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Ju. Yu. Krashanitsya, I. V. Shapochka (Uzhhorod National University)

ON INTEGRAL 3-ADIC REPRESENTATIONS OF THE CYCLIC GROUP OF ORDER 27

Let G be the cyclic group of the order 27, $\mathbb{Z}_3$ be the ring 3-adic integers, Γ and Δ be a matrix representations of the group G over the ring $\mathbb{Z}_3$ which have precisely three irreducible components. It' shown in the paper, that matrix representation Γ is generally equivalent to the matrix representation Δ if and only if Γ is equivalent to Δ.

Показано, що узагальнена еквівалентність матричних цілочислових 3-адичних зображень циклічної групи 27-го порядку, що мають точно три незвідні компоненти, співпадає з звичайною еквівалентністю цих зображень.

Let G be a finite group, R be a commutative ring with identity and $GL(n, R)$ be a general linear group over the ring R, $n \in \mathbb{N}$. We say (see [1, 2]), that a matrix representation $\Gamma : G \to GL(n, R)$ is generally equivalent to a matrix representation $\Delta : G \to GL(n, R)$ of the group G over the ring R if there exists an automorphism φ of the group G and a matrix $C \in GL(n, R)$ such that $C^{-1}\Gamma(g)C = \Delta(\varphi(g))$ for all $g \in G$. Obviously, if a matrix representation $\Gamma : G \to GL(n, R)$ is equivalent to a matrix representations $\Delta : G \to GL(n, R)$, than Γ is generally equivalent to Δ. However, the converse statement is not always true. Such as for two complex matrix representations $\Gamma : a \to i$, $\Delta : a \to -i$ of the cyclic group $\langle a \rangle$ of order 4, where i is the primitive 4th root of unity. In the other hand if G is a finite perfect group then any automorphism of the group G is inner. That's why the generalized equivalence of two matrix representation of the group G over a ring R should be simple equivalence of these representations. It's shown in papers [1, 2], that if G is the cyclic p-group of the order less then p^3, then two integral p-adic representations of the group G are generally equivalent if and only if they are simply equivalent. Here we consider the very special case of the problem of coincidence of generally equivalence and equivalence of matrix representations of a finite group.

Hereinafter let $G = \langle a \rangle$ be the cyclic group of order 27 and $\mathbb{Z}_3$ be the ring of 3-adic integers. Let ε, ξ, η be the primitive 3th, 9th and 27th roots of unity respectively. We denote by the $\tilde{\varepsilon}$, $\tilde{\xi}$, $\tilde{\eta}$ the Frobenius companion matrices of the cyclotomic polynomials $\Phi_3(x) = x^2 + x + 1$, $\Phi_9(x) = x^6 + x^3 + 1$ and $\Phi_{27}(x) = x^{18} + x^9 + 1$ respectively.

It's well known that an arbitrary irreducible matrix representations of the cyclic group G over the ring $\mathbb{Z}_3$ is equivalent to one of the following representations:

$$a \to 1, \qquad a \to \tilde{\varepsilon}, \qquad a \to \tilde{\xi}, \qquad a \to \tilde{\eta}.$$

It's also known (see [3]) that an arbitrary matrix representations of the cyclic group G over $\mathbb{Z}_3$ has a normal form

$$a \to \begin{pmatrix} \tilde{\eta}^{(n_3)} & * & * & * \\ 0 & \tilde{\xi}^{(n_2)} & * & * \\ 0 & 0 & \tilde{\varepsilon}^{(n_1)} & * \\ 0 & 0 & 0 & I_{n_0} \end{pmatrix}, \tag{1}$$

where n_0, n_1, n_2, n_3 are some natural numbers (may be zero, but $n_0+n_1+n_2+n_3 \neq 0$) and $A^{(n)} = I_n \otimes A$ is the Kronecker product of the identity $n \times n$-matrix I_n and a matrix A.

We say that the matrix representation (1) of the cyclic group G has precisely n irreducible components if $n_0 + n_1 + n_2 + n_3 = n$. Let k be the one of the $\{1, 2, 3, 4\}$. As well we say that the matrix representation (1) of the cyclic group G contains k pairwise nonequivalent irreducible components if only k of numbers n_0, n_1, n_2, n_3 are nonzero.

Theorem 1. *Let G be the cyclic group of the order 27, $\mathbb{Z}_3$ be the ring 3-adic integers, Γ and Δ be a matrix representations of the group G over the ring $\mathbb{Z}_3$ which have precisely three irreducible components. Matrix representation Γ is generally equivalent to the matrix representation Δ if and only if Γ is equivalent to Δ.*

Proof. Let $G = \langle a \rangle$ be the cyclic group of the order 27, $\mathbb{Z}_3$ be the ring 3-adic integers, Γ and Δ be a matrix representations of the group G over the ring $\mathbb{Z}_3$ which have precisely three irreducible components. If Γ and Δ are decomposable representations or they contain irreducible representation $a \to 1$ then proof of the theorem implies from [1, 2].

Now let Γ and Δ be an indecomposable $\mathbb{Z}_3$-representations of the group G which are not contain irreducible component $a \to 1$. Then by [4] each of them is equivalent to one of the followings:

$$\Lambda_{jk} : a \to \begin{pmatrix} \tilde{\eta} & (\tilde{\eta} - I_{18})^j \langle 1 \rangle_{18\times 6} & 0 \\ 0 & \tilde{\xi} & (\tilde{\xi} - I_6)^k \langle 1 \rangle_{6\times 2} \\ 0 & 0 & \tilde{\varepsilon} \end{pmatrix},$$

$$\Psi_{jl} : a \to \begin{pmatrix} \tilde{\eta} & (\tilde{\eta} - I_{18})^j \langle 1 \rangle_{18\times 6} & (\tilde{\eta} - I_{18})^l \langle 1 \rangle_{18\times 2} \\ 0 & \tilde{\xi} & 0 \\ 0 & 0 & \tilde{\varepsilon} \end{pmatrix},$$

$$\Upsilon_{kl} : a \to \begin{pmatrix} \tilde{\eta} & 0 & (\tilde{\eta} - I_{18})^l \langle 1 \rangle_{18\times 2} \\ 0 & \tilde{\xi} & (\tilde{\xi} - I_6)^k \langle 1 \rangle_{6\times 2} \\ 0 & 0 & \tilde{\varepsilon} \end{pmatrix},$$

$$\Upsilon_{kl} : a \to \begin{pmatrix} \tilde{\eta} & 0 & (\tilde{\eta} - I_{18})^l \langle 1 \rangle_{18\times 6} \\ 0 & \tilde{\xi} & (\tilde{\xi} - I_6)^k \langle 1 \rangle_{6\times 2} \\ 0 & 0 & \tilde{\varepsilon} \end{pmatrix},$$

$$\Theta^{xy}_{jkl} : a \to \begin{pmatrix} \tilde{\eta} & (\tilde{\eta} - I_{18})^j \langle 1 \rangle_{18\times 6} & (xI_{18} + y\tilde{\eta})(\tilde{\eta} - I_{18})^l \langle 1 \rangle_{18\times 2} \\ 0 & \tilde{\xi} & (\tilde{\xi} - I_6)^k \langle 1 \rangle_{6\times 2} \\ 0 & 0 & \tilde{\varepsilon} \end{pmatrix},$$

where 0 is a correspondent zero matrix, $\langle 1 \rangle_{mn}$ is the $m \times n$-matrix, which has only one nonzero element 1 in the first row and the last column, $j \in \{0, 1, 2, 3, 4, 5\}$, k, $l \in \{0, 1\}$, x, $y \in \{0, 1, 2\}$ and $(x, y) \neq (0, 0)$.

2 is the primitive root modulo 27. That's why for the proof of the theorem it's sufficient to show, that for any representation $\Xi : a \to \Xi(a)$ of the above list of indecomposable representations the representation $a \to \Xi(a^2)$ is equivalent to Ξ. The proof of each case is based on the methods, which are described in [3] and it

splits on several steps. Let us consider only one case $\Xi = \Lambda_{21}$. The proofs of other cases are analogous.

Let

$$A_\eta = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A_\xi = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}, \quad A_\varepsilon = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}, \quad C_1 = \begin{pmatrix} A_\eta & 0 & 0 \\ 0 & A_\xi & 0 \\ 0 & 0 & A_\varepsilon \end{pmatrix}.$$

Then the representation $a \to \Xi(a^2)$ is equivalent to the representation

$$\Xi_1 : a \to C_1^{-1}\Xi(a^2)C_1 = \begin{pmatrix} \tilde{\eta} & X_1 & 0 \\ 0 & \tilde{\xi} & Y_1 \\ 0 & 0 & \tilde{\varepsilon} \end{pmatrix},$$

where

$$X_1 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad Y_1 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix}.$$

The following steps of the proof consists of finding of matrices

$$C_2 = \begin{pmatrix} I_{18} & 0 & 0 \\ 0 & I_6 & Y_2 \\ 0 & 0 & I_2 \end{pmatrix}, \quad C_3 = \begin{pmatrix} I_{18} & 0 & 0 \\ 0 & \tilde{\alpha} & 0 \\ 0 & 0 & I_2 \end{pmatrix},$$

$$C_4 = \begin{pmatrix} I_{18} & X_2 & 0 \\ 0 & I_6 & 0 \\ 0 & 0 & I_2 \end{pmatrix}, \quad C_5 = \begin{pmatrix} \tilde{\beta} & 0 & 0 \\ 0 & I_6 & 0 \\ 0 & 0 & I_2 \end{pmatrix}, \quad C_6 = \begin{pmatrix} I_{18} & 0 & Z \\ 0 & I_6 & 0 \\ 0 & 0 & I_2 \end{pmatrix},$$

that

$$(C_2C_3)^{-1}\Xi_1(a)(C_2C_3) = \begin{pmatrix} \tilde{\eta} & * & * \\ 0 & \tilde{\xi} & (\tilde{\xi} - I_6)\langle 1\rangle_{6\times 2} \\ 0 & 0 & \tilde{\varepsilon} \end{pmatrix} = \Xi_2(a),$$

$$(C_4C_5)^{-1}\Xi_2(a)(C_4C_5) = \begin{pmatrix} \tilde{\eta} & (\tilde{\eta} - I_{18})^2\langle 1\rangle_{18\times 6} & * \\ 0 & \tilde{\xi} & (\tilde{\xi} - I_6)\langle 1\rangle_{6\times 2} \\ 0 & 0 & \tilde{\varepsilon} \end{pmatrix} = \Xi_3(a),$$

$$C_6^{-1}\Xi_3(a)C_6 = \begin{pmatrix} \tilde{\eta} & (\tilde{\eta} - I_{18})^2\langle 1\rangle_{18\times 6} & 0 \\ 0 & \tilde{\xi} & (\tilde{\xi} - I_6)\langle 1\rangle_{6\times 2} \\ 0 & 0 & \tilde{\varepsilon} \end{pmatrix} = \Xi(a).$$

Let us indicate these matrices by mention only Y_2, $\tilde{\alpha}$, X_2, $\tilde{\beta}$ and Z:

$$Y2 = \begin{pmatrix} 0 & -1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \tilde{\alpha} = I_6 + 3\tilde{\xi} + 3\tilde{\xi}^2 + 3\tilde{\xi}^3 + 3\tilde{\xi}^4 + \tilde{\xi}^5,$$

$$X_2 = \begin{pmatrix} 0 & 3 & 3 & 1 & 2 & 9 \\ 0 & 9 & 12 & 8 & 0 & -5 \\ 0 & 0 & 9 & 12 & 8 & 0 \\ 0 & 0 & 0 & 9 & 12 & 8 \\ 0 & 0 & 0 & 0 & 9 & 12 \\ 0 & 0 & 0 & 0 & 0 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 12 \\ 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -9 & -9 & -4 & 2 & 8 \\ 0 & -3 & -12 & -11 & -4 & 4 \\ 0 & 0 & -3 & -12 & -11 & -4 \\ 0 & 0 & 0 & -3 & -12 & -11 \end{pmatrix},$$

$$\tilde{\beta} = -161I_{18} - 170\tilde{\eta} - 172\tilde{\eta}^2 - 165\tilde{\eta}^3 - 150\tilde{\eta}^4 - 123\tilde{\eta}^5 - 87\tilde{\eta}^6 - 51\tilde{\eta}^7 -$$
$$- 15\tilde{\eta}^8 - 138\tilde{\eta}^9 - 105\tilde{\eta}^{10} - 69\tilde{\eta}^{11} - 33\tilde{\eta}^{12} + 3\tilde{\eta}^{13} + 44\tilde{\eta}^{14} + 85\tilde{\eta}^{15} + 118\tilde{\eta}^{16} + 144\tilde{\eta}^{17},$$

$$Z = 11614593457^{-1} \cdot \begin{pmatrix} -15739075749 & 4729595574 \\ 40239524169 & -15739075749 \\ -52637316189 & 40239524169 \\ 35103322056 & -52637316189 \\ -12506160480 & 35103322056 \\ 4199849118 & -12506160480 \\ -1854928758 & 4199849118 \\ 559659792 & -1854928758 \\ -914648361 & 559659792 \\ -9075810006 & 3814947213 \\ 20096348439 & -9075810006 \\ -25145717673 & 20096348439 \\ 15960183117 & -25145717673 \\ -4360375407 & 15960183117 \\ 2654268765 & -4360375407 \\ -5267939358 & 2654268765 \\ 4859519280 & -5267939358 \\ -4729595574 & 4859519280 \end{pmatrix}.$$

It means, that the representation $a \to \Xi(a^2)$ is equivalent to the representation Ξ.

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