Quasiclassical theory of the Dirac equation with a scalar-vector interaction and its applications in the physics of heavy-light mesons

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Abstract

We construct a relativistic potential quark model of D, D_s, B, and B_s mesons in which the light quark motion is described by the Dirac equation with a scalar-vector interaction and the heavy quark is considered a local source of the gluon field. Within the quasiclassical approximation, we obtain simple asymptotic formulas for the energy and mass spectra and for the mean radii of D, D_s, B, and B_s mesons, which ensure a high accuracy of calculations even for states with the radial quantum number $n_r \sim 1$. We show that the fine structure of P-wave states in heavy-light mesons is primarily sensitive to the choice of two parameters: the strong-coupling constant α_s and the coefficient λ of mixing of the long-range scalar and vector potentials $S_{\mathrm{l.r.}}(r)$ and $V_{\mathrm{l.r.}}(r)$.

1 Introduction

The heavy-light quark-antiquark ($Q\bar{q}$) systems, being QCD analogues of relativistic hydrogen-like atoms, are ideal objects for investigations and permit verifying quantum theory results very precisely experimentally. Theoretical description of the mass spectra and decay probabilities of such composite objects requires constructing a consistent theory of bound states, which should be based on the fundamental principles of local quantum field theory and use its apparatus [1]. But calculating these characteristics of composite systems directly in the local quantum field theory is not always possible, because the only known calculation method in this theory is still based on the perturbation theory, while the nature of creating a bound state of interacting particles must undoubtedly be determined by nonperturbative effects.

The Dirac equation with a mixed scalar-vector interaction plays an important role in the contemporary development of the relativistic theory of bound states. It is valuable because it provides an adequate mathematical model for a wide circle of problems in hadronic physics in which it is possible to transit consistently from a two-particle problem to the external field approximation. This equation indicates the presence of the spin and spin moment for the quark and antiquark, and the problems of describing fine and superfine structures in the energy spectra of heavy-light ($Q\bar{q}$) mesons, which are the QCD analogues of hydrogen-like atoms, arise naturally from this equation.

The mathematical theory of the Dirac equation with a scalar-vector interaction was developed in [2] (see [3, 4, 5] for a detailed bibliography). But in most cases, attempts to construct exact solutions of this equation for more or less realistic potentials encounter difficulties that have not yet been overcome. When constructing approximate methods for investigating bound states of the Dirac equation, nonperturbative methods, in which the expansion parameter in the potential is not considered small, are especially important. One of the most widely used among these methods is the method of asymptotic expansion in the Planck constant $\hbar$, which is called the quasiclassical approximation.

The construction of quasiclassical solutions of the spinor equation with a scalar-vector interaction was reported in [6, 7]. The scheme of quasiclassical quantization proposed in [7] allows to make clear a connection of quasiclassical asymptotic behavior in spectral problems for the Dirac equation in external scalar and vector fields with the Lorentz structure of interaction potentials corresponding to them.

2 Quasiclassical approximation for the Dirac equation with a vector and scalar interaction potential

The problem of describing the motion of a relativistic spin-1/2 particle in a central field composed of scalar and vector external fields after the separation of variables reduces to solving the system of radial Dirac equations ($c = 1$)

$$\left.\begin{aligned} &\hbar\frac{dF}{dr} + \frac{\tilde{k}}{r}F - \left[(E - V(r)) + (m + S(r))\right]G = 0, \\ &\hbar\frac{dG}{dr} - \frac{\tilde{k}}{r}G + \left[(E - V(r)) - (m + S(r))\right]F = 0. \end{aligned}\right\} \tag{1}$$

Hereafter, we use the notation $F(r) = rf(r)$ and $G(r) = rg(r)$, where $f(r)$ and $g(r)$ are the radial functions for the respective upper and lower components of the Dirac bispinor [8], E and m are the total energy and rest mass of the particle, $S(r)$ is the Lorentz-scalar potential, and the potential $V(r)$ up to a multiplier coincides with the zeroth (temporal) component of the four-vector potential $A_\mu = (A_0, \mathbf{A})$, where $\mathbf{A} = 0$, $V(r) = -eA_0(r)$, and $e > 0$. In system (1), $\tilde{k} = \hbar k$, where the quantum number $k = \mp(j + 1/2)$ for $l = j \mp 1/2$, j is the total angular moment of the fermion, and l is the orbital moment (for the upper component of $F(r)$), and hence $|k| = j + 1/2 = 1, 2, \ldots$.

In [7] the system (1) was consecutively solved using the known technique of left and right eigenvectors of the homogeneous system. For the effective potential (EP) of the barrier type (see Fig. 1)

$$U(r, E) = \frac{E}{m}V + S + \frac{S^2 - V^2}{2m} + \frac{k^2}{2m\,r^2}, \tag{2}$$

semiclassical expressions were obtained for the wave functions in the classically forbidden and permitted bands and also the quantization condition determining

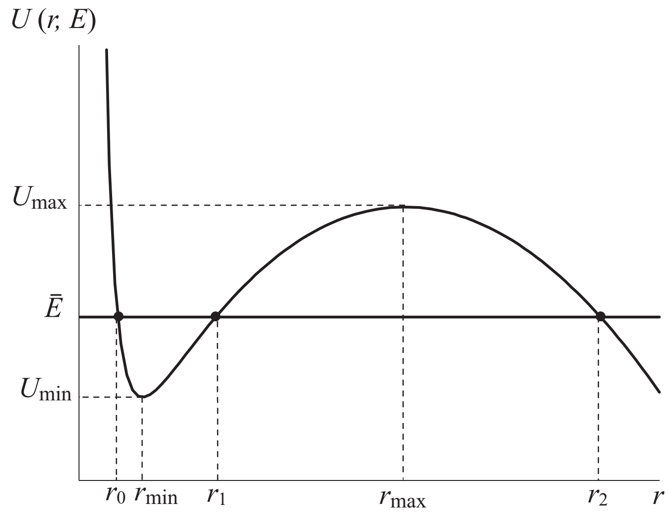

Figure 1: The form of the EP $U(r,E)$ of the barrier type; r_0, r_1, and r_2 are roots of the equation $p^2 = 0$.

the energy (position) of the bound state E in the mixture of the scalar and vector potentials:

$$\int\limits_{r_0}^{r_1} \left(p + \frac{k\,w}{p\,r}\right) dr = \left(n_r + \frac{1}{2}\right)\pi, \qquad w = \frac{1}{2}\left(\frac{V' - S'}{m + S + E - V} - \frac{1}{r}\right). \tag{3}$$

Here, $n_r = 0, 1, 2, \ldots$ is the radial quantum number, and

$$p(r) = \left[(E - V(r))^2 - (m + S(r))^2 - (k/r)^2\right]^{1/2} \tag{4}$$

is the semiclassical momentum for the radial motion of the particle in the potential well $r_0 < r < r_1$, where r_0 and r_1 are the turning points, i.e., the roots of the equation $p^2(r) = 0$.

The new quantization rule (3) differs from the standard Bohr-Sommerfeld quantization condition [27] by the relativistic expression for the momentum $p(r)$ and by the correction proportional to $w(r)$, which takes into account the spin-orbital interaction and results in the splitting of levels with different signs of the quantum number k.

3 The dependence of the EP $U(r, E)$ on the Lorentz structure of the external field

The simplest model of the interaction of a relativistic spin-1/2 particle simultaneously with both scalar and vector external fields, which we meet below when calculating the quasiclassical spectrum of relativistic bound states (see Sec. 4), is governed by the potentials

$$\begin{aligned} V(r) &\equiv V_{\mathrm{Coul}}(r) + V_{\mathrm{l.r.}}(r) = -\frac{\xi}{r} + \lambda v(r), \\ S(r) &\equiv S_{\mathrm{l.r.}}(r) = (1-\lambda)v(r), \quad v(r) = \sigma r + V_0, \end{aligned} \tag{5}$$

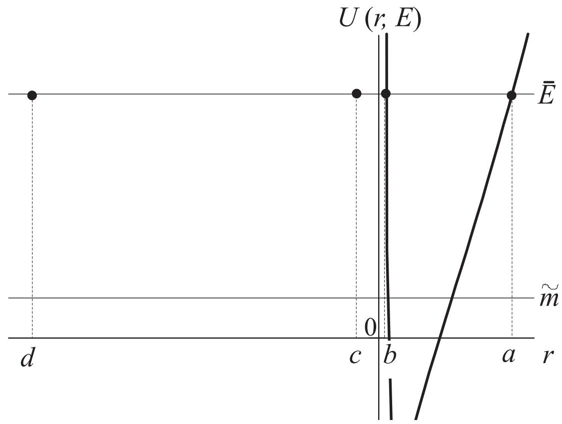

Figure 2: The EP $U(r,E)$ of Dirac system (1) with potential (5) in the case where $\lambda < 1/2$, $\sigma > 0$, and $\widetilde{E} > \widetilde{m}$; a, b, c and d are the quasimomentum roots in (8).

where V_0 is a real constant, ξ is the Coulomb coefficient, and λ is the parameter of mixing between the vector and scalar long-range potentials $V_{\text{l.r.}}(r)$ and $S_{\text{l.r.}}(r)$; $0 \leqslant \lambda \leqslant 1$. Below in this section, we do not restrict the value or even the sign of the parameter σ.

Our goal is to investigate the behavior of the EP $U(r,E)$ at large and small r. Substituting $V(r)$ and $S(r)$ of form (5) in (2) and keeping only the most singular terms when $r \to 0$ only the leading terms (in r) when $r \to \infty$, we obtain:

$$U\,(r,E) \sim \begin{cases} \dfrac{(1-2\lambda)\sigma^2}{2m}\, r^2 + \ldots,\ r \to \infty,\ \lambda \neq \dfrac{1}{2}, & (6a) \\ \dfrac{E+m}{2m}\,\sigma\, r + \ldots,\ r \to \infty, \qquad \lambda = \dfrac{1}{2}, & (6b) \\ \dfrac{\gamma^2}{2m\, r^2},\ r \to 0,\ \gamma^2 = k^2 - \xi^2. & (6c) \end{cases}$$

Let us now consider behavior of EP $U(r,E)$ and wave functions at large distances r in more detail. First note that only the quadratic term $(S^2 - V^2)/2m$ is essential in the asymptotic domain in formula (2) for $\lambda \neq 1/2$ and has the behavior $(1-2\lambda)\sigma^2 r^2/2m$ when $r \to \infty$. It is hence obvious that for any sign of the parameter σ, the EP $U(r,E)$ of model (5) under consideration (at sufficiently large distances) is an attractive potential for $\lambda > 1/2$ and a repulsive potential for $\lambda < 1/2$.

It is clear from what was said above that for $\lambda < 1/2$, the EP $U(r,E)$ of model (5) is an unboundedly increasing (as r increases) confining potential with only a discrete spectrum of energy levels; it is then essential that the quadratic dependence of the EP $U(r,E)$ on r (and hence the confinement property) appears because of the relativistic terms $(S^2 - V^2)/2m$. An example form of the EP $U(r,E)$ for $\lambda < 1/2$ is shown in Fig. 2.

But for $\lambda > 1/2$ and an arbitrary value of $\sigma \neq 0$, the effective Hamiltonian H of the squared Dirac equation in external field (5) has complex eigenvalues of energy because the EP $U(r,E)$ becomes negative in this case (at sufficiently large distances) and less than the effective particle energy $\bar{E} = (E^2 - m^2)/2m$, which corresponds to attraction. Therefore, for $\lambda > 1/2$, the EP $U(r, E)$ of model (5) has the form of a well separated from the external domain by a wide potential barrier (for $|\sigma| \ll 1$; see Fig. 1). It is obvious that the leading contribution to forming the barrier of the EP $U(r, E)$ comes from the Lorentz-vector component $V_{\rm l.r.}(r)$ of the long-range potential $v(r)$.

We point out one more important particular case realized at $\lambda = 1/2$. Substituting potentials (5) with the value $\lambda = 1/2$ in expression (2), we see that the quadratic dependence of the "tail" of $U(r, E)$ on r disappears and the long-range components $V_{\rm l.r.}(r)$ and $S_{\rm l.r.}(r)$ of the first two terms dominate EP (2) at large r, which results in a practically linear dependence of $U(r, E)$ on r (see (6b)).

4 Quasiclassical description of the energy spectrum of heavy-light quark-antiquark systems

To use the potential approach to describe properties of heavy-light mesons, we must construct the quark-antiquark interaction potential. As is known from QCD, because of the asymptotic freedom property, the Coulomb-type potential of the one-gluon exchange gives the leading contribution at small distances ($r < 0.25\,\mathrm{Fm}$).

As the distance increases, the long-range confining interaction (the confinement), whose actual form has not yet been established in the QCD framework, prevails. The confining potential may have a complicated Lorentz structure (see [10, 11, 12, 13]). Therefore, we assume that the $Q\bar{q}$ interaction is a combination of the following potentials:

a. the one-gluon exchange potential $V_{\rm Coul}(r) = -\xi/r$, where $\xi = 4/3\,\alpha_s$, α_s is the strong coupling constant $\alpha_s(Q) = 12\pi/[(33 - 2N_f)\log(Q^2/\Lambda^2)]$, N_f is the number of quark flavors, and $\Lambda = 360\,\mathrm{MeV}$ is the QCD parameter,

b. the long-range linear scalar confining potential $S_{\rm conf}(r) = (1-\lambda)\,v(r)$, where $v(r)$ is determined by expression (5), and

c. the long-range linear vector potential $V_{\rm conf}(r) = \lambda\,v(r)$.

The total effective quark-antiquark interaction is then described by a combination of the perturbative one-gluon exchange potential $V_{\rm Coul}(r)$ and the scalar and vector long-range confining potentials $S_{\rm conf}(r)$ and $V_{\rm conf}(r)$. Therefore, the potentials S and V are given by (5), where $\sigma = 0.18\,\mathrm{GeV}^2$ is the string tension, V_0 is the constant of the additive shift of the bond energy, and the coefficient λ of mixing between the vector and scalar confining potentials is the adjustable parameter, $0 \leqslant \lambda < 1/2$.

Choosing the mixing coefficient in the range $0 \leqslant \lambda < 1/2$ corresponds to the scalar confinement prevailing. In this case, the EP $U(r, E)$ of our model has the form of a standard oscillator well with a single minimum (at the point $r_{min} \approx \gamma^2/\widetilde{E}\xi$) and no maximums (see Fig. 2). The equation $p^2 = 2m(\bar{E} - U(r, E)) = 0$ determining the turning points then results in the complete fourth-degree algebraic equation $r^4 + f\,r^3 + g\,r^2 + h\,r + l = 0$ with the coefficients

$$f = \frac{2[\widetilde{m}\,(1-\lambda) + \widetilde{E}\lambda]}{(1-2\lambda)\,\sigma},\; g = -\frac{\widetilde{E}^2 - \widetilde{m}^2 - 2\xi\sigma\lambda}{(1-2\lambda)\,\sigma^2},$$
$$h = -\frac{2\widetilde{E}\xi}{(1-2\lambda)\,\sigma^2},\; l = \frac{\gamma^2}{(1-2\lambda)\,\sigma^2}, \tag{7}$$

where $\widetilde{E} = E - \lambda V_0$, $\widetilde{m} = m + (1-\lambda)V_0$ are the characteristic parameters with the respective meanings of the "shifted" energy and the "shifted" mass. This equation has four real roots $d < c < b < a$ determined by the equalities

$$a = -\frac{f}{4} + \frac{1}{2}\left(\Xi + \Delta_+\right),\; b = -\frac{f}{4} + \frac{1}{2}\left(\Xi - \Delta_+\right),$$
$$c = -\frac{f}{4} - \frac{1}{2}\left(\Xi - \Delta_-\right),\; d = -\frac{f}{4} - \frac{1}{2}\left(\Xi + \Delta_-\right). \tag{8}$$

Here, we use the notation

$$\Xi = \left[\frac{f^2}{4} - \frac{2g}{3} + \frac{u}{3}\left(\frac{2}{Z}\right)^{\frac{1}{3}} + \frac{1}{3}\left(\frac{Z}{2}\right)^{\frac{1}{3}}\right]^{\frac{1}{2}},\; \Delta_\pm = \sqrt{F \pm \frac{D}{4\Xi}},\; D = -f^3 + 4fg - 8h,$$
$$Z = v + \sqrt{-4u^3 + v^2}, \quad F = \frac{f^2}{2} - \frac{4g}{3} - \frac{u}{3}\left(\frac{2}{Z}\right)^{1/3} - \frac{1}{3}\left(\frac{Z}{2}\right)^{1/3},$$
$$u = g^2 - 3fh + 12l,\; v = 2g^3 - 9fgh + 27h^2 + 27f^2 l - 72gl.$$

For the potentials under consideration, the quasiclassical momentum is determined by equalities (4) and (5). Using formulas (8), we represent it in the form convenient for what follows ($\sigma > 0$ and $\sigma < 0$)

$$p(r) = |\sigma|\sqrt{1-2\lambda}\,\frac{R(r)}{r} = |\sigma|\sqrt{1-2\lambda}\frac{\sqrt{(a-r)(r-b)(r-c)(r-d)}}{r}. \tag{9}$$

We integrate in quantization condition (3) over the classically allowed domain between the two positive turning points $r_0 = b < r_1 = a$, while the other two turning points ($d < c < 0$) are in the nonphysical domain $r < 0$. Using formula (9), we transform quantization integrals (3) into the sum of the integrals

$$J_1 = \int_b^a p(r)dr = -|\sigma|\sqrt{1-2\lambda}\int_b^a \frac{\left(r^3 + fr^2 + gr + h + lr^{-1}\right)}{R}dr,$$
$$J_2 = \int_b^a \frac{k\,w}{p(r)r}dr = \frac{-k}{2|\sigma|\sqrt{1-2\lambda}}\left[\int_b^a \frac{dr}{(r-\lambda_+)\,R} + \int_b^a \frac{dr}{(r-\lambda_-)\,R}\right], \tag{10}$$

where we introduce the notation

$$\lambda_\pm = -\frac{\widetilde{E} + \widetilde{m} \mp \sqrt{(\widetilde{E} + \widetilde{m})^2 - 4\sigma\xi(1-2\lambda)}}{2\sigma\,(1-2\lambda)}.$$

The particle energy spectrum is determined by quantization condition (3), which, after quantization integrals (10) are evaluated (see Appendix in [14]), becomes the transcendental equation

$$-\frac{2\sqrt{1-2\lambda}}{\sqrt{(a-c)(b-d)}}\left\{\frac{|\sigma|(b-c)^2}{\Re}\left[N_1F(\chi)+N_2E(\chi)+N_3\Pi(\nu,\chi)+N_4\Pi\left(\frac{c}{b}\nu,\chi\right)\right]\right.$$
$$\left.+\frac{k}{2(1-2\lambda)|\sigma|}\left[(b-c)\left(N_5\Pi(\nu_+,\chi)+N_6\Pi(\nu_-,\chi)\right)+N_7F(\chi)\right]\right\}=\left(n_r+\frac{1}{2}\right)\pi, \quad (11)$$

where $F(\chi)$, $E(\chi)$, and $\Pi(\nu,\chi)$ are the complete elliptic integrals of the respective first, second, and third kind (see [15, 16]). The expressions for ν, χ, $\nu_\pm$, $\Re$, and N_i $(i=1,\dots,7)$ are collected in Appendix because they are rather cumbersome.

Finding an "exact" solution of Eq. (11) in the general case is, of course, impossible. To construct the asymptotic behavior of the heavy-light meson energy levels E_{n_rk}, it is necessary to apply asymptotic methods of calculation of quantization integrals just as in Sec. 4 of [14]. This imposes some restrictions in calculation of shifts of quasistationary levels and its widths for both small values of intensity s of radial-constant (scalar-vector) long-range field and not too large ones. Namely, the quantity $\widetilde{m}=m+(1-\lambda)V_0$ divides the range of $\widetilde{E}=E-\lambda V_0$ into the two domains, in which the energy levels has a various asymptotic behavior. Consider some of the most typical situations connected with the relative values of the energy $\widetilde{E}$ and level $\widetilde{m}$.

Case A: Let $\sigma>0$ and the conditions $\sigma\ll\xi\widetilde{m}^2$ and $\widetilde{E}<\widetilde{m}$ be satisfied. Estimating expressions (8) for the turning points in the approximation $\sigma/\xi\widetilde{m}^2\ll 1$, we can easily obtain

$$a,\,b\approx\frac{\widetilde{E}\xi\pm\theta}{\mu^2}\left[1-\frac{\widetilde{E}\xi\pm\theta}{\mu^4}\left(\eta_1\pm\frac{\widetilde{m}\xi\eta_2}{\mu}\right)\sigma\right],$$
$$c\approx-\frac{\widetilde{m}-\widetilde{E}}{\sigma}-\frac{\xi}{\widetilde{m}-\widetilde{E}},\; d\approx-\frac{\widetilde{m}+\widetilde{E}}{\sigma(1-2\lambda)}+\frac{\xi}{\widetilde{m}+\widetilde{E}}. \quad (12)$$

Hereafter, we use the notation

$$\theta=\sqrt{(\widetilde{E}\,k)^2-(\widetilde{m}\,\gamma)^2},\; \mu=\sqrt{\widetilde{m}^2-\widetilde{E}^2},\; \eta_1=(1-\lambda)\widetilde{m}+\lambda\widetilde{E},\; \eta_2=\lambda\widetilde{m}+(1-\lambda)\widetilde{E}. \quad (13)$$

It is also obvious from (12) that for small positive values of σ, the turning points c and d are sufficiently far from the two points a and b and tend to $-\infty$ in the limit as $\sigma\to 0$.

In this case, the derivation of asymptotic expansions of quantization integrals (3) in a small parameter $\sigma/\xi\widetilde{m}^2$ is carried out just as in item A of Sec. 4 of [14] and gives the expression for the level energy

$$E_{n_rk}=\widetilde{E}_0+\lambda V_0+\frac{\sigma}{2\xi\widetilde{m}^2}\left[\left(\frac{\xi^2\widetilde{m}^2}{\mu_0^2}-k^2\right)\eta_{10}+\left(\frac{2\xi^2\widetilde{m}\widetilde{E}_0}{\mu_0^2}-k\right)\eta_{20}\right]+O\left(\left(\frac{\sigma}{\xi\widetilde{m}^2}\right)^2\right), \quad (14)$$

where $\widetilde{E}_0 = \widetilde{m}\left[1+\xi^2/\left(n_r'+\gamma\right)^2\right]^{-1/2}$, $n_r' = n_r + (1+\operatorname{sgn} k)/2$, and the quantities μ_0, η_{10}, and η_{20} are obtained from μ, η_1, and η_2 by substituting $\widetilde{E}$ for $\widetilde{E}_0$. The previously accepted condition $\sigma > 0$ is unnecessary here because this result remains applicable also in the case of negative values of the parameter σ.

Case B: In the domain $\widetilde{E} > \widetilde{m}$ and $\sigma > 0$, which is of actual importance for the physics of heavy-light mesons, a small dimensionless parameter $\sigma\gamma/\widetilde{E}^2$ appears in the spectral problem. Imposing the condition $\sigma\gamma/\widetilde{E}^2 \ll 1$, we can easily obtain the approximate expressions for the turning points from exact formulas (8):

$$a \approx \frac{\widetilde{E}-\widetilde{m}}{\sigma} + \frac{\xi}{\widetilde{E}-\widetilde{m}}, \quad b,\, c \approx \frac{-\widetilde{E}\xi \pm \theta}{\widetilde{E}^2-\widetilde{m}^2}, \quad d \approx -\frac{\widetilde{E}+\widetilde{m}}{\sigma(1-2\lambda)} + \frac{\xi}{\widetilde{E}+\widetilde{m}}. \tag{15}$$

As can be seen from these formulas, the turning points a and b are rather distant from each other.

Further we shall give only the recipe of evaluation of the quantization integrals $J_{1,2}$. Just as in the item B of Sec. 4 of [14], we now find a point $\widetilde{r}$ that divides the integration domain $b \leqslant r \leqslant a$ into the domain $b \leqslant r \leqslant \widetilde{r}$ where the Coulomb potential prevails and the domain $\widetilde{r} \leqslant r \leqslant a$ where the long-range potential $v(r)$ prevails. The most natural seems to find a point $\widetilde{r}$ where the long-range potential $v(r)$ is equal to the Coulomb potential. From this requirement, we have $\widetilde{r} \approx (\widetilde{E}\xi/\eta_1\sigma)^{1/2}$. We calculate integrals (10) by expanding the quasimomentum $p(r)$ in a power series in the parameters $r/a \ll 1$ and $r/|d| \ll 1$ in the domain $b \leqslant r \leqslant \widetilde{r}$ and in the small parameters $b/r \ll 1$ and $|c|/r \ll 1$ in the domain $\widetilde{r} \leqslant r \leqslant a$. When we add the asymptotic expansions of integrals over $b \leqslant r \leqslant \widetilde{r}$ and $\widetilde{r} \leqslant r \leqslant a$, the final result will not contain the quantity $\widetilde{r}$. So, we have obtained the equation

$$\begin{aligned}&\frac{\eta_1\sqrt{\widetilde{E}^2-\widetilde{m}^2}}{2\sigma(2\lambda-1)} - \eta\left(\frac{\eta_2^2}{2\sigma(2\lambda-1)} + \lambda\,\xi\right) - \frac{\widetilde{E}\xi}{\sqrt{\widetilde{E}^2-\widetilde{m}^2}}\log\left(\frac{\sigma\,\eta_2\theta}{4\,e\,(\widetilde{E}^2-\widetilde{m}^2)^2}\right)\\ &-\gamma\arccos\left(\frac{-\widetilde{E}\xi}{\theta}\right) - \frac{\operatorname{sgn} k}{2}\arccos\left(\frac{-\widetilde{m}\xi}{\theta}\right) = \left(n_r+\frac{1}{2}\right)\pi.\end{aligned} \tag{16}$$

where $\eta = (1-2\lambda)^{-1/2}\arccos(\eta_1/\eta_2)$. If we expand the left-hand side of (16) in $\widetilde{m}/\widetilde{E} \ll 1$ up to the terms of the third order, then we obtain the transcendental equation for the level energy $E_{n_r k}$ which we solve by the method of consecutive iterations. Thus, we arrive at the expression for the energy (within $O(\sigma\gamma/\widetilde{E}^2)$)

$$\begin{aligned}E_{n_r k}^{\mathrm{WKB(as)}} = \zeta^{-1}\Bigg\{&B + \bigg(B^2 + \zeta\Big[2\sigma(1-2\lambda)\Big(\xi\log\frac{\sigma|k|(1-\lambda)}{4{\widetilde{E}^{(0)}}^2} + 3\xi + \lambda\xi A + \pi N\Big)\\ &+\lambda\widetilde{m}^2(1-\lambda A)\Big]\bigg)^{1/2}\Bigg\} + \lambda V_0,\end{aligned} \tag{17}$$

where

$$\zeta = (1-\lambda)^2 A - \lambda - \frac{2\sigma\xi(1-2\lambda)}{\widetilde{E}^{(0)\,2}}, \quad B = (1-\lambda)(1-\lambda A)\widetilde{m} - \frac{4\sigma\xi(1-2\lambda)}{\widetilde{E}^{(0)}},$$
$$A = \frac{\arccos\left(\frac{\lambda}{1-\lambda}\right)}{\sqrt{1-2\lambda}}, \quad N = n_r + \frac{1}{2} + \frac{\operatorname{sgn} k}{4} + \frac{1}{\pi}\left(\gamma \arccos\left(-\frac{\xi}{|k|}\right) - \xi\right),$$

and $\widetilde{E}^{(0)} = E^{(0)} - \lambda V_0$. Here, $E^{(0)}$ is the zeroth approximation for the energy on which the quantity $E_{n_r k}$ depends rather weakly, and we can set $E^{(0)} \approx E_{n_r k}(\xi)|_{\xi=0}$ in most cases.

We have obtained formula (17) for the energy levels $E_{n_r k}$, which depend nonanalytically on the string tension σ and which therefore cannot be obtained in the perturbation theory framework.

The results of calculating the energy levels $E^{\mathrm{WKB}}_{n_r k}$ and $E^{\mathrm{WKB(as)}}_{n_r k}$ based on transcendental equation (11) and asymptotic formula (17) together with the exact values of $E_{n_r k}$ obtained by solving the Dirac equation numerically are presented in Table 1 for $n_r = 0, 1, 2$ and $k = \pm 1, \pm 2$. In these calculations, we set the values of α_s, λ, V_0, $m_{u,d}$, and m_s to those used in QCD to describe the states of $B(b\overline{u}$ or $b\overline{d})$ and $B_s(b\overline{s})$ mesons. As can be seen in Table 1, the quasiclassical values $E^{\mathrm{WKB}}_{n_r k}$ and $E^{\mathrm{WKB(as)}}_{n_r k}$ ensure the respective 1 % and 2 % accuracies (except the energy of states with the radial quantum number $n_r = 0$, for which the accuracy of both formulas is about 8 %). The accuracy of determining $E_{n_r k}$ from quasiclassical formula (17) is therefore such that the first-order approximation usually suffices for practical purposes.

5 The mass spectrum of heavy-light quark systems

In the leading order in $1/m_Q$, the mass spectrum of meson states with one heavy quark is given by the expression [3, 17, 18, 19]

$$M^{\mathrm{theor}}_{n_r k}(Q\bar{q}) = E_{n_r k} + \sqrt{E^2_{n_r k} - m_q^2 + m_Q^2}, \tag{18}$$

where m_Q and m_q are the masses of the heavy quark Q and the light quark $\bar{q}$ constituting the $Q\bar{q}$ meson. Calculating the mass spectrum of $Q\bar{q}$ mesons therefore reduces to consistently calculating the energy eigenvalues of Dirac equation (1) in composite field (5) whose source here is the heavy quark Q.

Because the Hamiltonian of Eq. (1) does not contain terms describing the interaction of the spin of the Q quark with the orbital and spin moments $\vec{l}$ and $\vec{s}_q$ of the light antiquark, both the spin moment $\vec{S}_Q$ of the heavy quark Q and the total moment $\vec{j} = \vec{s}_q + \vec{l}$ of the light antiquark $\bar{q}$ are two separate integrals of motion. This allows classifying the states by the quantum numbers $j = \frac{1}{2}, \frac{3}{2}, \ldots$ of the operator of the total moment of the light antiquark $\bar{q}$, while the states of the total moment of the composite $Q\bar{q}$ system $\vec{J} = \vec{j} + \vec{S}_Q$ are degenerate with respect to the orientation of the spin $\vec{S}_Q$ of the heavy quark Q. Two almost degenerate states of the composite $Q\bar{q}$-system with $J = j \pm 1/2$ in the spin symmetry approximation

Table 1: The results of calculating the level energies $E_{n_r k}^{\text{WKB}}$ (based on transcendental equation (11)) and $E_{n_r k}^{\text{WKB(as)}}$ (based on quasiclassical expression (17)) and also the exact values of $E_{n_r k}$ calculated at the parameter values $\alpha_s = 0.3$, $\lambda = 0.3$, $V_0 = -0.45\,\text{GeV}$ and $m_{u,d} = 0.33\,\text{GeV}$, $m_s = 0.5\,\text{GeV}$ (the energies are measured in GeV).

		$b\overline{u}$, $b\overline{d}$				$b\overline{s}$	
L_j	(n_r, k)	$E_{n_r k}$	$E_{n_r k}^{\text{WKB}}$	$E_{n_r k}^{\text{WKB(as)}}$	$E_{n_r k}$	$E_{n_r k}^{\text{WKB}}$	$E_{n_r k}^{\text{WKB(as)}}$
$S_{1/2}$	$(0, -1)$	0.4327	0.4408	0.4729	0.5248	0.5322	0.5623
	$(1, -1)$	0.8796	0.8838	0.8943	0.9750	0.9791	0.9912
	$(2, -1)$	1.1978	1.2009	1.2066	1.2946	1.2976	1.3049
$P_{3/2}$	$(0, -2)$	0.7355	0.7373	0.7504	0.8376	0.8392	0.8460
	$(1, -2)$	1.0880	1.0892	1.0947	1.5790	1.5900	1.1927
	$(2, -2)$	1.3658	1.3667	1.3699	1.4650	1.4659	1.4685
$P_{1/2}$	$(0, 1)$	0.7249	0.7293	0.7030	0.8235	0.8278	0.7985
	$(1, 1)$	1.0701	1.0733	1.0594	1.1696	1.1728	1.1572
	$(2, 1)$	1.3470	1.3496	1.3405	1.4466	1.4492	1.4390
$D_{3/2}$	$(0, 2)$	0.9661	0.9671	0.9343	1.0655	1.0665	1.0315
	$(1, 2)$	1.2588	1.2596	1.2385	1.3583	1.3591	1.3369
	$(2, 2)$	1.5058	1.5066	1.4914	1.6052	1.6059	1.5901

[20] therefore correspond to each state of the Dirac equation with the given j and with the spatial parity $P = (-1)^{l+1}$.

The values $l = 0$ (S states in the quark-antiquark model) and $j = 1/2^-$ correspond to the ground state of the $Q\bar{q}$ meson. This doublet consists of two states $J^P = (0^-, 1^-)$. In the case $l = 1$ (the P state in the quark model), we have two states with $j = 1/2^+$ and $j = 3/2^+$ and two corresponding doublets $J^P = (0^+, 1^+)$ and $J^P = (1^+, 2^+)$.

In actual $Q\bar{q}$ systems, the degeneracy of doublet states corresponding to different moments $J = j \pm 1/2$ at the given j is broken primarily because of the $\vec{s}_q\,\vec{S}_Q$ interaction. Therefore, to be able to compare our theoretical predictions with experimental data, we present the observation values for the centers of masses of the hyperfine structure (HFS) multiplets in Tables 2, 3; these centers of masses are calculated by the known formula

$$M_{\text{exp}} = \frac{\sum_J (2J+1)\, M_J}{\sum_J (2J+1)}, \tag{19}$$

where M_J is the experimental value of the mass of state with the given J.

Based on these observations, we have tried to describe the spectra of masses of low-lying states of the heavy-light $B(b\bar{u}\,\text{or}\,b\bar{d})$, $B_s(b\bar{s})$, $D(c\bar{u}\,\text{or}\,c\bar{d})$, and $D_s(c\bar{s})$ mesons considering σ and λ to be universal quantities and setting the values of the parameters α_s and V_0 constant in every family of heavy-light mesons allowing them to vary slightly only when passing from one family to another.

We use only one a priori restriction: the value of the coefficient λ must lie in the interval $0 \leqslant \lambda < 1/2$. Comparing the results of calculations based on formulas (11) and (18) with the experimental data [21, 22], we find that the best agreement is reached at $\lambda = 0.3$ and for the parameter choices

$$\sigma = 0.18\,\mathrm{GeV}^2,\ \alpha_s(c\bar{u}\,\mathrm{or}\,c\bar{d}) = 0.386,\ \alpha_s(b\bar{u}\,\mathrm{or}\,b\bar{d}) = 0.3,$$
$$V_0(c\bar{u}\,\mathrm{or}\,c\bar{d}) = -375\,\mathrm{MeV},\quad V_0(b\bar{u}\,\mathrm{or}\,b\bar{d}) = -450\,\mathrm{MeV}.$$

For the masses of u, d, s, c, and b quarks, we used their constituent masses $m_{u,d} = 330\,\mathrm{MeV}$, $m_s = 500\,\mathrm{MeV}$, $m_c = 1550\,\mathrm{MeV}$, and $m_b = 4880\,\mathrm{MeV}$. The mass spectra of D and D_s mesons calculated in this approximation are presented in Table 2.

Table 2: The mass spectrum of D and D_s mesons obtained in the WKB approximation for potentials (5) (masses are expressed in MeV).

		D			D_s			
L_j	(n_r, k)	M_{num}	M_{theor}	M_{exp}	M_{num}	M_{theor}	M_{exp}	
$S_{1/2}$	(0, -1)	1989.1	2001.5	1971.1	2057.2	2069.0	2072	
	(1, -1)	2624.5	2632.3	< 2637	2729.4	2737.4	–	
$P_{3/2}$	(0, -2)	2440.1	2443.2	2447.3	2550.1	2552.1	2559.2(I)	2530.7(II)
	(1, -2)	2979.7	2981.9	–	3105.2	3107.2	–	–
$P_{1/2}$	(0, 1)	2395.2	2403.7	2407.8	2499.7	2508.5	2423.8(I)	2480.9(II)
	(1, 1)	2926.8	2933.4	–	3051.7	3058.5	–	–

The agreement between the model and experiment is in the 3–5% range, except for the masses of states $\mathrm{P}_{3/2}$ and $\mathrm{P}_{1/2}$ of the $c\bar{s}$ system for which the mismatch depends on the interpretation of the $D_{s1}(2536)^{\pm}$ meson with the mass 2535.35±0.34±0.5 MeV and is 10% if we consider it to be the vector state $J^P = 1^+$ belonging to the doublet $j = 3/2^+$ (values (I) for M_{exp} in Table 2) (see [23, 24, 25, 26, 27]) or 4% if we consider it to be the state $J^P = 1^+$ of the doublet $j = 1/2^+$ (see [28, 29]) (values (II) in Table 2). We note that our calculations agree better with the second possibility.

For $b\bar{u}$ and $b\bar{s}$ systems, we obtained a good agreement of our results with the experimental data for the ground state with $j = 1/2^-$ and for the P state with $j = 3/2^+$ (see Table 3). For states in the doublet $j = 1/2^+$, we have only theoretical predictions of other authors. For the $b\bar{u}$ system, our results agree with the data obtained in [30], and a remarkable agreement with the results in [24, 25, 31] was obtained for the $b\bar{s}$ system.

In the leading approximation (in $1/m_Q$), the wave functions and excitation energies of the strange quark in the field of a heavy c or b quark reproduce the corresponding characteristics of heavy-light mesons with light u and d quarks with high accuracy. Therefore, up to an additive upward shift of masses on the value

Table 3: The mass spectrum of B and B_s mesons obtained in the WKB approximation for potentials (5) (masses are expressed in MeV).

		B			B_s		
L_j	(n_r, k)	M_{num}	M_{theor}	M_{exp}	M_{num}	M_{theor}	M_{exp}
$S_{1/2}$	(0, -1)	5320.7	5329.5	5313.5	5407.4	5415.6	5404.8
	(1, -1)	5827.3	5832.2	–	5926.3	5931.2	–
$P_{3/2}$	(0, -2)	5659.6	5661.6	< 5698	5763.7	5765.6	< 5853
	(1, -2)	6076.9	6078.4	–	6185.5	6186.8	–
$P_{1/2}$	(0, 1)	5647.4	5652.4	5751.6 [24]	5747.2	5752.2	5751.8 [24]
				5624 [30]			5753.3 [25]
							5700.5 [30]
							5755.0 [31]
	(1, 1)	6055.1	6059.0	–	6162.8	6166.8	–

of the current mass of the strange quark

$$m_s \approx M[D_s] - M[D] \approx M[B_s] - M[B] \approx 0.1\,\text{GeV}$$

the level systems for D_s and B_s mesons coincides with the respective level systems for D and B mesons if we do not take the level splitting depending on the spin of the heavy quark into account. Further, the spin-orbital splitting of lower states of D_s and B_s mesons for the levels $\mathrm{P}_{3/2}$ and $\mathrm{P}_{1/2}$ is 35% larger than that of the D and B mesons.

Summary

The most important results of investigation performed can be summarized as follows:

1. The relativistic potential quark model of $Q\bar{q}$-mesons in which the light quark motion is described by the Dirac equation with a scalar-vector interaction and the heavy quark is considered a local source of the gluon field is constructed. The effective interquark interaction is described by a combination of the perturbative one-gluon exchange potential $V_{\text{Coul}}(r) = -\xi/r$ and the long-range Lorentz-scalar and Lorentz-vector linear potentials $S_{\text{l.r.}}(r) = (1-\lambda)(\sigma r + V_0)$ and $V_{\text{l.r.}}(r) = \lambda(\sigma r + V_0)$. It is established that the quark confinement arises always when the Lorentz-scalar part $S_{\text{l.r.}}$ of the long-range interquark interaction prevails the Lorentz-vector one $V_{\text{l.r.}}$.

2. Approximative analytical expressions for energy spectrum of heavy-light mesons obtained within quasiclassical approach at $\sigma\gamma/\widetilde{E}^2 \ll 1$ are asymptotically exact in the limit $n_r \to \infty$ and ensure a high accuracy of calculations even for states with the radial quantum number $n_r \sim 1$.

3. In the framework of the considered model we have obtained the satisfactory description of the mass spectrum of D-, D_s-, B-, and B_s-mesons. We show

that the fine structure of P-wave states in heavy-light mesons is primarily sensitive to the choice of two parameters: the strong-coupling constant α_s and the coefficient λ of mixing of the long-range scalar and vector potentials $S_{\mathrm{l.r.}}(r)$ and $V_{\mathrm{l.r.}}(r)$. There is the best agreement between the theoretical predictions and experimental data when the mixing coefficient $\lambda = 0.3$.

Appendix

The quantities introduced in transcendental equation (11) have the form

$$\nu = \frac{a-b}{a-c}, \quad \nu_\pm = \frac{\lambda_\pm - c}{\lambda_\pm - b}\nu, \quad \Re = (1-\nu)\left(\chi^2-\nu\right),$$

$$N_1 = \frac{\chi^2(b-c)}{4} - \frac{3\aleph(b-c)}{8(1-\nu)} - \frac{\left(\chi^2-\nu\right)}{2}(f+3c)$$

$$+\frac{\Re}{(b-c)^2}\left(c^3+c^2f+cg+h+l/c\right), \quad \aleph = \chi^2(3-2\nu)+\nu(\nu-2),$$

$$N_2 = -\frac{\nu}{2}\left[f+3c+\frac{3}{4}\frac{(b-c)\aleph}{\Re}\right],$$

$$N_3 = \frac{1}{2}\left[\frac{3}{4}\frac{(b-c)\aleph^2}{\Re} + \frac{2\Re}{(b-c)}\left(3c^2+2cf+g\right)\right.$$

$$\left.+(b-c)\left(\left(1+\chi^2\right)\nu-3\chi^2\right)+\aleph(f+3c)\right],$$

$$N_4 = -\frac{\Re}{(b-c)}\frac{l}{bc}, \quad N_5 = [(b-\lambda_+)(\lambda_+-c)]^{-1}, \quad N_6 = [(b-\lambda_-)(\lambda_--c)]^{-1},$$

$$N_7 = \frac{2}{(\lambda_+-c)(\lambda_--c)}\left(c+\frac{\widetilde{E}+\widetilde{m}}{2(1-2\lambda)\sigma}\right).$$

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