УДК 519.8

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About one new principle of rational collective choice

A new principle of rational group choice is proposed for the group choice problem in the Arrow's formulation. The question of consistence of some well-known collective choice rules was explored according to this principle.

Key Words: group decision making, Condorcet's principle, the group choice rules.

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Про один новий принцип раціонального колективного вибору

Для задачі колективного вибору в постановці К.Ерроу запропоновано новий принцип раціонального колективного вибору. Вивчено питання обгрунтованості за цим принципом деяких відомих правил колективного вибору.

Ключові слова: колективне прийняття рішень, принцип Кондорсе, правила колективного вибору.

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Introduction. One of the main problems of group decision making is the problem of best alternative choosing. At the end of XVIII century French philosoper and matematician Condorcet was the first to draw attention to the failure of best alternative definition according to relative majority rule [1]. In return he offered his rule of the best alternative definition: the best is that alternative (Condorcet's alternative, the best alternative according to Condorcet, Condorcet winner), which outperforms all other alternatives according to relative majoity rule. The author himself noticed the existence [illegible] profiles for which there is no Condorcet winner. [illegible] until now the same name principle proposed by [illegible] in one way or other (in various variations) is included in all rational models of rational group decision making [1-3], and the rules that comply with this principle are called reasonable (wealthy) Condorcet. That is why one of the most critical problems is the problem to generalize the concept of reasonability according to Condorcet.

In this paper we propose a new principle for the best alternative choice in the group decision making, which is consistent with the already known generalizations of Condorcet principle. The research results of some well-known a group choice rules as to their consistency according to the proposed principle are also offered.

A group choice problem in the Arrow's formulation. Let a finite set of alternatives be $A=\{a_1,...,a_{n_A}\}$ (n_A – number of alternatives), and also a set (profile) $\Pi=\{P_1,...,P_{n_E}\}$ of estimations of alternatives set A, where P_l is strict linear order on a set A, given by individual l of collective (group) and which corresponds to their individual preferences on a set of alternatives A ($l \in N_E=\{1,...,n_E\}$, n_E – quantity of individuals). The problem of collective (group, resulting, etc.) order definition which in the "best" possible way displays the preferences on set A of group of individuals in general is set.

[illegible]he only possible and correct solution on the set [illegible] the two alternatives is a collective choice [illegible]rding to the relative majority rule: the best alternative is that one which was given a strong preference for at least half of individuals. On the set of three or more alternatives arises a problem of constructing of collective choice rule, which would be adequate continuation of voting according to majority principle for a couple of alternatives.

Definition 1. Let's denote for any profile of individual $c_{ij}=Card\left(\left\{l \in N_E:(a_i,a_j)\in P_l\right\}\right)$, $\forall\, i,j \in N_A=\{1,...,n_A\}$. Value c_{ij} – is the number of individuals which gave strong prefernence to alternative a_i over alternative a_j in corresponding profile.

Definition 2. For any profile of individual preferences value $m_{ij}=c_{ij}-c_{ji}$ is called majority margin of alternatives a_i over aternative a_j,

$\forall\, i, j \in N_A$, and matrix $M = \left(m_{ij}\right)_{i,j=1,\ldots,n_A}$ is called matrix of majority margin.

In the futher if you need to point out correspondence of mentioned notations to certain profile $\Pi \in \mathrm{P}$ (P – set of all possible profiles), then we will write down $c_{ij}(\Pi)$, $m_{ij}(\Pi)$ and $M(\Pi)$.

Definition 3. Relation of simpe majority (majority relation), which was generated by profile $\Pi \in \mathrm{P}$, is called connected binary relation $R_{M(\Pi)}$ on a set A, which is defined as following

$$(a_i, a_j) \in R_{M(\Pi)} \Leftrightarrow m_{ij}(\Pi) \ge 0, \quad \forall\, a_i, a_j \in A.$$

Prudent principle. For each alternative $a_i \in A$, $i \in N_A$ we introduce a set of indexes:

$$N_i^- = \{z \in N_A : m_{iz} < 0, z \ne i\},$$
$$N_i^+ = \{z \in N_A : m_{iz} > 0, z \ne i\}.$$

Taking into consideation the definition 1, 2 for each alternative a_i, $\forall\, i \in N_A$ the following interpretations are justified:

- m_{ij}, $j \in N_i^+$ – number of winning units or just winning of alternative a_i from alternative a_j;
- $|m_{ij}| = -m_{ij}$, $j \in N_i^-$ – number of loss units or just loss of alternative a_i from alternative a_j;
- $S_i^+ = \sum_{z \in N_i^+} m_{iz}$ – total winning of alternative a_i;
- $-S_i^- = -\sum_{z \in N_i^-} m_{iz}$ – total loss of alternative a_i.

The following principle of rational choice is suggested in the context of above mentioned interpretations.

- *if winning of alternative a_i from alternative a_j is more than its total loss, then in collective preference alternative a_i should be better than alternative a_j;*
- *if loss of alternative a_i from alternative a_j is more than its total winning, then in collective preference alternative a_i should be worse then alternative a_j.*

It is easy to see that the above mentioned principle is quite prudent and is not contrary to the well-known principle of Condorcet.

Let's consider the rules that are used in group decision making, most of which are classic, well-known, and often used in practice, others, according to some authors [4], are only theoretical. We formulate some of them on the language of binary relations which are necessary for further their analyzing concerning the satisfaction of principles suggested by us.

Kemeny rule. Let $R_1, R_2 \in \Omega$ (Ω –set of all connected, asymmetric, transitive binary relations) are two arbitrary strict linear orders. Let's define the distance between theese two relations as the distance beween the sets:

$$\delta(R_1, R_2) = \frac{Card\left((R_1 \setminus R_2) \cup (R_2 \setminus R_1)\right)}{2}.$$

Binary relation R_{Kemeny} is called collective order of Kemeny (Kemeny's median) if and only if $R_{Kemeny} = arg \min_{R \in \Omega} \sum_{l=1}^{n_E} \delta(R, P_l)$.

Egalitarian Simpson's rule. To win according to this rule it is necessary that alternative does not collect against it a large majority according to this rule. Simpson's score of alternative $a_i \in A$ is called value

$$S(a_i) = \min_{z \in N_i^-} m_{iz}. \qquad (1)$$

Simpson's collective order is called ordening $R_{Simpson}$, which is defined as follows:

$$(a_i, a_j) \in R_{Simpson} \Leftrightarrow S(a_i) \ge S(a_j), \ \forall\, a_i, a_j \in A.$$

Utilitarian Tideman's rule. As alternative to the rule (1) it is possible to define the rule which is based on an utilitarian criterion. Utilitarian score of alternative $a_i \in A$ is called value $U(a_i) = \sum_{z \in N_i^-} m_{iz}$.

Utilitarian collective order is called ordening R_{UC}, which is defined as follows:

$$(a_i, a_j) \in R_{UC} \Leftrightarrow U(a_i) \ge U(a_j), \quad \forall\, a_i, a_j \in A.$$

<u>*Prudent order*</u>. Suppose $\lambda \in \{-n_E, -n_E + 2, \ldots, n_E - 2, n_E\}$ and let's define relation $R_{>\lambda}$: $(a_i, a_j) \in R_{>\lambda} \Leftrightarrow m_{ij} > \lambda$, $\forall\, i, j \in N_A, i \ne j$. Prudent oreder is called strict linear order R_{PO}, which completes acyclic relation $R_{>\beta}$, that is $R_{>\beta} \subseteq R_{PO}$,

$$\beta = min\{\lambda \in \{-n_E, -n_E + 2, \ldots, n_E - 2, n_E\} : R_{>\lambda} - \text{acyclic}\}.$$

Theorem 1. *Strict Kemeny's median, egalitarian Simpson's rule and utilitarian Tideman's rule comply with the prudent principle. There is always a prudent order, which complies with the prudent principle.*

Proof. The problem of collective strict linear order finding in the form of Kemeny's median is equivalent to the problem of linear ordering of alternatives. [5] That is why the validity of the strict Kemeny's median according to the prudent principle is a special case of a more general result [6].

Let for some fixed (but arbitrary) profile $m_{ij} > -S_i^-$. Let's assume opposite, that is let $\min\limits_{z\in N_j^-} m_{jz} \ge \min\limits_{z\in N_i^-} m_{iz}$. From inequality $m_{ij} > -S_i^- = -\sum\limits_{z\in N_i^-} m_{iz}$ and property of matrix of majority margin it follows that $-m_{ji} = m_{ij} > -m_{iz}$, $\forall\, z \in N_i^-$. That is $m_{ji} < m_{iz} < 0$, $\forall\, z\in N_i^-$. Then inequality $\min\limits_{z\in N_j^-} m_{jz} \le m_{ji} < m_{iz^*} = \min\limits_{z\in N_i^-} m_{iz}$ is obvious, where $z^* \in Arg \min\limits_{z\in N_i^-} m_{iz}$, which contradicts our assumption. For utilitarian Tideman's rule the opposite assumption would mean in particular that $0 > \sum\limits_{z\in N_j^-} m_{jz} \ge \sum\limits_{z\in N_i^-} m_{iz}$. Then from condition $m_{ij} > -S_i^- > 0$ and properties of matrix of majority margin it follows $m_{ji} < 0$, that is $i \in N_j^-$. Then from proposition we get $0 > m_{ji} + \sum\limits_{z\in N_j^- \setminus\{i\}} m_{jz} \ge \sum\limits_{z\in N_i^-} m_{iz}$, whence it follows that $0 \ge \sum\limits_{z\in N_j^- \setminus\{i\}} m_{jz} \ge \sum\limits_{z\in N_i^-} m_{iz} - m_{ji}$ or $0 \ge \sum\limits_{z\in N_j^- \setminus\{i\}} m_{jz} \ge S_i^- + m_{ij} > 0$. So we have $0 > 0$ – contradiction.

Let $m_{ij} > \beta$. Then for any prudent order: $(a_i, a_j) \in R_{>\beta}$ (it follows directly from the rule). And since every acyclic relation is asymmetric [1], then $(a_j, a_i) \notin R_{>\beta}$. Let's consider a case when $m_{ij} \le \beta$. Then from $m_{ij} > -S_i^- \ge 0$ and $m_{ij} = -m_{ji}$ it follows $m_{ji} < 0 < m_{ij} \le \beta$, whence $(a_i, a_j) \notin R_{>\beta}$ and $(a_j, a_i) \notin R_{>\beta}$. Let's take arbitrary $k \in N_A$, $k \ne i$, $k \ne j$. Let's schow that $(a_k, a_i) \notin R_{>\beta}$. Having assumed $(a_k, a_i) \in R_{>\beta}$ then we get $m_{ki} > \beta > 0$. From property $m_{ki} = -m_{ik}$ it follows that $k \in N_i^-$. Then estimation $m_{ij} \ge -S_i^- \ge -m_{ik} > \beta$ is valid, which contradicts the situation considered by us. So, in this case it is always possible to add strict partial order $R_{>\beta}$ to strict linear order $R_>$ so that $(a_i, a_j) \in R_>$, in particular, in order that a_i to be the best. The theorem is proved.

Borda's rule puts in order alternatives according to the sum of the ranks of alternatives in a profile of individual preferences. We use equivalent method of points calculation which is based on the majority margin. Borda's score of alternative $a_i \in A$ is called value $B(a_i) = \sum\limits_{z=1}^{n_A} w_{iz}$. Borda's collective order is called ordening R_{Borda}, which is defined as follows:

$$(a_i, a_j) \in R_{Borda} \Leftrightarrow B(a_i) \ge B(a_j),\ \forall\, a_i, a_j \in A.$$

Copland's rule. In order to defeat Copland's rule it is necessary to win on the basis of simple majority from the greatest number of other alternatives. Copland's score of alternative $a_i \in A$ is called value

$$C(a_i) = 2Card\big(\{z \in N_A \setminus \{i\} : w_{iz} > 0\}\big) + \\ + Card\big(\{z \in N_A \setminus \{i\} : w_{iz} = 0\}\big).$$

Copland's collective rule is called ordening $R_{Copeland}$, which is defined as follows:

$$(a_i, a_j) \in R_{Copeland} \Leftrightarrow C(a_i) \ge C(a_j),\ \forall\, a_i, a_j \in A.$$

Slater's rule is to find collective orders, which are closest to the corresponding relation of simple majority. Let R_M be relation of simple majority generated by some profile. Binary relation R_{Slater} is called Slater's collective order if and only if

$$R_{Slater} = arg \min_{R\in\Omega} \delta(R, R_M). \qquad (2)$$

Lemma 1 [7]. *Let given matrix* $M = \left(m_{ij}\right)_{i,j=1,\dots,n_A}$ *is such, that* $m_{ij} + m_{ji} = 0$, m_{ij} – *is odd (even),* $\forall\, i, j \in N_A$, $i \ne j$. *Then there exists a profile* $\Pi \in \mathrm{P}$ *of strict linear orders* n_A *of alternatives, for which* M *is matrix of majority margin.*

Theorem 2. *Prudent principle is violated at the definition of collective preference according to Borda's rule, if* $n_A \ge 3$*; according to Copland's and Slater's rules, if* $n_A \ge 4$*. On the set of the three alternatives there is always a strict linear order, constructed according to Slater's rule that complies with the prudent principle.*

Proof. Let's construct integral matrix M, requiring odd or parity (in paticular equal zero) of all elements, according to the following rule:

$$\begin{gathered} m_{12} = r,\ m_{13} = -u,\ m_{23} = g, \\ r > u > 0,\ g > 2r - u, \\ m_{1j} = m_{2j} = x \ge 0,\ 3 < j \le n_A, \\ m_{ij} - \text{arbitrary integer},\quad 3 < i < j \le n_A. \end{gathered}$$

According to lemma 1 there exists a profile of individual preferenes, for which matrix M will be matrix of majority margin. For this profile of inequality $r = m_{12} > -S_1^- = u$ it is necessary to require compliance with the prudent principle for alternative a_1 relative to alternative a_2. On the other hand we have the following Borda's scores for alternatives a_1 и a_2: $B(a_1) = r - u + (n_A - 3)x$, $B(a_2) = g - r + (n_A - 3)x$, whence because of introduced restrictions on the value g, r, and u, we have $B(a_1) < B(a_2)$, that is $(a_1, a_2) \notin R_{Borda}$.

Let's consider matrix M with following elements:

$$\begin{gathered} m_{12} = r,\ m_{21} = -r, \\ m_{1j} = -x,\ m_{2j} = x,\ x \ge 1,\ 3 \le j \le n_A, \\ r > (n_A - 2)x, \\ m_{ij} - \text{arbitrary integer},\quad 3 \le i < j \le n_A. \end{gathered}$$

For the corresponding profile from inequality $r = m_{12} > -S_1^- ==(n_A - 2)x$ it is necessary to require compliance with the prudent principle for alternative a_1 relative to alternative a_2. The following Copland's scores for alternative a_1 and a_2: $C(a_1) = 2$, $C(a_2) = 2(n_A - 2)$ are valid, whence due to restriction $n_A \ge 4$ we have $(a_1, a_2) \notin R_{Copeland}$.

Let's define matrix M as follows:

$$\begin{gathered} m_{12} = r,\ m_{13} = -u,\ m_{14} = -g, \\ r > u + g > 0,\ u > 0,\ g > 0, \\ m_{23} > 0,\ m_{24} > 0,\ m_{34} > 0, \\ m_{1j} > 0,\ m_{2j} > 0,\ m_{3j} > 0,\ m_{4j} > 0,\ 4 < j \le n_A, \\ m_{ij} - \text{arbitrary integer},\quad 4 < i < j \le n_A. \end{gathered}$$

For alternative a_1 relative to alternative a_2 due to inequality $r = e_{12} > -S_1^- = u + g$ it is necessary to require compliance with the prudent principle. For the profile which corresponds to matrix M which is under consideration we have the following matrix of majority relations:

	a_1	a_2	a_3	a_4	a_5	$\cdots$	a_{n_A}
a_1	0	1	0	0	1	$\cdots$	1
a_2	0	0	1	1	1	$\cdots$	1
a_3	1	0	0	1	$\vdots$	$\ddots$	$\vdots$
a_4	1	0	0	0	1	$\cdots$	1
a_5	0	0	$\cdots$	0			
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$		$\tilde{R}_M$	
a_{n_A}	0	0	$\cdots$	0			

where $\tilde{R}_M$ is matrix of majority relations on a set of alternatives $\{a_5, \dots a_{n_A}\}$. The results of work [5] at $n_E = 1$ remain valid also for problem (2), according to which problem (2) is equivalent linear ordening problem of alternatives with the following matrix prices:

	a_1	a_2	a_3	a_4	a_5	$\cdots$	a_{n_A}
a_1	0	1	−1	−1	1	$\cdots$	1
a_2	−1	0	1	1	1	$\cdots$	1
a_3	1	−1	0	1	$\vdots$	$\ddots$	$\vdots$
a_4	1	−1	−1	0	1	$\cdots$	1
a_5	−1	−1	$\cdots$	−1			
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$		$\tilde{E}$	
a_{n_A}	−1	−1	$\cdots$	−1			

where $\tilde{E}$ is matrix prices on a set of alternatives $\{a_5, \dots a_{n_A}\}$. According to decompositional procedures [6] for linear ordening problem of alternatives with matrix is needed decomposition $\{N^{(1)}, N_A \setminus N^{(1)}\}$, where $N^{(1)} = \{1,2,3,4\}$. The only solution for such problem on the set of indexes $N^{(1)}$ will be variant $p^* = (2,3,4,1)$, whence due to definition of necessity concept of analyzed decomposition [6], in particular we get $(a_1, a_2) \notin R_{Slater}$, $\forall R_{Slater} \in Arg \min\limits_{R \in \Omega} \delta(R, R_M)$.

Let $n_A = 3$. Let's take arbitrary profile of individual preferences on a set of three alternatives.

Let's choose arbitrary alternative a_i, $i \in N_A = \{1,2,3\}$. If alternative a_i is the best according to Condorcet, then a_i is also the best according to Slater because Slater's rule is reasonable according to Condorcet [2]. Let's consider a case when a_i is not the best according to Condorcet and let $m_{ij} > -S_i^- > 0$, $j \in N_A \setminus \{i\}$. In this case following situations for matrix of majority margin are possible.

$a)$	a_i	a_j	a_k
a_i	0	r	$-u$
a_j	$-r$	0	g
a_k	u	$-g$	0

$b)$	a_i	a_j	a_k
a_i	0	r	$-u$
a_j	$-r$	0	0
a_k	u	0	0

$c)$	a_i	a_j	a_k
a_i	0	r	$-u$
a_j	$-r$	0	$-g$
a_k	u	g	0

at arbitrary even (odd) integers $r > 0$, $u > 0$, $g > 0$. We have the following matrixes of majority relations for these profiles.

$a)$	a_i	a_j	a_k
a_i	0	1	0
a_j	0	0	1
a_k	1	0	0

$b)$	a_i	a_j	a_k
a_i	0	1	0
a_j	0	0	1
a_k	1	1	0

$c)$	a_i	a_j	a_k
a_i	0	1	0
a_j	0	0	0
a_k	1	1	0

Whence we get the following strict preferences according to Slater.

$a)$

$a_i \succ a_j \succ a_k$

$a_j \succ a_k \succ a_i$

$a_k \succ a_i \succ a_j$

$b)$

$a_k \succ a_i \succ a_j$

$a_k \succ a_j \succ a_i$

$c)$

$a_k \succ a_i \succ a_j$

The theorem is proved.

Conclusions. The new principle of rational collective choice is proposed in this paper for the problem of collective choice in the classical formulation of Arrow. If there is a strong Condorcet winner for some profile of individual preferences, this principle coincides with the very principle of Condorcet, and in the case of its absence it is its reasonable substitution (continuation). Study of the consistency question according to this principle of some well-known rules of collective choice proves once again the complexity and paradox of the theory of collective decision-making.

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Надійшла до редколегії 12.03.2013