian processes had been used.

The space $\mathbf{F}_\psi(\Omega)$ was introduced by Yermakov and Ostrovsky in the paper [5]. The paper [7] is devoted to studying the properties of such spaces and there had been found the conditions of fulfilling the condition $\mathbf{H}$ in this spaces. The condition $\mathbf{H}$ is necessary for finding the reliability and accuracy when we calculate integrals by Monte Carlo methods.

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The paper is organized as follows. Section 2 contains the needed definitions and results from the theory of spaces $\mathbf{F}_\psi(\Omega)$. In Section 3, we find the estimates for distribution of norms in $L_p(T)$ of the stochastic processes from the spaces $\mathbf{F}_\psi(\Omega)$. The reliability and accuracy for the calculation of integrals by the Monte Carlo method are found in Section 4. Section 5 is devoted to finding the reliability and accuracy in the space $L_p(T)$ for the calculation of integrals depending on a parameter.

2 $\mathbf{F}_\psi(\Omega)$ space

Definition 2.1 ([8]). Let $\psi(u) > 0$, $u \geq 1$ be a monotonically increasing and continuous function for which $\psi(u) \to \infty$ as $u \to \infty$. A random variable ξ belongs to the space $\mathbf{F}_\psi(\Omega)$ if

$$\sup_{u\geq 1} \frac{(E|\xi|^u)^{1/u}}{\psi(u)} < \infty.$$

A similar definition was formulated in the paper by Yermakov and Ostrovskii [5]. But there it was required that $E\xi = 0$ as $\xi \in \mathbf{F}_\psi(\Omega)$. Moreover, random variables were considered for which $E|\xi|^u = \infty$ for some $u > 0$. It had been proved in [5] that $\mathbf{F}_\psi(\Omega)$ is a Banach space with the norm

$$\|\xi\|_\psi = \sup_{u\geq 1} \frac{(E|\xi|^u)^{1/u}}{\psi(u)}.$$

Let us provide some examples of random variables from the space $\mathbf{F}_\psi(\Omega)$.

Example 2.2. The random variable ξ satisfying the condition $|\xi| < C$ with probability one, where $C > 0$ is some constant, belongs to every space $\mathbf{F}_\psi(\Omega)$. Herewith

$$\|\xi\|_\psi = \sup_{u\geq 1} \frac{(E|\xi|^u)^{1/u}}{\psi(u)} \leq \sup_{u\geq 1} \frac{(C^u)^{1/u}}{\psi(u)} = \sup_{u\geq 1} \frac{C}{\psi(u)} = \frac{C}{\psi(1)}.$$

Example 2.3. The random variable with Laplace distribution (its density function is $p(x) = \frac{1}{2}e^{-|x|}$) belongs to the space $\mathbf{F}_\psi(\Omega)$, where $\psi(u) = u$. This follows from the equivalence $\sqrt[k]{E|\xi|^k} = \sqrt[k]{k!} \sim k$ as $k \geq 1$.

Example 2.4. The normally distributed random variable $\xi = N(0, 1)$ belongs to the space $\mathbf{F}_\psi(\Omega)$, where $\psi(u) = u^{1/2}$ since $\sqrt[2l]{E|\xi|^{2l}} = \sqrt[2l]{(2l)!/(2^l l!)} \sim l^{1/2}$ as $l \geq 1$.

Theorem 2.5 ([8]). *If a random variable ξ belongs to the space $\mathbf{F}_\psi(\Omega)$, then for any $\varepsilon > 0$, the following inequality holds true:*

$$P\{|\xi| > \varepsilon\} \leq \inf_{u\geq 1} \frac{\|\xi\|_\psi^u (\psi(u))^u}{\varepsilon^u}.$$

Theorem 2.6 ([8]). *If a random variable ξ belongs to the space $\mathbf{F}_\psi(\Omega)$ and $\psi(u) = u^\alpha$, where $\alpha > 0$, then for any $\varepsilon \geq e^\alpha \|\xi\|_\psi$, the following inequality is true:*

$$P\{|\xi| > \varepsilon\} \leq \exp\left\{-\frac{\alpha}{e}\left(\frac{\varepsilon}{\|\xi\|_\psi}\right)^{1/\alpha}\right\}.$$

Theorem 2.7 ([8]). *If a random variable ξ belongs to the space $\mathbf{F}_\psi(\Omega)$ and $\psi(u) = e^{au^\beta}$, where $a > 0$, $\beta > 0$, then for any $\varepsilon \geq e^{a(\beta+1)}\|\xi\|_\psi$, the following is true:*

$$P\{|\xi| > \varepsilon\} \leq \exp\left\{-\frac{\beta}{a^{1/\beta}}\left(\frac{1}{\beta+1}\ln\frac{\varepsilon}{\|\xi\|_\psi}\right)^{(\beta+1)/\beta}\right\}.$$

Theorem 2.8 ([8]). *Let the random variable ξ belong to the space $\mathbf{F}_\psi(\Omega)$ where $\psi(u) = (\ln(u+1))^\lambda$, $\lambda > 0$. Then, for any $\varepsilon \geq (e\ln 2)^\lambda \|\xi\|_\psi$, the following inequality holds true:*

$$P\{|\xi| > \varepsilon\} \leq e^\lambda \exp\left\{-\lambda \exp\left\{\left(\frac{\varepsilon}{\|\xi\|_\psi}\right)^{1/\lambda}\frac{1}{e}\right\}\right\}.$$

Definition 2.9 ([7]). We say that the condition **H**, for the Banach spaces $B(\Omega)$ of random variables, is fulfilled if there exists an absolute constant C_B such that for any centered and independent random variables $\xi_1, \xi_2, \ldots, \xi_n$ from $B(\Omega)$, the following is true:

$$\left\|\sum_{i=1}^{n} \xi_i\right\|^2 \leq C_B \sum_{i=1}^{n} \|\xi_i\|^2.$$

The constant C_B is called a scale constant for the space $B(\Omega)$. For the space $\mathbf{F}_\psi(\Omega)$, we shall denote the constants $C_{\mathbf{F}_\psi(\Omega)}$ as C_ψ.

Theorem 2.10 ([11]). *For the space $\mathbf{F}_\psi(\Omega)$, where $\psi(u) = u^\alpha$, $\alpha \geq 1/2$, the condition **H** is fulfilled and the following inequality is true:*

$$\left\|\sum_{i=1}^{n} \xi_i\right\|_\psi^2 \leq 4 \cdot 9^\alpha \sum_{i=1}^{n} \|\xi_i\|_\psi^2.$$

*Note, that when $\alpha < 1/2$, then the condition **H** is not fulfilled for this space.*

Theorem 2.11 ([10]). *Let $\mathbf{F}_\psi(\Omega)$ be the space defined by the function $\psi(u) = e^{au^\beta}$, where $a > 0$, $0 < \beta < 1$. If $1/(2a\beta)^{1/\beta} = 1$, then the condition **H** is fulfilled for the space $\mathbf{F}_\psi(\Omega)$ with the constant $C_\psi = 4e^{2^\beta a}$. And if $1/(2a\beta)^{1/\beta} > 1$, then for $\mathbf{F}_\psi(\Omega)$ the condition **H** is true with the constant*

$$C_\psi = \frac{4e^{a(2^\beta+1)-1/2\beta}}{(2a\beta)^{1/2\beta}}.$$

Lemma 2.12 ([8]). *Let $\xi \in \mathbf{F}_\psi(\Omega)$, $p \geq 1$. Then,*

$$\|E|\xi|\|_\psi \leq \|\xi\|_\psi.$$

3 Estimates in the norm $L_p(T)$ for the stochastic processes from the spaces $\mathbf{F}_\psi(\Omega)$

Let us present a lemma that will be used further on.

Lemma 3.1. *Let ξ be a random variable belonging to the space $\mathbf{F}_\psi(\Omega)$. Then, for $p \geq 1$,*

$$\|\xi\|_\psi \leq \frac{\psi(p)}{\psi(1)} \sup_{u \geq p} \frac{(E|\xi|^u)^{1/u}}{\psi(u)}.$$

Proof. It follows from the Lyapunov inequality that

$$\sup_{1 \leq u \leq p} \frac{(E|\xi|^u)^{1/u}}{\psi(u)} \leq \frac{\psi(p)}{\psi(1)} \frac{(E|\xi|^p)^{1/p}}{\psi(p)}.$$

So,

$$\begin{aligned}\|\xi\|_\psi &= \max\left(\sup_{1 \leq u \leq p} \frac{(E|\xi|^u)^{1/u}}{\psi(u)}, \sup_{u \geq p} \frac{(E|\xi|^u)^{1/u}}{\psi(u)}\right) \\ &\leq \max\left(\frac{\psi(p)}{\psi(1)} \frac{(E|\xi|^p)^{1/p}}{\psi(p)}, \sup_{u \geq p} \frac{(E|\xi|^u)^{1/u}}{\psi(u)}\right) \leq \frac{\psi(p)}{\psi(1)} \sup_{u \geq p} \frac{(E|\xi|^u)^{1/u}}{\psi(u)}.\end{aligned}$$

The last inequality implies the statement needed. □

Theorem 3.2. *Let ν be the σ-finite measure on the compact metric space (T, ρ) and $X = \{X(t), t \in T\}$ be a measurable stochastic process from the space $\mathbf{F}_\psi(\Omega)$. If for some $p \geq 1$ the following condition is true:*

$$\int_T \|X(t)\|_\psi^p \, d\nu(t) < \infty,$$

then:

(1) *the integral* $\int_T |X(t)|^p \, d\nu(t)$ *exists with probability one and the following inequality holds true:*

$$\left\| \left(\int_T |X(t)|^p \, d\nu(t) \right)^{1/p} \right\|_\psi \leq \frac{\psi(p)}{\psi(1)} \left(\int_T \|X(t)\|_\psi^p \, d\nu(t) \right)^{1/p}. \tag{3.1}$$

(2) *For any* $\varepsilon > 0$, *the following inequality holds:*

$$P\left\{ \left(\int_T |X(t)|^p \, d\nu(t) \right)^{1/p} > \varepsilon \right\} \leq \inf_{u \geq 1} \frac{1}{\varepsilon^u} \left(\frac{\psi(p)}{\psi(1)} \right)^u \left(\int_T \|X(t)\|_\psi^p \, d\nu(t) \right)^{u/p} (\psi(u))^u. \tag{3.2}$$

Proof. Since

$$E \int_T |X(t)|^p \, d\nu(t) = \int_T E|X(t)|^p \, d\nu(t) \leq \int_T (\psi(p))^p \|X(t)\|_\psi^p \, d\nu(t) < \infty,$$

it follows that $\int_T |X(t)|^p \, d\nu(t)$ exists with probability one. It follows from the generalized Minkowski inequality that for $u \geq p$,

$$\begin{aligned} E\left(\int_T |X(t)|^p \, d\nu(t) \right)^{u/p} &= \left(\left(E\left(\int_T |X(t)|^p \, d\nu(t) \right)^{u/p} \right)^{p/u} \right)^{u/p} \\ &\leq \left(\int_T (E|X(t)|^u)^{p/u} \, d\nu(t) \right)^{u/p} \leq \left(\int_T \|X(t)\|_\psi^p (\psi(u))^p \, d\nu(t) \right)^{u/p} \\ &\leq (\psi(u))^u \left(\int_T \|X(t)\|_\psi^p \, d\nu(t) \right)^{u/p}. \end{aligned} \tag{3.3}$$

From Lemma 3.1 and inequality (3.3), we obtain

$$\begin{aligned} \left\| \left(\int_T |X(t)|^p \, d\nu(t) \right)^{1/p} \right\|_\psi &\leq \frac{\psi(p)}{\psi(1)} \sup_{u \geq p} \frac{(E|\int_T |X(t)|^p \, d\nu(t)|^{u/p})^{1/u}}{\psi(u)} \\ &\leq \frac{\psi(p)}{\psi(1)} \sup_{u \geq p} \frac{\psi(u)(\int_T \|X(t)\|_\psi^p \, d\nu(t))^{1/p}}{\psi(u)} = \frac{\psi(p)}{\psi(1)} \left(\int_T \|X(t)\|_\psi^p \, d\nu(t) \right)^{1/p}. \end{aligned}$$

Inequality (3.1) has been proved and inequality (3.2) follows from Theorem 2.5. □

Example 3.3. Consider the space $\mathbf{F}_\psi(\Omega)$, where $\psi(u) = u^\alpha$, $\alpha > 0$. It follows from Theorems 3.2 and 2.6 that for $\varepsilon \geq (ep)^\alpha (\int_T \|X(t)\|_\psi^p \, d\nu(t))^{1/p}$,

$$P\left\{ \left(\int_T |X(t)|^p \, d\nu(t) \right)^{1/p} > \varepsilon \right\} \leq \exp\left\{ -\frac{\alpha}{ep} \left(\frac{\varepsilon}{(\int_T \|X(t)\|_\psi^p \, d\nu(t))^{1/p}} \right)^{1/\alpha} \right\}.$$

Example 3.4. Consider the space $\mathbf{F}_\psi(\Omega)$, where $\psi(u) = e^{au^\beta}$, $a > 0$, $\beta > 0$. It follows from Theorems 3.2 and 2.7 that if

$$\varepsilon \geq e^{a(\beta + p^\beta)} \left(\int_T \|X(t)\|_\psi^p \, d\nu(t) \right)^{1/p},$$

then

$$P\left\{ \left(\int_T |X(t)|^p \, d\nu(t) \right)^{1/p} > \varepsilon \right\} \leq \exp\left\{ -\frac{\beta}{a^{1/\beta}} \left(\frac{1}{\beta + 1} \ln \frac{\varepsilon}{e^{a(\beta + p^\beta)} (\int_T \|X(t)\|_\psi^p \, d\nu(t))^{1/p}} \right)^{(\beta+1)/\beta} \right\}.$$

Example 3.5. Consider the space $\mathbf{F}_\psi(\Omega)$, where $\psi(u) = (\ln(u+1))^\lambda$, $\lambda > 0$. According to Theorems 3.2 and 2.8, for $\varepsilon \geq (\log_2(p+1))^\lambda (\int_T \|X(t)\|_\psi^p \, d\nu(t))^{1/p}$, we can affirm that

$$P\left\{ \left(\int_T |X(t)|^p \, d\nu(t) \right)^{1/p} > \varepsilon \right\} \leq e^\lambda \exp\left\{ -\lambda \exp\left\{ \frac{\ln 2}{e \ln(p+1)} \left(\frac{\varepsilon}{(\int_T \|X(t)\|_\psi^p \, d\nu(t))^{1/p}} \right)^{1/\lambda} \right\} \right\}.$$

Theorem 3.6. *Let ν be a σ-finite measure on a compact metric (T,ρ) and $Y = \{Y(t), t \in T\}$ be the stochastic process from the space $\mathbf{F}_\psi(\Omega)$ and assume that the condition* **H** *is fulfilled for this space with the constant C_ψ. Let*

$$EY(t) = m(t) \quad \text{and} \quad Z_n(t) = \frac{1}{n}\sum_{k=1}^{n}(Y_k(t) - m(t)),$$

where $Y_k(t)$ are independent copies of $Y(t)$. Then, the following inequality holds for all $p \geq 1$:

$$\left\| \left(\int_T |Z_n(t)|^p \, d\nu(t) \right)^{1/p} \right\|_\psi \leq \frac{2\sqrt{C_\psi}}{\sqrt{n}} \cdot \frac{\psi(p)}{\psi(1)} \left(\int_T \|Y(t)\|_\psi^p \, d\nu(t) \right)^{1/p}, \tag{3.4}$$

and for every $\varepsilon > 0$, the following estimate is true:

$$P\left\{ \left(\int_T |Z_n(t)|^p \, d\nu(t) \right)^{1/p} > \varepsilon \right\} \leq \inf_{u\geq 1} \frac{1}{\varepsilon^u} \left(\frac{2\sqrt{C_\psi}}{\sqrt{n}} \cdot \frac{\psi(p)}{\psi(1)} \right)^u \left(\int_T \|Y(t)\|_\psi^p \, d\nu(t) \right)^{u/p} (\psi(u))^u. \tag{3.5}$$

Proof. It follows from Definition 2.9 and Lemma 2.12 that

$$\|Z_n(t)\|_\psi^2 \leq \frac{1}{n^2} C_\psi \sum_{k=1}^{n} \|Y_k(t) - m(t)\|_\psi^2 = \frac{1}{n} C_\psi \|Y(t) - m(t)\|_\psi^2 \leq \frac{1}{n} C_\psi (\|Y(t)\|_\psi + \|m(t)\|_\psi)^2 \leq \frac{4}{n} C_\psi \|Y(t)\|_\psi^2.$$

Theorem 3.2 implies the interrelations

$$\begin{aligned} \left\| \left(\int_T |Z_n(t)|^p \, d\nu(t) \right)^{1/p} \right\|_\psi &\leq \frac{\psi(p)}{\psi(1)} \left(\int_T \|Z_n(t)\|_\psi^p \, d\nu(t) \right)^{1/p} \\ &\leq \frac{\psi(p)}{\psi(1)} \left(\int_T \left(\frac{2\sqrt{C_\psi}}{\sqrt{n}} \|Y(t)\|_\psi \right)^p d\nu(t) \right)^{1/p} = \frac{2\sqrt{C_\psi}}{\sqrt{n}} \cdot \frac{\psi(p)}{\psi(1)} \left(\int_T \|Y(t)\|_\psi^p \, d\nu(t) \right)^{1/p}. \end{aligned}$$

Therefore, inequality (3.4) holds true and inequality (3.5) follows from Theorem 2.5. □

Example 3.7. Let us consider the space $\mathbf{F}_\psi(\Omega)$, where $\psi(u) = u^\alpha$, $\alpha > 0$. Then, it follows from Theorems 3.6 and 2.6 that if

$$\varepsilon \geq (ep)^\alpha \frac{2\sqrt{C_\psi}}{\sqrt{n}} \left(\int_T \|Y(t)\|_\psi^p \, d\nu(t) \right)^{1/p},$$

then

$$P\left\{ \left(\int_T |Z_n(t)|^p \, d\nu(t) \right)^{1/p} > \varepsilon \right\} \leq \exp\left\{ -\frac{\alpha}{ep} \left(\frac{\varepsilon}{\frac{2\sqrt{C_\psi}}{\sqrt{n}} (\int_T \|Y(t)\|_\psi^p \, d\nu(t))^{1/p}} \right)^{1/\alpha} \right\}.$$

Example 3.8. Consider the space $\mathbf{F}_\psi(\Omega)$, where $\psi(u) = e^{au^\beta}$, $a > 0$, $\beta > 0$, then according to Theorems 3.6 and 2.7, for

$$\varepsilon \geq e^{a(\beta + p^\beta)} \frac{2\sqrt{C_\psi}}{\sqrt{n}} \left(\int_T \|Y(t)\|_\psi^p \, d\nu(t) \right)^{1/p},$$

we can conclude that

$$P\left\{ \left(\int_T |Z_n(t)|^p \, d\nu(t) \right)^{1/p} > \varepsilon \right\} \leq \exp\left\{ -\frac{\beta}{a^{1/\beta}} \left(\frac{1}{\beta+1} \ln \frac{\varepsilon}{\frac{2e^{a(\beta+p^\beta)}\sqrt{C_\psi}}{\sqrt{n}} (\int_T \|Y(t)\|_\psi^p \, d\nu(t))^{1/p}} \right)^{(\beta+1)/\beta} \right\}.$$

Example 3.9. Consider the space $\mathbf{F}_\psi(\Omega)$, where $\psi(u) = (\ln(u+1))^\lambda$, $\lambda > 0$, then according to Theorems 3.6 and 2.8, for $\varepsilon \geq (e\ln(p+1))^\lambda 2{C_\psi}^{1/2} n^{-1/2} (\int_T \|Y(t)\|^p \, d\nu(t))^{1/p}$, we get the estimate

$$P\left\{ \left(\int_T |Z_n(t)|^p \, d\nu(t) \right)^{1/p} > \varepsilon \right\} \leq e^\lambda \exp\left\{ -\lambda \exp\left\{ \frac{\ln 2}{e\ln(p+1)} \left(\frac{\varepsilon}{\frac{2\sqrt{C_\psi}}{\sqrt{n}} (\int_T \|Y(t)\|^p \, d\nu(t))^{1/p}} \right)^{1/\lambda} \right\} \right\}.$$

4 Accuracy and reliability for the calculation of integrals by the Monte Carlo method

Let $\{\mathcal{S}, \mathcal{A}, \mu\}$ be a measurable space, μ be a σ-finite measure and $p(s) \geq 0$, $s \in \mathcal{S}$, be a measurable function such that $\int_{\mathcal{S}} p(s)\, d\mu(s) = 1$. Let $P(A)$, $A \in \mathcal{A}$ be the measure $P(A) = \int_A p(s)\, d\mu(s)$. The measure $P(A)$ is a probability measure and the space $\{\mathcal{S}, \mathcal{A}, P\}$ is a probability space.

Let $f(s)$ be a measurable function on $\{\mathcal{S}, \mathcal{A}, \mu\}$. Consider $\int_{\mathcal{S}} f(s)p(s)\, d\mu(s) = I$. Suppose, that this integral exists.

Remark 4.1. We can consider the integral of the form $\int_{\mathcal{S}} \varphi(s)\, d\mu(s)$. If $p(s) > 0$ is a probability density function in the space $\{\mathcal{S}, \mathcal{A}, \mu\}$, then

$$\int\limits_{\mathcal{S}} \varphi(s)\, d\mu(s) = \int\limits_{\mathcal{S}} \frac{\varphi(s)}{p(s)} p(s)\, d\mu(s) = \int\limits_{\mathcal{S}} f(s)p(s)\, d\mu(s),$$

where $f(s) = \varphi(s)/p(s)$.

We can consider $f(s) = \xi$ as random variables on $\{\mathcal{S}, \mathcal{A}, P\}$ and $\int_{\mathcal{S}} f(s)p(s)\, d\mu(s) = \int_{\mathcal{S}} f(s)\, dm(s) = E\xi$.

Let ξ_i, $i = 1, \ldots, n$, be the independent copies of the random variable ξ and $Z_n = \frac{1}{n}\sum_{i=1}^n \xi_i$. Then, according to the strong law of large numbers $Z_n \to E\xi_1 = I$ with probability one. We consider Z_n as an estimate for I.

Definition 4.2. We state that Z_n approximates I with reliability $1 - \delta$ $(0 < \delta < 1)$ and accuracy $\varepsilon > 0$ if the following inequality holds:

$$P\{|Z_n - I| > \varepsilon\} \leq \delta. \tag{4.1}$$

Theorem 4.3. *Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent and identically distributed random variables from the space $\mathbf{F}_\psi(\Omega)$ which fulfills the condition $\mathbf{H}$. Let $Y_n = \frac{1}{\sqrt{n}}\sum_{i=1}^n(\xi_i - I)$, where $I = E\xi_1$. Then, for any $\varepsilon > 0$, the following inequality holds true:*

$$P\{|Y_n| > \varepsilon\} \leq \inf_{u\geq 1} \frac{L^u(\psi(u))^u}{\varepsilon^u}, \tag{4.2}$$

where $L = \|\xi_1 - I\|_\psi \sqrt{C_\psi}$ and C_ψ is the constant from Definition 2.9.

Proof. It follows from Definition 2.9 that

$$\|Y_n\|_\psi^2 = \left\|\frac{1}{\sqrt{n}} \sum_{i=1}^n (\xi_i - I)\right\|_\psi^2 = \frac{1}{n}\left\|\sum_{i=1}^n (\xi_i - I)\right\|_\psi^2 \leq \frac{1}{n} C_\psi \sum_{i=1}^n \|\xi_i - I\|_\psi^2 = C_\psi \|\xi_1 - I\|_\psi^2.$$

Inequality (4.2) follows from Theorem 2.5. □

Corollary 4.4. *Assume that the conditions of Theorem 4.3 are true. Then, for any $\varepsilon > 0$, the following inequality holds:*

$$P\left\{\left|\frac{1}{n}\sum_{i=1}^n \xi_i - I\right| > \varepsilon\right\} \leq \inf_{u\geq 1} \frac{L^u(\psi(u))^u}{(\sqrt{n}\varepsilon)^u}.$$

Proof. The proof of Corollary 4.4 is similar to the proof of [6, Corollary 3.2]. □

Remark 4.5. It is evident that

$$\|\xi_1 - I\|_\psi \leq 2\|\xi_1\|_\psi.$$

Indeed, from Lemma 2.12, it follows that

$$\|\xi_1 - E\xi_1\|_\psi \leq \|\xi_1\|_\psi + \|E\xi_1\|_\psi \leq 2\|\xi_1\|_\psi.$$

Corollary 4.6. *Let all the conditions of Theorem 4.3 hold. Then, for any $\varepsilon > 0$, the following inequality is true:*

$$P\left\{\left|\frac{1}{n}\sum_{i=1}^n \xi_i - I\right| > \varepsilon\right\} \leq \inf_{u\geq 1} \frac{2^u\widetilde{L}^u(\psi(u))^u}{(\sqrt{n}\varepsilon)^u},$$

where $\widetilde{L} = \|\xi_1\|_\psi \sqrt{C_\psi}$.

Proof. Corollary 4.6 follows from Corollary 4.4 and Remark 4.5. □

Example 4.7. Consider the space $\mathbf{F}_\psi(\Omega)$, where $\psi(u) = u^\alpha$, $\alpha > 1/2$. Then, taking into account Example 2.10, we get that for this space the condition **H** is fulfilled with the constant $C_\psi = 4 \cdot 9^\alpha$. Then, Corollary 4.6 and Theorem 2.6 imply that if $\varepsilon \geq 4(3e)^\alpha \|\xi_1\|_\psi/\sqrt{n}$, then

$$P\left\{\left|\frac{1}{n}\sum_{i=1}^{n}\xi_i - I\right| > \varepsilon\right\} \leq \exp\left\{-\frac{\alpha}{3e}\left(\frac{\sqrt{n}\varepsilon}{4\|\xi_1\|_\psi}\right)^{1/\alpha}\right\}.$$

Example 4.8. Consider the space $\mathbf{F}_\psi(\Omega)$, where $\psi(u) = e^{au^\beta}$, $a > 0$, $0 < \beta < 1$. According to Theorem 2.11 we have two choices. In the first case when $1/(2a\beta)^{1/\beta} = 1$ the condition **H** is fulfilled for the space $\mathbf{F}_\psi(\Omega)$ with the constant $C_\psi = 4e^{2^\beta a}$. Then, Corollary 4.6 and Theorem 2.7 imply

$$P\left\{\left|\frac{1}{n}\sum_{i=1}^{n}\xi_i - I\right| > \varepsilon\right\} \leq \exp\left\{-\frac{\beta}{a^{1/\beta}}\left(\frac{1}{\beta+1}\ln\frac{\sqrt{n}\varepsilon}{4e^{2^{\beta-1}a}\|\xi_1\|_\psi}\right)^{(\beta+1)/\beta}\right\},$$

for $\varepsilon \geq 4e^{a(2^{\beta-1}+\beta+1)}\|\xi_1\|_\psi/\sqrt{n}$.

For the second case $1/(2a\beta)^{1/\beta} > 1$, the condition **H** is fulfilled for the space $\mathbf{F}_\psi(\Omega)$ with the constant $C_\psi = 4e^{a(2^\beta+1)-1/2\beta}/(2a\beta)^{1/2\beta}$. Then, it follows from Corollary 4.6 and Theorem 2.7 that

$$P\left\{\left|\frac{1}{n}\sum_{i=1}^{n}\xi_i - I\right| > \varepsilon\right\} \leq \exp\left\{-\frac{\beta}{a^{1/\beta}}\left(\frac{1}{\beta+1}\ln\frac{\sqrt{n}\varepsilon(2a\beta)^{1/4\beta}}{4e^{\frac{a}{2}(2^\beta+1)-\frac{1}{4\beta}}\|\xi_1\|_\psi}\right)^{(\beta+1)/\beta}\right\},$$

where

$$\varepsilon \geq \frac{4e^{a(2^{\beta-1}+\beta+3/2)-\frac{1}{4\beta}}\|\xi_1\|_\psi}{\sqrt{n}(2a\beta)^{1/4\beta}}.$$

Theorem 4.9. *Let $I = \int_{\mathcal{S}} f(s)p(s)\,d\mu(s)$, $\xi(s)$ be a random variable, $s \in \{\mathcal{S}, \mathcal{A}, \mu\}$, $p(s)$ be a density for ξ, ξ_i, $i = 1, 2, \dots, n$, be independent copies of the random variable ξ and $Z_n = \frac{1}{n}\sum_{i=1}^{n}\xi_i$. If the random variable ξ belongs to the space $\mathbf{F}_\psi(\Omega)$ satisfying the condition **H** with the constant C_ψ and n is such that*

$$\inf_{u\geq 1}\frac{2^u\tilde{L}^u(\psi(u))^u}{(\sqrt{n}\varepsilon)^u} \leq \delta, \tag{4.3}$$

then Z_n approximates I with reliability $1-\delta$ and accuracy ε ($\tilde{L}$ in estimate (4.3) is determined in Corollary 4.6).

Proof. Theorem follows from Corollary 4.6 and inequality (4.1). □

Example 4.10. Consider the space $\mathbf{F}_\psi(\Omega)$, where $\psi(u) = u^\alpha$, $\alpha \geq 1/2$. Taking into account Example 4.7 and Theorem 4.9, when $\varepsilon \geq 4(3e)^\alpha\|\xi\|_\psi/\sqrt{n}$ we have that

$$\inf_{u\geq 1}\frac{2^u\tilde{L}^u(\psi(u))^u}{(\sqrt{n}\varepsilon)^u} = \exp\left\{-\frac{\alpha}{3e}\left(\frac{\sqrt{n}\varepsilon}{4\|\xi\|_\psi}\right)^{1/\alpha}\right\}.$$

So, inequality (4.3) holds true if

$$\exp\left\{-\frac{\alpha}{3e}\left(\frac{\sqrt{n}\varepsilon}{4\|\xi\|_\psi}\right)^{1/\alpha}\right\} \leq \delta.$$

Then,

$$n \geq \left(\frac{4\|\xi\|_\psi}{\varepsilon}\right)^2\left((-\ln\delta)\frac{3e}{\alpha}\right)^{2\alpha}$$

and

$$n \geq \left(\frac{4(3e)^\alpha\|\xi\|_\psi}{\varepsilon}\right)^2\max\left(1, \left(\frac{-\ln\delta}{\alpha}\right)^{2\alpha}\right).$$

Example 4.11. Let us consider the space $\mathbf{F}_\psi(\Omega)$, where $\psi(u) = e^{au^\beta}$, $a > 0$, $0 < \beta < 1$. Taking into account Example 4.8 and Theorem 4.9, we shall consider two cases. First, if $1/(2a\beta)^{1/\beta} = 1$, then inequality (4.3) holds when the following inequality is true:

$$\exp\left\{-\frac{\beta}{a^{1/\beta}}\left(\frac{1}{\beta+1}\ln\frac{\sqrt{n}\varepsilon}{4e^{2^{\beta-1}a}\|\xi\|_\psi}\right)^{(\beta+1)/\beta}\right\}\le\delta.$$

Then,

$$n\ge\left(\frac{4e^{2^{\beta-1}a}\|\xi\|_\psi}{\varepsilon}\right)^2\exp\left\{2(\beta+1)\left((-\ln\delta)\frac{a^{1/\beta}}{\beta}\right)^{\beta/(\beta+1)}\right\}$$

and

$$n\ge\left(\frac{4e^{2^{\beta-1}a}\|\xi\|_\psi}{\varepsilon}\right)^2\max\left(e^{2(\beta+1)},\exp\left\{2(\beta+1)\left((-\ln\delta)\frac{a^{1/\beta}}{\beta}\right)^{\beta/(\beta+1)}\right\}\right).$$

Second, if $1/(2a\beta)^{1/\beta}>1$, then inequality (4.3) is fulfilled when

$$\exp\left\{-\frac{\beta}{a^{1/\beta}}\left(\frac{1}{\beta+1}\ln\frac{\sqrt{n}\varepsilon(2a\beta)^{1/4\beta}}{4e^{\frac{a}{2}(2^\beta+1)-\frac{1}{4\beta}}\|\xi\|_\psi}\right)^{(\beta+1)/\beta}\right\}\le\delta.$$

Then,

$$n\ge\left(\frac{4e^{\frac{a}{2}(2^\beta+1)-\frac{1}{4\beta}}\|\xi\|_\psi}{\varepsilon(2a\beta)^{1/4\beta}}\right)^2\exp\left\{2(\beta+1)\left((-\ln\delta)\frac{a^{1/\beta}}{\beta}\right)^{\beta/(\beta+1)}\right\}$$

and

$$n\ge\left(\frac{4e^{\frac{a}{2}(2^\beta+1)-\frac{1}{4\beta}}\|\xi\|_\psi}{\varepsilon(2a\beta)^{1/4\beta}}\right)^2\max\left(e^{2a(\beta+1)},\exp\left\{2(\beta+1)\left((-\ln\delta)\frac{a^{1/\beta}}{\beta}\right)^{\beta/(\beta+1)}\right\}\right).$$

Example 4.12. Consider the integral of the following type:

$$\int_0^{+\infty}\int_0^{+\infty}c(x,y)(x+y)^{\nu-1}e^{-px}e^{-qy}\,dx\,dy,$$

where $|c(x,y)|\le1$, $\nu>3/2$. We use the following notation:

$$I=\frac{1}{pq}\int_0^{+\infty}\int_0^{+\infty}c(x,y)(x+y)^{\nu-1}pe^{-px}qe^{-qy}\,dx\,dy.$$

Let ξ and η be independent random variables distributed exponentially

$$P\{\xi<x\}=\begin{cases}1-e^{-px}, & x>0,\\ 0, & x<0,\end{cases}$$

$$P\{\eta<y\}=\begin{cases}1-e^{-qx}, & y>0,\\ 0, & y<0,\end{cases}$$

where $p(x)=pe^{-px}$ and $p(y)=qe^{-qy}$. So,

$$I=\frac{1}{pq}\int_0^{+\infty}\int_0^{+\infty}c(x,y)(x+y)^{\nu-1}pe^{-px}qe^{-qy}\,dx\,dy=\frac{1}{pq}Ec(\xi,\eta)(\xi+\eta)^{\nu-1}.$$

Assume that the function $\psi(u)=u^{\nu-1}$. Then, since $\nu>3/2$, we get

$$\sup_{u\ge1}\frac{(E(c(\xi,\eta)(\xi+\eta)^{\nu-1})^u)^{1/u}}{u^{\nu-1}}\le\sup_{u\ge1}\frac{(E(|c(\xi,\eta)|(\xi+\eta)^{u(\nu-1)})^{\frac{1}{u(\nu-1)}})^{\nu-1}}{u^{\nu-1}}\le\sup_{u\ge1}\frac{(E((\xi+\eta)^{u(\nu-1)})^{\frac{1}{u(\nu-1)}})^{\nu-1}}{u^{\nu-1}}$$
$$\le\sup_{u\ge1}\frac{((E\xi^{u(\nu-1)})^{\frac{1}{u(\nu-1)}}+(E\eta^{u(\nu-1)})^{\frac{1}{u(\nu-1)}})^{\nu-1}}{u^{\nu-1}}.$$

Consider

$$E\xi^{u(v-1)} = \int_0^{+\infty} x^{u(v-1)} p e^{-px}\, dx.$$

Let us make the substitution of variables in this integral, namely $px = t$. Then,

$$\int_0^{+\infty} x^{u(v-1)} p e^{-px}\, dx = \frac{1}{p^{u(v-1)}} \int_0^{+\infty} e^{-t} t^{u(v-1)}\, dt = \frac{1}{p^{u(v-1)}} \Gamma(u(v-1)+1).$$

So,

$$(E\xi^{u(v-1)})^{1/(u(v-1))} = \frac{1}{p}(\Gamma(u(v-1)+1))^{1/(u(v-1))}.$$

Similarly, we can find

$$(E\eta^{u(v-1)})^{1/(u(v-1))} = \frac{1}{q}(\Gamma(u(v-1)+1))^{1/(u(v-1))}.$$

Thereby,

$$\sup_{u\ge 1} \frac{(E((\xi+\eta)^{v-1})^u)^{1/u}}{u^{v-1}} \le \sup_{u\ge 1} \frac{(\Gamma(u(v-1)+1))^{1/u}(1/p+1/q)^{v-1}}{u^{v-1}}.$$

Since $\Gamma(z) \le e^{-z} z^{z-1/2} C_z$, where $C_z = \sqrt{2\pi} e^{1/(12z)}$, we have that

$$(\Gamma(u(v-1)+1))^{1/u} \le e^{-(v-1+1/u)} (u(v-1)+1)^{v-1+1/2u} (C_z)^{1/u},$$

where $z = u(v-1)+1$. Note that $C_z \le S = \sqrt{2\pi} e^{1/18}$. This implies that

$$\begin{aligned}
&\sup_{u\ge 1} \frac{(E((\xi+\eta)^{v-1})^u)^{1/u}}{u^{v-1}} \\
&\quad\le \sup_{u\ge 1} \frac{e^{-(v-1)} e^{-1/u} (u(v-1)+1)^{v-1} (u(v-1)+1)^{1/(2u)} (S)^{1/u} (1/p+1/q)^{v-1}}{u^{v-1}} \\
&\quad\le e^{-(v-1)} \left(\frac{1}{p}+\frac{1}{q}\right)^{v-1} \sup_{u\ge 1} \left(\frac{S}{e}\right)^{1/u} \frac{u^{v-1}(v-1+1/u)^{v-1} u^{1/(2u)} (v-1+1/u)^{1/(2u)}}{u^{v-1}} \\
&\quad\le e^{-(v-1)} \left(\frac{1}{p}+\frac{1}{q}\right)^{v-1} \sup_{u\ge 1} \left(\frac{S}{e}\right)^{1/u} u^{1/(2u)} \left(v-1+\frac{1}{u}\right)^{v-1+1/(2u)} \le e^{-(v-1)+1/(2e)} v^{v-1/2} \left(\frac{1}{p}+\frac{1}{q}\right)^{v-1}.
\end{aligned}$$

That is,

$$\left\| c(\xi,\eta) \frac{1}{pq} (\xi+\eta)^{v-1} \right\|_\psi \le \frac{1}{pq} e^{-(v-1)+1/(2e)} v^{v-1/2} \left(\frac{1}{p}+\frac{1}{q}\right)^{v-1}.$$

Taking into account Example 4.10, we get the following:

$$n \ge \left(\frac{4(3)^{v-1} e^{1/(2e)} v^{v-1/2} (1/p+1/q)^{v-1}}{pq\varepsilon}\right)^2 \max\left(1, \left(-\frac{\ln\delta}{v-1}\right)^{2(v-1)}\right).$$

5 Reliability and accuracy in the space $L_p(T)$ for the calculation of integrals depending on a parameter

Let us consider the integral $\int_{\mathbb{S}} f(s,t) p(s)\, d\mu(s) = I(t)$ assuming that it exists. Let the function $f(s,t)$ depend on the parameter $t \in T$, where (T, ρ) is some compact set and assume that the function $f(s,t)$ is continuous with regard to t.

Suppose $f(s,t)$ is a stochastic process on $\{\mathbb{S}, \mathcal{A}, P\}$ and which we denote as $\xi(s,t) = \xi(t)$ and

$$I(t) = \int_{\mathbb{S}} f(s,t) p(s)\, d\mu(s) = \int_{\mathbb{S}} f(s,t)\, dm(s) = E\xi(t).$$

Let $\xi_i(t)$, $i = 1, 2, \dots, n$, be the independent copies of the stochastic process $\xi(t)$ and $Z_n(t) = \frac{1}{n}\sum_{i=1}^{n}\xi_i(t)$. So, according to the strong law of large numbers $Z_n(t) \to E\xi(t) = I(t)$ with probability one for any $t \in T$.

Definition 5.1. We say that $Z_n(t)$ approximates $I(t)$ in the space $L_p(T)$ with reliability $1 - \delta > 0$ and accuracy $\varepsilon > 0$ if the following inequality holds true:

$$P\left\{\left(\int_T |Z_n(t) - I(t)|^p \, d\mu(t)\right)^{1/p} > \varepsilon\right\} \le \delta.$$

Theorem 5.2. *Let $I(t) = E\xi(t) = \int_S f(s,t)p(s)\,d\mu(s)$, $\xi(t)$ be the stochastic process which belongs to the space $\mathbf{F}_\psi(\Omega)$ satisfying the condition $\mathbf{H}$ with the constant C_ψ. Let also*

$$\widetilde{Z}_n(t) = \frac{1}{n}\sum_{i=1}^{n}(\xi_i(t) - I(t)),$$

where $\xi_i(t)$ are the independent copies of the stochastic process $\xi(t)$. Then, for all $p \ge 1$, the following inequality holds true:

$$\left\|\left(\int_T |\widetilde{Z}_n(t)|^p \, d\mu(t)\right)^{1/p}\right\| \le \frac{2\sqrt{C_\psi}}{\sqrt{n}} \cdot \frac{\psi(p)}{\psi(1)}\left(\int_T \|\xi(t)\|_\psi^p \, d\mu(t)\right)^{1/p},$$

and $\widetilde{Z}_n(t)$ approximates $I(t)$ with reliability $1 - \delta$ and accuracy ε in the space $L_p(T)$ for such n that

$$\inf_{u \ge 1}\frac{1}{\varepsilon^u}\left(\frac{2\sqrt{C_\psi}}{\sqrt{n}} \cdot \frac{\psi(p)}{\psi(1)}\right)^u \left(\int_T \|\xi(t)\|^p \, d\mu(t)\right)^{u/p} (\psi(u))^u \le \delta. \tag{5.1}$$

Proof. The theorem follows from Theorem 3.6 if inequality (5.1) is fulfilled. □

Example 5.3. Consider the space $\mathbf{F}_\psi(\Omega)$, where $\psi(u) = u^\alpha$, $\alpha > 1/2$. Then, Example 2.10 implies that the condition $\mathbf{H}$ is fulfilled for this space with the constant $C_\psi = 4 \cdot 9^\alpha$. It follows from Example 3.7 and Theorem 5.2 that if

$$\varepsilon \ge \frac{4(3pe)^\alpha\left(\int_T \|\xi(t)\|_\psi^p \, d\mu(t)\right)^{1/p}}{\sqrt{n}},$$

then

$$\inf_{u \ge 1}\frac{1}{\varepsilon^u}\left(\frac{2\sqrt{C_\psi}}{\sqrt{n}} \cdot \frac{\psi(p)}{\psi(1)}\right)^u \left(\int_T \|\xi(t)\|^p \, d\mu(t)\right)^{u/p} (\psi(u))^u \le \exp\left\{-\frac{\alpha}{e}\left(\frac{\sqrt{n}\varepsilon}{4(3pe)^\alpha\left(\int_T \|\xi(t)\|_\psi^p \, d\mu(t)\right)^{1/p}}\right)^{1/\alpha}\right\}.$$

So, inequality (5.1) holds, if it is true that

$$\exp\left\{-\frac{\alpha}{e}\left(\frac{\sqrt{n}\varepsilon}{4(3pe)^\alpha\left(\int_T \|\xi(t)\|_\psi^p \, d\mu(t)\right)^{1/p}}\right)^{1/\alpha}\right\} \le \delta$$

for

$$n \ge \left(\frac{4(3pe)^\alpha\left(\int_T \|\xi(t)\|_\psi^p \, d\mu(t)\right)^{1/p}}{\varepsilon}\right)^2 \left((-\ln\delta)\frac{e}{\alpha}\right)^{2\alpha}.$$

Then,

$$n \ge \left(\frac{4(3p)^\alpha\left(\int_T \|\xi(t)\|_\psi^p \, d\mu(t)\right)^{1/p}}{\varepsilon}\right)^2 \max\left(1, \left(-\frac{\ln\delta}{\alpha}\right)^{2\alpha}\right).$$

Example 5.4. Consider the space $\mathbf{F}_\psi(\Omega)$, where $\psi(u) = e^{au^\beta}$ and $a > 0$, $0 < \beta < 1$. Then, it follows from Theorem 2.11 that two cases are possible. In the first case, when $1/(2a\beta)^{1/\beta} = 1$ the condition $\mathbf{H}$ is satisfied for the space $\mathbf{F}_\psi(\Omega)$ with the constant $C_\psi = 4e^{2^\beta a}$. Therefore, Example 3.8 and Theorem 5.2 imply that

$$\begin{aligned}
&\inf_{u \ge 1}\frac{1}{\varepsilon^u}\left(\frac{2\sqrt{C_\psi}}{\sqrt{n}} \cdot \frac{\psi(p)}{\psi(1)}\right)^u \left(\int_T \|\xi(t)\|^p \, d\mu(t)\right)^{u/p} (\psi(u))^u \\
&\quad \le \exp\left\{-\frac{\beta}{a^{1/\beta}}\left(\frac{1}{\beta+1}\ln\frac{\sqrt{n}\varepsilon}{4e^{a(2^{\beta-1}+p^\beta-1)}\left(\int_T \|\xi(t)\|_\psi^p \, d\mu(t)\right)^{1/p}}\right)^{(\beta+1)/\beta}\right\}.
\end{aligned}$$

So, inequality (5.1) holds if

$$\exp\left\{-\frac{\beta}{a^{1/\beta}}\left(\frac{1}{\beta+1}\ln\frac{\sqrt{n}\varepsilon}{4e^{a(2^{\beta-1}+p^{\beta}-1)}(\int_T\|\xi(t)\|_\psi^p\,d\mu(t))^{1/p}}\right)^{(\beta+1)/\beta}\right\}\le\delta.$$

Assuming that

$$n\ge\left(\frac{1}{\varepsilon}4e^{a(2^{\beta-1}+p^{\beta}-1)}\left(\int_T\|\xi(t)\|_\psi^p\,d\mu(t)\right)^{1/p}\right)^2\exp\left\{2(\beta+1)\left((-\ln\delta)\frac{a^{1/\beta}}{\beta}\right)^{\beta/(\beta+1)}\right\},$$

then

$$n\ge\left(\frac{1}{\varepsilon}4e^{a(2^{\beta-1}+p^{\beta}-1)}\left(\int_T\|\xi(t)\|_\psi^p\,d\mu(t)\right)^{1/p}\right)^2\max\left(e^{a(\beta+1)},\exp\left\{2(\beta+1)\left((-\ln\delta)\frac{a^{1/\beta}}{\beta}\right)^{\beta/(\beta+1)}\right\}\right).$$

In the second case, when $1/(2a\beta)^{1/\beta}>1$, we get that

$$\begin{aligned}&\inf_{u\ge1}\frac{1}{\varepsilon^u}\left(\frac{2\sqrt{C_\psi}}{\sqrt{n}}\cdot\frac{\psi(p)}{\psi(1)}\right)^u\left(\int_T\|\xi(t)\|^p\,d\mu(t)\right)^{u/p}(\psi(u))^u\\&\quad\le\exp\left\{-\frac{\beta}{a^{1/\beta}}\left(\frac{1}{\beta+1}\ln\frac{\sqrt{n}\varepsilon(2a\beta)^{\frac{1}{4\beta}}}{\frac{4e^{a(2^{\beta-1}+p^{\beta}-1/2)-\frac{1}{4\beta}}}{(2a\beta)^{1/4\beta}}(\int_T\|\xi(t)\|_\psi^p\,d\mu(t))^{1/p}}\right)^{(\beta+1)/\beta}\right\}\end{aligned}$$

and inequality (5.1) holds if the following estimate is true:

$$\exp\left\{-\frac{\beta}{a^{1/\beta}}\left(\frac{1}{\beta+1}\ln\frac{\sqrt{n}\varepsilon(2a\beta)^{\frac{1}{4\beta}}}{\frac{4e^{a(2^{\beta-1}+p^{\beta}-1/2)-\frac{1}{4\beta}}}{(2a\beta)^{1/4\beta}}(\int_T\|\xi(t)\|_\psi^p\,d\mu(t))^{1/p}}\right)^{(\beta+1)/\beta}\right\}\le\delta$$

for

$$n\ge\left(\frac{\frac{4e^{a(2^{\beta-1}+p^{\beta}-1/2)-\frac{1}{4\beta}}}{(2a\beta)^{1/4\beta}}(\int_T\|\xi(t)\|_\psi^p\,d\mu(t))^{1/p}}{(2a\beta)^{\frac{1}{4\beta}}\varepsilon}\right)^2\exp\left\{2(\beta+1)\left((-\ln\delta)\frac{a^{1/\beta}}{\beta}\right)^{\beta/(\beta+1)}\right\}.$$

Then,

$$n\ge\left(\frac{\frac{4e^{a(2^{\beta-1}+p^{\beta}-1/2)-\frac{1}{4\beta}}}{(2a\beta)^{1/4\beta}}(\int_T\|\xi(t)\|_\psi^p\,d\mu(t))^{1/p}}{(2a\beta)^{\frac{1}{4\beta}}\varepsilon}\right)^2\max\left(e^{a(\beta+1)},\exp\left\{2(\beta+1)\left((-\ln\delta)\frac{a^{1/\beta}}{\beta}\right)^{\beta/(\beta+1)}\right\}\right).$$

6 Conclusion

Estimates for the distribution of norms in $L_p(T)$ of the stochastic processes from $\mathbf{F}_\psi(\Omega)$ spaces, accuracy and reliability for the calculation of integrals by Monte Carlo methods and reliability and accuracy in $L_p(T)$ space for calculation of integrals dependent on the parameter have been found in this paper.

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