DERIVATION OF THE DIRAC AND MAXWELL EQUATIONS FROM THE FIRST PRINCIPLES OF RELATIVISTIC CANONICAL QUANTUM MECHANICS

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Abstract – **The derivation of the Dirac and Maxwell equations from the first principles of relativistic canonical quantum mechanics (RCQM) is demonstrated. The formalism of the RCQM is constructed on the basis of a self-consistent system of axioms. The important conclusion, that an arbitrary physical and mathematical information, which contains in the model of RCQM, directly and unambiguously is translated into the corresponding information in the model given by the Dirac equation, is formulated.**

I. INTRODUCTION

The significance of the Dirac equation

$$(\partial_0 + iH_{\mathrm{D}})\psi(x) = 0;\ H_{\mathrm{D}} \equiv \vec{\alpha}\cdot\vec{p} + \beta m,\ \partial_\mu \equiv \frac{\partial}{\partial x^\mu},\ \vec{\alpha} = \gamma^0\vec{\gamma},\ \beta = \gamma^0,\ \hbar = c = 1, \quad (1)$$

$$\gamma^\mu:\ \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu},\ \ g = \mathrm{diag}(1,-1,-1,-1), \quad (2)$$

and its generalizations in the different models of theoretical physics (QED, QHD, etc.) is well known.

Let us recall only that the first analysis of this equation enabled Dirac to give the theoretical prediction of the positron, which was discovered experimentally by Anderson in 1932. Taking into account the significance of the Dirac equation for theoretical physics, it is useful to demonstrate not the historical but logistically-scientific analysis of the roots of this equation appearance.

In many handbooks and monographs of quantum field theory the declaration is presented that the Dirac equation is the result of linearization of the Klein – Gordon – Fock (KGF) equation

$$(\Box + m^2)\varphi(x) \equiv (\partial_t^2 - \Delta + m^2)\varphi(x) = 0 \quad (3)$$

with respect to the derivatives ∂_μ. Indeed, it follows from the properties (2) of γ^μ matrices that

$$(i\gamma^\mu\partial_\mu - m)^2 = -(\partial_t^2 - \Delta + m^2) = 0. \quad (4)$$

It means that each solution of the Dirac equation (1) satisfies ***identically*** the KGF equation (3). Sometimes it is asserted in this connection that the KGF equation itself "is derived" from the relativistic relation $E^2 = \vec{p}^2 + m^2$ between the energy E, momentum $\vec{p}$ and rest mass m. And the Schrödinger equation for a single particle "follows" from the non-relativistic form $T = \vec{p}^2/2m$ of the particle kinetic energy.

Note in this connection that each of three equations Schrödinger, KGF and Dirac are ***independent and basic axioms*** of corresponding models in theoretical physics.

Therefore, for the modern analysis of physically-heuristic content of the Dirac equation, the following question is actual. ***Whether there exists "a more fundamental model" of a spin s=1/2 particle-antiparticle***

doublet, from which the Dirac equation (and its content) would follow as a consequence directly and unambiguously.

In this report the positive answer on this question is given. Here it is shown that axiomatically formulated relativistic canonical quantum mechanics (RCQM) of a particle-antiparticle doublet of spin *s*=1/2 should be chosen as such a model (the illustration on the example of an electron-positron doublet, $e^{-}e^{+}$-doublet, is presented).

II. Relativistic canonical quantum mechanics

The ***axioms of the model*** are formulated on the level of the well-known von Neumann's monograph. Requirements of such physically verified principles as ***the principle of relativity with respect to the tools of cognition (PRTC), the principle of heredity (PH)*** with both classical mechanics of single mass point and non-relativistic quantum mechanics (and ***the principle of correspondence (PC)*** with these theories), and also ***the Einstein principle of relativity (EPR)***, are taken into consideration. The last principle requires first of all ***the special relativity*** **(SR)** to be taken into account.

The ***basic axioms*** of the model (we present here the brief consideration) as the mathematical assertions have the form.

1. ***On the space of states***. The space of states of isolated $e^{-}e^{+}$-doublet in arbitrarily-fixed inertial frame of reference (IFR) in its $\vec{x}$-realization is the Hilbert space

$$\mathrm{H}^{3,4} = \mathrm{L}_2(\mathrm{R}^3)\otimes \mathrm{C}^{\otimes 4};\ \mathrm{L}_2(\mathrm{R}^3) = \{f \equiv (f^{\alpha}):\\ \mathrm{R}^3 \to \mathrm{C}^{\otimes 4};\ \int d^3x \left| f(t,\vec{x})\right|^2 < \infty\},\ \alpha = 1,2,3,4, \tag{5}$$

(similarly, in momentum $\vec{p}$-realization), where $\vec{x}$ and $\vec{p}$ are the operators of canonically conjugated dynamical variables of $e^{-}e^{+}$-doublet, and the vectors f, $\tilde{f}$ in $\vec{x}$ and $\vec{p}$ realizations are linked by the 3-dimensional Fourier transformation (the variable *t* is the parameter of time-evolution).

2. ***The mathematical correctness of the consideration*** demands the application of the rigged Hilbert space $\mathrm{S}^{3,4} \subset \mathrm{H}^{3,4} \subset {}^{\times}\mathrm{S}^{3,4}$. Here the Schwartz test function space $\mathrm{S}^{3,4} \subset \mathrm{H}^{3,4}$, which is the verified tool of the PRTK realization, is kernel (i. e., it is dense both in $\mathrm{H}^{3,4}$ and in the space ${}^{\times}\mathrm{S}^{3,4}$ of the generalized Schwartz functions). Such application allows us to fulfill, without any loss of generality, all necessary derivations in the space $\mathrm{S}^{3,4}$ on the level of correct differential and integral calculus.

3. ***On the time evolution of the space vectors.*** The time dependence of the state vectors $f \in \mathrm{H}^{3,4}$ (time *t* is the parameter of evolution) is given either in the integral form by the unitary operator

$$u(t_0,t) = \exp[-i\widehat{\omega}(t-t_0)];\ \widehat{\omega} \equiv \sqrt{-\Delta + m^2}, \tag{6}$$

(below is put $t_0 = 0$), or in the differential form by the Schrödinger – Foldy equation of motion

$$(\partial_0 + i\widehat{\omega})f(x) = 0;\ x \in \mathrm{M}(1,3), \tag{7}$$

where operator $\widehat{\omega}$ is the relativistic analog of the energy operator (Hamiltonian) of non-relativistic quantum mechanics. The Minkowski space-time M(1,3) is pseudo Euclidean with metric $g = \mathrm{diag}(1,-1,-1,-1)$.

4. ***On the fundamental dynamical variables.*** The dynamical variable $\vec{x} \in \mathrm{R}^3 \subset \mathrm{M}(1,3)$ (as well as the variable $\vec{k} \in \mathrm{R}^3_{\vec{k}}$) represents the external degrees of freedom of $e^{-}e^{+}$-doublet. The spin $\vec{s}$ of $e^{-}e^{+}$-doublet is the first in the list of the carriers of the internal degrees of freedom. Taking into account the Pauli principle and the fact, that experimentally positron is observed as the mirror reflection of the electron, the operators of e^-e^+ - doublet spin and the charge sign are taken in the form

$$\vec{s} = \frac{1}{2}\begin{vmatrix}\vec{\sigma} & 0 \\ 0 & -C\vec{\sigma}C\end{vmatrix},\ g = -\gamma^0 \equiv \begin{vmatrix}-\mathrm{I}_2 & 0 \\ 0 & \mathrm{I}_2\end{vmatrix},\ \mathrm{I}_2 \equiv \begin{vmatrix}1 & 0 \\ 0 & 1\end{vmatrix}, \tag{8}$$

where $\vec{\sigma}$ are the standard Pauli matrices, C is the operator of complex conjugation.

5. ***On the algebra of observables.*** Using the operators of canonically conjugated coordinate $\vec{x}$ and momentum $\vec{p}$ ($[x^j, p^\ell] = i\delta_{j\ell}$ in $\mathrm{H}^{3,4}$), being completed by the operators $\vec{s}$ and g, we construct the algebra of observables (according to the PH) as the Hermitian functions of 10 $(\vec{x}, \vec{p}, \vec{s}, \text{-}\gamma^0)$ generating elements of the algebra.

6. ***On the relativistic invariance of the theory.*** This invariance (realization of the SR) is ensured by the proof of the equation (7) invariance with respect to the induced representations of the universal covering $\mathscr{P} \supset \mathscr{L} = \mathrm{SL(2,C)}$ of proper ortochronous Poincare group $\mathrm{P}_+^\uparrow \supset \mathrm{L}_+^\uparrow = \mathrm{SO}(1,3)$.

7. ***On the Clifford – Dirac algebra.*** The operators (4) belong to the physically verified quantummechanical representation of the 29-dimensional extended real Clifford – Dirac algebra [1]. This representation is generating by the quantummechanical representation of the γ^{A} matrices

$$\begin{aligned}&\widehat{\gamma}^1 = \gamma^1 C,\ \widehat{\gamma}^2 = \gamma^0\gamma^2 C,\ \widehat{\gamma}^3 = \gamma^3 C,\ \widehat{\gamma}^4 = \gamma^0\gamma^4 C,\\ &\widehat{\gamma}^5 = \gamma^1\gamma^3 C,\ \widehat{\gamma}^6 = -i\gamma^2\gamma^4 C,\ \widehat{\gamma}^7 = i,\end{aligned} \tag{9}$$

where γ^μ are the Dirac matrices in standard Pauli – Dirac representation, and $\widehat{\gamma}^{\mathrm{A}}\widehat{\gamma}^{\mathrm{B}} + \widehat{\gamma}^{\mathrm{B}}\widehat{\gamma}^{\mathrm{A}} = -2\delta^{\mathrm{AB}}$.

II. The derivation of the Dirac equation

The Dirac equation for the spinor field directly and unambiguously follows from the Schrödinger – Foldy equation (7), both in the Foldy – Wouthuysen (FW) [2] and in the standard Pauli – Dirac representations, on the basis of the following operator equalities:

$$v(\partial_0 + i\widehat{\omega})v = (\partial_0 + i\gamma^0\widehat{\omega}),\quad V^+v(\partial_0 + i\widehat{\omega})vV^- = \partial_0 + i(\vec{\alpha}\cdot\vec{p} + \beta m), \tag{10}$$

where operators v and $V^\pm$ have the form

$$v \equiv \begin{vmatrix}\mathrm{I} & 0 \\ 0 & C\mathrm{I}\end{vmatrix},\ V^\pm \equiv \frac{\pm i\gamma^\ell\partial_\ell + \widehat{\omega} + m}{\sqrt{2\widehat{\omega}(\widehat{\omega} + m)}}. \tag{11}$$

An arbitrary solution ψ of the Dirac equation (1) directly and unambiguously follows from the corresponding solution f of the Schrödinger – Foldy equation (7) in RCQM:

$$\psi^{\text{Dirac}}(x) = V^+\phi^{\text{Foldy-Wouthuysen}}(x) = V^+vf^{\text{Schrodinger-Foldy}}(x), \tag{12}$$

Here $f(x)$ is the solution of the Schrödinger – Foldy equation (3) and $\phi(x)$ is the solution of the FW equation [2]

$$(\partial_0 + i\gamma^0 \widehat{\omega})\phi(x) = 0,\ \widehat{\omega} \equiv \sqrt{-\Delta + m^2}. \tag{13}$$

It means that the Dirac equation is derived from the first principles of the RCQM.

III. CONCLUSIONS

The transition from the equation of RCQM to the Dirac equation is fulfilled by the non-singular operator

$$V \equiv \frac{i\gamma^\ell \partial_\ell + \widehat{\omega} + m}{\sqrt{2\widehat{\omega}(\widehat{\omega}+m)}} \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & C & 0 \\ 0 & 0 & 0 & C \end{vmatrix},\ V^{-1} \equiv \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & C & 0 \\ 0 & 0 & 0 & C \end{vmatrix} \frac{-i\gamma^\ell \partial_\ell + \widehat{\omega} + m}{\sqrt{2\widehat{\omega}(\widehat{\omega}+m)}},\quad VV^{-1} = V^{-1}V = \mathrm{I}.. \tag{14}$$

It is evident that ***non-singular transformation of the main object*** (and of associated with this object observables of physical quantities) ***can not insert any "new physics"*** (new physical consequences). Therefore, the following ***assertion*** is valid.

An arbitrary physical and mathematical information, which contains in the model of RCQM, directly and unambiguously is translated into the corresponding information in the model of the Dirac equation.

In order to derive the Maxwell equations from the Schrödinger – Foldy equation (7) of RCQM one has to substitute into the Dirac equation $(i\gamma^\mu \partial_\mu - m)\psi = 0$ instead of ψ the following 4-column combination of electromagnetic $(\vec{E},\vec{H})$ and scalar (E^0, H^0) fields:

$$\psi = \text{column}\left| E^3 + iH^0, E^1 + iE^2, E^0 + iH^3, -H^2 + iH^1 \right| \tag{15}$$

or to multiply the operators (14) by the nonsingular operator W and W^{-1} respectively

$$W = \frac{1}{2}\begin{vmatrix} 0 & (C+1) & 0 & (C-1) \\ 0 & i(C-1) & 0 & i(C+1) \\ (C+1) & 0 & (C-1) & 0 \\ (C-1) & 0 & (C+1) & 0 \end{vmatrix},\ W^{-1} = \frac{1}{2}\begin{vmatrix} 0 & 0 & (C+1) & (C-1) \\ (C+1) & i(C+1) & 0 & 0 \\ 0 & 0 & (C-1) & (C+1) \\ (C-1) & i(C-1) & 0 & 0 \end{vmatrix},\ WW^{-1} = W^{-1}W = 1, \tag{16}$$

which transforms the Dirac 4-component ψ and γ^μ matrices into the following expressions $\tilde{\psi} = W\psi \equiv \begin{vmatrix} \vec{E} - i\vec{H} \\ E^0 - iH^0 \end{vmatrix}$, $\tilde{\gamma}^\mu = W\gamma^\mu W^{-1}$. Of course one must put *m* equal to zero in the resulting formulae (and to put $E^0 = H^0 = 0$, if the free Maxwell equations are under consideration). For the complete set of 8 transformations like (15) see our publications on m=0 case, e. g. [3] and references therein.

Nevertheless, we prefer another derivation of the Maxwell equations from the RCQM, in which the equation (7) is considered together with the specific Maxwell-type representation of the γ^{A} matrices (9).

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