The WKB method for the Dirac equation
with the vector and scalar potentials

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The WKB approximation is developed for the Dirac equation with the spherically symmetrical vector and scalar potentials. The relativistic wavefunctions are constructed, new quantization rule containing the spin-orbital interaction is obtained. For spherically symmetrical model of the Stark effect the quasi-classical spectrum of relativistic hydrogen-like atom is calculated. Application of the WKB method to the mass spectrum of the hydrogen-like quark systems was done.

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1 Introduction

Except for very special potential, there is no analytic solution to the Dirac equation and, thus, numerical or asymptotic methods must be applied. In many theoretical and applied problems just a possibility to obtain an asymptotic solution allows to carry out fullest analysis of the problem. Therefore there is hardly a necessity to explain importance of creating and investigating asymptotic methods of solving the Dirac equation in detail.

The WKB method is one of the basic and universal approximative methods for the problems of the quantum mechanics and mathematical physics [1–4] which can not be solved exactly or have very complicated analytical solution. Unlike the perturbation theory the given approach is not connected with a smallness of interaction and consequently has a wider applicability region allowing to study qualitative legitimacies in the behaviour and properties of quantum mechanical systems. In the case of the Coulomb field the WKB method has a good accuracy even for small quantum numbers [1–4]. In particular the WKB method has been successfully used for the hydrogen atom in an external electric or magnetic field [1, 5], for the modeling potentials [6], in the nonrelativistic two-Coulomb-centre problem (the molecular hydrogen ion H_2^+) [7]. Discussion of contemporary situation of the method, its various versions and applications in the nonrelativistic theory of atoms and molecules, quantum chemistry, in problems of the theory of collisions etc. can be found in the monograph [8]. New region of application of the WKB approximation can be the

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low-energy sector of QCD (energy spectrum of hadrons and the widths of hadronic decays), where standard approaches based on perturbation theory are not usable because the interaction between quarks is not small.

Successful applications of the WKB method for various nonrelativistic problems have stimulated employing its relativistic extension to the Dirac equation. The passage from the Dirac equation in an external field to the Hamilton–Jacobi equation for a classic relativistic particle was considered for the first time by Pauli [9], and later by the authors [10, 11] in more detail. Usually, the WKB method has been used in the case of bound states [12–15], more rarely for the calculation of wavefunctions of continuum spectrum and in the scattering theory [16]. Some particularities of the quasi-classical discrete spectrum at binding energies that have the same order as the rest energy mc^2 were mentioned in [15]. Nevertheless, many interesting problems of the atomic collisions theory, nuclear physics and elementary particles physics [18–20] are reduced to solving the Dirac equation with spherically symmetrical barrier potentials for which the energy levels are quasistationary. The properties of these states represent the interest for study of one- and two-electron processes with a rearrangement in the collisions of heavy multiply charged ions with other particles (ions, atoms, molecules) [21], for describing the effects of spontaneous creation of positrons at slow collisions of heavy nuclei [22], at consideration of the vacuum shell of the supercritical atom [22, 23], and from the point of view of the study of the S-level ionization (with binding energy of order mc^2) of heavy atoms under influence of external fields.

The earlier applications [13, 14] of the WKB method to the strong external field were based on squaring the Dirac equation (the effective potential method [12, 15]). This approach is good at $E > -mc^2$. However, at $E < -mc^2$ the substitution $\chi(r) = (mc^2 + E - V(r))^{1/2} F(r)$ used in that approach becomes singular at $r = r_{\mathrm{g}}$, where $V(r_{\mathrm{g}}) = mc^2 + E$ (the attractive potential $V(r) < 0$, $0 < r < \infty$ was considered) and usual quasi-classical formulae fail at $r \approx r_{\mathrm{g}}$ due to divergence of the phase integral ($\int p\mathrm{d}r$). Various authors overcome this difficulty differently, sometimes rather originally and wittily, but no common method was proposed, and therefore probably the mentioned version of the WKB method was not used for that problem hereinafter.

To rescue the situation, the new variant of the quasi-classical analysis of the Dirac equation in a strong spherically symmetric external field [17] was developed at the end of 1970s. It turns out that the mentioned difficulty has a formal character because the original Dirac equation is not singular at $r = r_{\mathrm{g}}$. The singularity does not arise if we apply the WKB method to the initial system of radial Dirac equations for the radial wavefunctions F and G which correspond to the upper and lower components of the Dirac bispinor instead of the second order differential equation for the function $\chi(r)$. The simple quasi-classical expressions obtained in this way for the number of levels which are immersed into lower continuum have numerous application to the theory of supercritical atoms. So it is natural to try to apply the WKB method to two problems: the spherically symmetrical Stark effect and quark potential models. These two problems represent a peculiar test for possibilities of the WKB method.

The problems connected with ionization of atoms and ions in external fields have gained specific topicality after the discovery of lasers. In 1960s the quasi-classical theory of the ionization in an electric field was set up and, besides, below-barrier motion of the electron was considered as nonrelativistic one, which is fulfilled for the valence electrons in all atoms from the hydrogen to the uranium. But, in the case of the ionization of K-shell of heavy atoms the relativistic effects represent the serious interest. Thus the generalization of the quasi-classical approximation for the relativistic case is necessary. This generalization can be useful in the relativistic nuclear physics and quantum chromodynamics.

2   The WKB method for the Dirac equation in spherically symmetrical field

The problem which will be discussed here is finding the WKB-type approximation to the solution of the Dirac radial wave equation. A resume of the differential equations which are involved and of the usual WKB approximation to their solution will be given first.

The Dirac radial equation wave equation arise in discussing the motion of a particle in a spherically symmetric a scalar potential $S(r)$ and a vector potential $V(r)$ simultaneously present. We now consider the Dirac Hamiltonian

$$\hat{H} = \alpha\hat{\mathbf{p}} + \beta(m + S(r)) + V(r), \qquad c = 1. \tag{1}$$

$\hat{H}$ acts on bispinor

$$\Psi = r^{-1} \begin{pmatrix} F(r)\Omega_{jlM}(\mathbf{n}) \\ iG(r)\Omega_{jl'M}(\mathbf{n}) \end{pmatrix}, \tag{2}$$

where Ω is the spherical spinors are eigenfunctions of the operators J^2 and J_z with eigenvalues $j(j+1)$ and M l is the orbital moment $(l + l' = 2j)$, $\mathbf{n} = \mathbf{r}/r$.

After separation of angular variables in the Dirac equation (1), system of equation for the radial wavefunctions $F(r)$ and $G(r)$ can be written in the matrix form

$$\chi' = \frac{1}{\hbar}D\chi, \quad \chi = \begin{Bmatrix} F \\ G \end{Bmatrix}, \tag{3}$$

$$D = \begin{pmatrix} -\hbar k/r & E - V(r) + m + S(r) \\ -E + V(r) + m + S(r) & \hbar k/r \end{pmatrix}, \tag{4}$$

where the prime means the derivative with respect to r, $k = \mp(j + 1/2)$ denotes the eigenvalues of operator $\hat{K} = \beta(\sigma\mathbf{L} + \hbar)$ which commutes with $\hat{H}$. By analogy with the WKB treatment of one-dimensional Dirac problem as developed by [9] we look for the solution of Eq. (3) in a formal power series

$$\chi(r) = \varphi(r)\exp\left\{\int^r y(r)\,\mathrm{d}r\right\}, \tag{5}$$

$$y(r) = \tfrac{1}{\hbar}y_{-1}(r) + y_0(r) + \hbar y_1(r) + \hbar^2 y_2(r) + \dots,$$
$$\varphi(r) = \sum_{n=0}^{\infty} \hbar^n \varphi^{(n)}(r), \quad \text{as} \quad \hbar \longrightarrow 0. \tag{6}$$

Here φ and φ_n are two-component column-vectors. The upper component correspond to F and lower component correspond to G.

Substituting (5) through (6) into Eq. (3) and equating the coefficients of equal power of $\hbar$, we obtain equations to determine successively y_n and $\varphi^{(n)}$:

$$(D - y_{-1}I)\varphi^{(0)} = 0, \tag{7}$$

$$(D - y_{-1}I)\varphi^{(n+1)} = \varphi^{(n)\prime} + \sum_{k=0}^{n} y_{n-k}\varphi^{(k)}, \quad n = 0, 1, 2, \dots \tag{8}$$

where I is the identity 2×2 matrix. We see that every $\varphi^{(n+1)}$ is determined in turn by the previous equations. The equation of (7) determines eigenvalues $y_{-1} \equiv \lambda_i$ and eigenvectors $\varphi^{(0)} \equiv \varphi_i(r)$ of the matrix D:

$$y_{-1} \equiv \lambda_i = \pm q, \quad q = \sqrt{(m+S)^2 - (E-V)^2 + (k/r)^2}, \tag{9}$$

$$\varphi_i = A_1 \begin{pmatrix} m + S + E - V \\ \lambda_i + kr^{-1} \end{pmatrix} = A_2 \begin{pmatrix} \lambda_i - kr^{-1} \\ m + S - E + V \end{pmatrix}, \tag{10}$$

here and further $\hbar = 1$, the index $i = \pm$, A_1 and A_2 are normalizing factors which will be determined later.

Since the matrix D is not symmetric we may introduce left eigenvectors $\check{\varphi}_i$ which are not the same as the right eigenvectors φ_i. The left eigenvectors $\check{\varphi}_i$ are defined by the following equations

$$\check{\varphi}_i(D - \lambda_i I) = 0 \tag{11}$$

$$\check{\varphi}_i = B_1(m + S - E + V, \quad \lambda_i + kr^{-1})$$
$$= B_2(\lambda_i - kr^{-1}, \quad m + S + E - V). \tag{12}$$

In addition a pair of vectors $\check{\varphi}_i$ and φ_i are orthogonal:

$$(\check{\varphi}_i, \varphi_j) = \sum_{\alpha=1}^{2} (\check{\varphi}_i)_\alpha (\varphi_j)_\alpha = \text{const}\, \delta_{ij}. \tag{13}$$

Let us now determine y_0. To do so, we take the first equation of the set (8), substitute $\varphi_0 = \varphi_i$ and multiply this equation by $\check{\varphi}_i$ (on the left). Then by Eq. (11) the left-hand side of considered equation is zero and we obtain

$$y_0(r) = -\frac{(\check{\varphi}_i, \varphi_i')}{(\check{\varphi}_i, \varphi_i)}. \tag{14}$$

Now we choose the normalizing factors A_1, A_2, B_1, and B_2 in Eq. (10) and (12) so as to satisfy the equation

$$(\check{\varphi}_i, \varphi_i') = (\check{\varphi}_i', \varphi_i). \tag{15}$$

In this case

$$\int^r y_0(r)\mathrm{d}r = \ln[(\check\varphi_i, \varphi_i)^{-1/2}] \tag{16}$$

and we obtain

$$\chi_i = (\check\varphi_i, \varphi_i)^{-1/2} \exp\left\{\int^r \lambda_i(r)\mathrm{d}r\right\}\varphi_i, \tag{17}$$

that is similar to usual expression for quasi-classical wavefunction in nonrelativistic quantum mechanics

$$\psi \sim p^{-1/2}\exp\left\{i\int^r p(r)\mathrm{d}r\right\}.$$

In similar manner we may find all the other terms $y_1, y_2, \ldots, \varphi^{(1)}, \varphi^{(2)}, \ldots$ in the expansions (6). However, the formulae are very complicated, so we restrict our computing to the order $O(\hbar)$. In principle this is connected with the circumstance that generally the expansion (6) in power $\hbar$ does not converge and represent so-called asymptotic series. Since the parameter of expansion is the Planck constant $\hbar$ which is small enough we obtain a good approximation for χ if we take finite number of terms of the expansions (6).

The condition (15) can always be satisfied. Substituting the expression (10) and (12) into (15) yields

$$\frac{A_1 B_1' - A_1' B_1}{A_1 B_1} = -\frac{(m+S)V' + (E-V)S'}{q(q \pm kr^{-1})}. \tag{18}$$

Then we have

$$\chi_\pm = \left[2q\left(q \pm \frac{k}{r}\right)\right]^{-1/2}$$

$$\times \exp\left\{\pm\int^r q\mathrm{d}r + \frac{1}{2}\int^r \frac{(m+S)\,V' + (E-V)\,S'}{q\,(q \pm kr^{-1})}\mathrm{d}r\right\}\begin{pmatrix} m+S+E-V \\ kr^{-1} \pm q \end{pmatrix}. \tag{19}$$

Using the second way of writing the eigenvectors φ_i and $\check\varphi_i$ with factors A_2 and B_2 in (10) and (12) and doing analogous calculation we obtain the following formula

$$\chi_\pm = \left[2q\left(q \mp \frac{k}{r}\right)\right]^{-1/2}$$

$$\times \exp\left\{\pm\int^r q\mathrm{d}r - \frac{1}{2}\int^r \frac{(m+S)\,V' + (E-V)\,S'}{q\,(q \mp kr^{-1})}\mathrm{d}r\right\}\begin{pmatrix} \pm q - kr^{-1} \\ m+S-E+V \end{pmatrix}. \tag{20}$$

Let us discuss the meaning of these formulae. First we note that y_{-1} and y_0 contain no imaginaries in the below-barrier region $r_- < r < r_+$. Here q is (to

within a factor i) the radial momentum of the relativistic particle, $q(r)^2 > 0$ for $r_- < r < r_+$. The signs $+(-)$ in Eqs. (19) and (20) correspond to the solutions increasing (decreasing) with increasing r. For decreasing solution (sign $-$) Eq. (19) must be used for $k < 0$ and Eq. (20) for $k > 0$, and the reverse for the increasing solution. The choice of a convenient form for the solution is determined in such a way as the quantity $Q = q + |k|/r$ is positive in the below-barrier region. One then gets formulae for F and G which are free from singularities. Accordingly, the version of the WKB method given here completely eliminates the difficulty with the effective potential which was mentioned in Sect. 1. For another choice, singularities can appear (at $Q = 0$) for F and G. But these singularities are fictitious because in this case the numerator is zero too. To get F and G which are free from singularities we need to reveal this indeterminacy that is connected with additional evaluations.
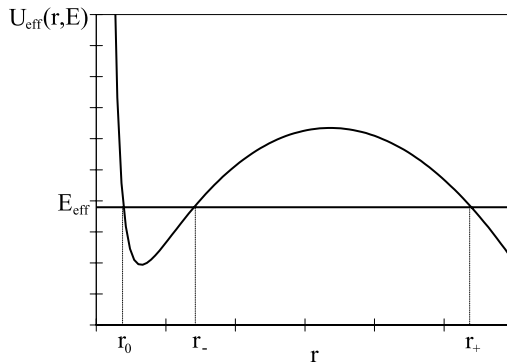

Fig. 1. The type of the effective potential $U_{\text{eff}}(r, E)$.

We now rewrite q in form $q = \sqrt{2m(U_{\text{eff}} - E_{\text{eff}})}$, where $E_{\text{eff}} = (E^2 - m^2)/2m$ is a binding energy and

$$U_{\text{eff}}(r, E) = \frac{E}{m}V + S + \frac{S^2 - V^2}{2m} + \frac{k^2}{2mr^2} \tag{21}$$

is an effective potential. We consider the case when effective potential has barrier shape (see Fig.1).

Then the wavefunctions take different form in the three regions: 1. $r_0 < r < r_-$ (potential well, $q^2 < 0$); 2. $r_- < r < r_+$ (below-barrier region, $q^2 > 0$); 3. $r > r_+$ (classically accessible region, $q^2 > 2$, continuous spectrum). Here r_0, r_-, and r_+ are the turning points. The behaviour of wavefunctions is described in the following section.

3   The wavefunction of the Dirac particle in classical allowed and forbidden regions

The region $r_0 < r < r_-$ is classically allowed region. Then the wavefunctions (19) and (20) oscillate, we have

$$F(r) = C_1^{\pm} \left[\frac{E - V + m + S}{p(r)} \right]^{1/2} \cos\Theta_1,$$

$$G(r) = C_1^{\pm} \operatorname{sgn} k \left[\frac{E - V - m - S}{p(r)} \right]^{1/2} \cos\Theta_2, \tag{22}$$

where $p(r)$ is the quasiclassical momentum for radial motion of the Dirac particle

$$p(r) = [(E - V)^2 - (m + S)^2 - (k/r)^2]^{1/2} \tag{23}$$

and the phases Θ_1 and Θ_2 are given by

$$\Theta_1(r) = \int\limits_{r_-}^{r} \left(p + \frac{kw}{pr} \right) \mathrm{d}r + \frac{\pi}{4}, \quad \Theta_2(r) = \int\limits_{r_-}^{r} \left(p + \frac{k\tilde{w}}{pr} \right) \mathrm{d}r + \frac{\pi}{4},$$

$$w = \frac{1}{2} \left(\frac{V' - S'}{m + S + E - V} - \frac{1}{r} \right), \quad \tilde{w} = \frac{1}{2} \left(\frac{V' + S'}{m + S - E + V} + \frac{1}{r} \right), \tag{24}$$

and $C_1^{\pm}$ are normalization constants. Signs $\pm$ correspond to $k > 0$ and $k < 0$, respectively.

Considering that the level width Γ is small (it is justified by the result) we can neglect penetration into classically forbidden regions $r < r_0$ and $r > r_-$, and normalize the wavefunction to single particle located in region 1:

$$\int\limits_{r_0}^{r_-} (F^2 + G^2) \mathrm{d}r = 1.$$

Thus contribution from exponential "trails" of the function F and G are neglected in regions $r < r_0$ and $r > r_-$. Here $\cos^2\Theta_i(r)$ can be replaced with average value $1/2$:

$$|C_1^{\pm}| = \left\{ \int\limits_{r_0}^{r_-} \frac{E - V(r)}{p(r)} \mathrm{d}r \right\}^{-1/2} = \left(\frac{2}{T} \right)^{1/2}, \tag{25}$$

where T is the period of the executed periodic motion of the relativistic particle. We note that in the turning points r_0 and r_-

$$E - V = \left[(m + S)^2 + \frac{k}{r^2} \right]^{1/2},$$

and $E - V > m + S$ in the region 1.

The below-barrier region $r_- < r < r_+$ is classically forbidden. Here $p = iq$ but quantities q, y_{-1}, and y_0 are real. As we know [1], the wavefunction must decrease exponentially with increasing r. For the state with $k > 0$ we have

$$\chi = \frac{C_2^+}{\sqrt{qQ}} \exp \left\{ - \int_{r_+}^r \left[q + \frac{(m+S)V' + (E-V)S'}{2qQ} \right] dr \right\} \begin{pmatrix} -Q \\ m+S-E+V \end{pmatrix}$$

(26)

and for $k < 0$

$$\chi = \frac{C_2^-}{\sqrt{qQ}} \exp \left\{ - \int_{r_+}^r \left[q - \frac{(m+S)V' + (E-V)S'}{2qQ} \right] dr \right\} \begin{pmatrix} m+S+E-V \\ -Q \end{pmatrix},$$

(27)

where $Q = q + |k| r^{-1}$.

The most interesting results are for a continuous spectrum ($r > r_+$). For $r > r_+$ the wavefunction corresponding to the quasistationary state is divergent (outgoing particle):

$$\chi = \frac{C_3^+}{\sqrt{pP}} \exp \left\{ \int_{r_+}^r \left[ip + \frac{(m+S)V' + (E-V)S'}{2pP} \right] dr \right\} \begin{pmatrix} iP \\ m+S-E+V \end{pmatrix},$$

(28)

where $k > 0$, and

$$\chi = \frac{C_3^-}{\sqrt{pP}} \exp \left\{ \int_{r_+}^r \left[ip - \frac{(m+S)V' + (E-V)S'}{2pP} \right] dr \right\} \begin{pmatrix} m+S+E-V \\ iP \end{pmatrix},$$

(29)

for $k < 0$. Here $P = p + i|k| r^{-1}$.

The formulae obtained determine quasiclassical asymptotics of solution to the Dirac equation (3) as $\hbar \to 0$ and cover almost the entire range of values of r, except the neighborhoods of the turning points r_- and r_+. To go around these points we can use the usual method which enable us to match the solutions on both sides of the turning point and to obtain relation between the normalization constants. Near the points r_- and r_+ the Dirac equation is reduced to the Schrödinger-like equation with an effective potential depending linearly on $r - r_\pm$. Then the solution is represented by the Airy function. One can also use the Zwaan method [1, 3]. The connection formulae are of the form

$$C_2^\pm = -iC_3^\pm = \mp \frac{C_1^\pm}{2} \left[\frac{E - V(r_-) + m + S(r_-)}{|k| r_-^{-1}} \right]^{\pm 1/2}$$

$$\times \exp \left\{ - \int_{r_-}^{r_+} \left[q \pm \frac{(m+S)V' + (E-V)S'}{2qQ} \right] dr \right\}.$$

(30)

As applications of these formulae we can solve a number of problems. For example the position E_r and width of quasistationary level $E = E_r - \mathrm{i}\Gamma/2$. In the case of exponentially small barrier Eqs. (22) lead to the quantization rule

$$\int_{r_0}^{r_-} \left(p + \frac{k\,w}{p\,r} \right) \mathrm{d}r = \left(n_r + \frac{1}{2} \right) \pi, \qquad n_r = 0, 1, 2, \dots, \tag{31}$$

which determines the level energy. Here n_r is the radial quantum number. Eq. (31) differs from the standard Bohr–Sommerfeld quantization rule by the relativistic expression for the momentum and in the term $\sim w(k)$ which takes into account the spin-orbit interaction and splits the levels with different signs of k.

The particle current at $r \to \infty$ determines the width of the level, i.e., the probability of tunneling

$$\Gamma = -2\,\mathrm{Im}[G^*(r)F(r)]. \tag{32}$$

Joining the solutions (28) and (29) and using (25) and (30) we obtain

$$\Gamma = \frac{1}{T}\,\exp\left\{ -2 \int_{r_-}^{r_+} \left(q - \frac{k\,w}{q\,r} \right) \mathrm{d}r \right\}. \tag{33}$$

4 An analytical examination of the quantization condition

Any approximative theory is never complete without a proper examination of its applicability to concrete problems. To examine the validity of the quantization condition (31) we apply it to four problems whose analytic solutions are well known. If one applies the eigenvalue condition (31) to the Coulomb problem where $V(r) = -\alpha Z/r$ ($\alpha = 1/137$, $S(r) = 0$) then

$$p(r) = \left[\left(E + \frac{\alpha Z}{r} \right)^2 - m^2 - \left(\frac{k}{r} \right)^2 \right]^{1/2}, \qquad w = \frac{1}{2}\left[\frac{\dfrac{\alpha Z}{r^2}}{E + \dfrac{\alpha Z}{r} + m} - \frac{1}{r} \right].$$

The equation $p^2(r) = 0$ has two positive roots

$$r_{0,-} = \frac{E\alpha Z \mp ((E\alpha Z)^2 - (m^2 - E^2)\gamma^2)^{1/2}}{m^2 - E^2}, \quad \gamma = \sqrt{k^2 - (\alpha Z)^2}.$$

Having calculated the integral from r_0 to r_- in quantization condition (31), we arrive at the expression

$$\frac{E\alpha Z}{\sqrt{m^2 - E^2}} = n + \gamma, \quad n = n_r + \frac{1 + \mathrm{sgn}\,k}{2} = \begin{cases} 0, 1, 2 \dots \text{ for } k < 0, \\ 1, 2 \dots \text{ for } k > 0, \end{cases}$$

whence we get expression, which is similar to the Sommerfeld–Dirac fine structure formula

$$E = m\left[1 + \left(\frac{\alpha Z}{n + \gamma} \right)^2 \right]^{-1/2}.$$

The Dirac equation also possesses an exact solution for superposition of a vector and scalar potential. Let us put $V(r) = -\alpha/r$, $S(r) = -\alpha'/r$ [24]. The vector Coulomb potential is derived from the exchange of massless photons between nucleus and the leptons orbiting around it, the scalar potential is created by exchange of massless scalar mesons.

In the subsequent calculation we follow a very similar procedure used in the previous case. We consider

$$p(r) = \left[\left(E + \frac{\alpha}{r}\right)^2 - \left(m - \frac{\alpha'}{r}\right)^2 - \left(\frac{k}{r}\right)^2\right]^{1/2},$$

and

$$w = \frac{1}{2}\left[\frac{\dfrac{\alpha - \alpha'}{r^2}}{E + \dfrac{\alpha - \alpha'}{r + m}} - \frac{1}{r}\right].$$

The roots of $p^2(r) = 0$ are

$$r_{0,-} = \frac{E\alpha + m\alpha' \mp ((E\alpha + m\alpha')^2 - (m^2 - E^2)\gamma^2)^{1/2}}{m^2 - E^2}.$$

Performing the integration between limits r_0 and r_- in (31), we obtain

$$\frac{E\alpha + m\alpha'}{\sqrt{m^2 - E^2}} = n + \gamma, \qquad \gamma = \sqrt{k^2 - \alpha^2 + \alpha'^2},$$

which leads to

$$E = m\left\{\frac{-\alpha\alpha'}{\alpha^2 + (n + \gamma)^2} \mp \left[\left(\frac{\alpha\alpha'}{\alpha^2 + (n + \gamma)^2}\right)^2 - \frac{\alpha'^2 - (n + \gamma)^2}{\alpha^2 + (n + \gamma)^2}\right]^{1/2}\right\}.$$

The result obtained is identical with result in [24].

In papers [25] the problem of bound states of the Dirac equation with $S(r) = V(r) = a\,r^2/4$ ($a > 0$) has been studied. In this case (23) and (24) take form

$$p(r) = \left[E^2 - m^2 - \frac{1}{2}(E + m)\,a\,r^2 - \left(\frac{k}{r}\right)^2\right]^{1/2}, \quad w(r) = -1/2r.$$

The limits of the integral in (31) are

$$r_{0,-} = \frac{1}{\sqrt{a}}\left[E - m \mp \sqrt{(E - m)^2 - 2\,a\,k^2(E + m)^{-1}}\right]^{1/2}.$$

Carrying out the integration of the left-hand side of (31) we obtain equation for E

$$\frac{E - m}{2}\sqrt{\frac{E + m}{2a}} - \frac{|k|}{2} - \frac{1}{4}\mathrm{sgn}\,k = n_r + \frac{1}{2},$$

input $\mathcal{K} = |k| + (1 + \text{sgn } k)/2$, get

$$(E - m)\sqrt{2(E + m)} = (4n_r + 2\mathcal{K} + 1)\sqrt{a}.$$

The last equation for the energy is cubic, its real solution is

$$E_{n_r,\mathcal{K}} = \tfrac{1}{6}\left(2m + 8 \times 2^{2/3}m^2 A^{-1/3} + 2^{1/3}A^{1/3}\right),$$

where $A = -B + \sqrt{B^2 - 1024m^6}$, $B = 32m^3 - 27a(1 + 2\mathcal{K} + 4n_r)^2$. This result totally coincides with results obtained in [25].

Let's consider the motion of a massless fermion in the scalar field $S(r) = -\alpha'/r + \sigma r$, $\sigma > 0$; $V(r) = 0$. Integration in quantization condition (31) from r_0 to r_- gives following result:

$$\frac{E^2 + 2\sigma(\alpha' - \gamma)}{4\sigma} - \frac{k}{\sigma(r_0 + r_-)\pi}\left[2r_0\left(\frac{\Pi(\nu_+^2,\xi)}{r_0^2 - P_+^2} + \frac{\Pi(\nu_-^2,\xi)}{r_0^2 - P_-^2}\right)\right.$$
$$\left. - \left(\frac{1}{r_0 + P_+} + \frac{1}{r_0 + P_-}\right)F(\xi)\right] = n_r + \frac{1}{2}, \tag{34}$$

where

$$r_{0,-} = \frac{1}{2^{1/2}\sigma}\sqrt{E^2 + 2\alpha'\sigma \mp \sqrt{(E^2 + 2\alpha'\sigma)^2 - (2\sigma\gamma)^2}}, \quad \gamma = \sqrt{k^2 + \alpha'^2},$$

$$P_\pm = \frac{1}{2\sigma}(-E \pm \sqrt{E^2 + 4\alpha'\sigma}), \quad \xi = \sqrt{\frac{E^2 + 2\sigma(\alpha' - \gamma)}{E^2 + 2\sigma(\alpha' + \gamma)}}, \quad \nu_\pm^2 = \xi\frac{P_\pm + r_0}{P_\pm - r_0},$$

$F(\xi)$ and $\Pi(\nu_\pm^2,\xi)$ are the complete elliptic integrals of the first and third kind, respectively.

In (34) taking into account the asymptotics of the elliptic integrals [26] at $\sigma\gamma/E_{n_r,k}^2 \ll 1$, one obtains the following spectrum

$$\frac{E_{n_r,k}^2}{4\sigma} = n_r + \frac{1}{2} + \frac{\gamma - \alpha'}{2} + \frac{k}{4\gamma} + \frac{\sigma k}{2\pi E_{n_r,k}^2}\left(0.38 + \ln\frac{E_{n_r,k}^2}{\sigma\gamma}\right) + O\left(\left(\frac{\sigma\gamma}{E_{n_r,k}^2}\right)^2\right). \tag{35}$$

The form (35) is exact for small $(\sigma\gamma/E_{n_r,k}^2)^2$, and the accuracy can be tested by comparison with the exact formula (34); two sets of lowest levels with $k = \pm 1$ coincide within 2% and for higher levels the precision is less than 1%. At $\alpha' = 0$ expression (35) is reduced to Simonov's result [27].

In case of $E_{n_r,k}^2 \gg \sigma\gamma$ in formula (35) expression $\left(0.38 + \ln E_{n_r,k}^2/\sigma\gamma\right)\Big/\pi$ tends to 1 and one can obtain the following expression for energies:

$$\varepsilon_{n_r,k} = \frac{E_{n_r,k}}{\sqrt{\sigma}} = \pm\sqrt{N - \alpha' + [(N - \alpha')^2 + 2k]^{1/2}}, \quad N = 2n_r + 1 + \gamma + \frac{k}{2\gamma}. \tag{36}$$

5 Spherical model of the Stark effect of the hydrogen-like atom

In this section we study atomic level of the hydrogen-like atom in spherically symmetric field. Let us consider the potential

$$V(r) = -\frac{Z\alpha}{r} - Fr, \qquad S(r) = 0, \tag{37}$$

where Z is the nuclear charge, $\alpha = 1/137$, F is the electric field strength. Now $\hbar = c = m = 1$, and ε is the energy of an electron in units of mc^2. For a small electric field there is region for which the distances from a hydrogen-like ion are much greater than the dimension of a ion $(r \gg 2Z\alpha/\lambda)$ and much smaller than F^{-1}. It means that in this region we can neglect the penetration through the potential barrier $r_- < r < r_+$ and normalize the wavefunction to one electron localized in broad potential well $r_0 < r < r_-$ of an ion.

In order to find an analytical form of constants $C_1^{\pm}$ it is necessary to solve equation $p^2(r) = 0$. Taking into account that F can be neglected in surroundings of a nucleus then we obtain for the turning points

$$r_0 = r_1^{(0)} \approx \frac{\varepsilon Z\alpha - \nu}{\lambda^2}, \quad r_- = r_2^{(0)} \approx \frac{\varepsilon Z\alpha + \nu}{\lambda^2},$$
$$\nu = \sqrt{(\varepsilon Z\alpha)^2 - (\lambda\gamma)^2}, \quad \lambda = \sqrt{1 - \varepsilon^2}. \tag{38}$$

The substitution (37), (38) into (25) and calculation of the corresponding integral yield

$$(C_1^{\pm})^2 = \frac{\lambda^3}{\pi Z\alpha}. \tag{39}$$

We note that the value $|C_1^{\pm}|\alpha^{-1} = \pi^{-1/2}ZN^{-3/2}$ $(N = Z\alpha/\lambda)$ is the relativistic analogue of the corresponding nonrelativistic formula for normalized constant $a = \pi^{-1/2}Zn^{-3/2}$ of the H-like atom in the Schrödinger theory.

There is another method for determination of the constants $C_1^{\pm}$. We join the solution in region $2Z\alpha/\lambda^2 \ll r \ll F^{-1}$ and asymptotics $(r \to \infty)$ of unperturbed atomic wavefunctions

$$\left. \begin{array}{c} F_{as}(r) \\ G_{as}(r) \end{array} \right\} = \pm A\sqrt{1 \pm \varepsilon}\,[1 + B_{\pm}r^{-1} + \ldots]\,r^{\varepsilon Z\alpha/\lambda}\mathrm{e}^{-\lambda r},$$

where

$$A = \lambda(2\lambda)^{\varepsilon Z\alpha\lambda} \left[\frac{\dfrac{Z\alpha}{\lambda} - k}{2Z\alpha\,\Gamma\left(\varepsilon\dfrac{Z\alpha}{\lambda} - \gamma + 1\right)\Gamma\left(\varepsilon\dfrac{Z\alpha}{\lambda} + \gamma + 1\right)} \right]^{1/2},$$

$$B_{\pm} = \frac{1}{2\lambda}\left(k + \frac{Z\alpha}{\lambda}\right)\left(k - \frac{Z\alpha}{\lambda} \pm 1\right). \tag{40}$$

In this case

$$C_1^{\pm} = 2A\lambda^{1/2} \left(\frac{\nu}{7\lambda^2 e}\right)^{\varepsilon Z\alpha/\lambda} \left(\frac{Z\alpha + k\lambda}{Z\alpha - k\lambda}\right)^{1/4} \left(\frac{\varepsilon Z\alpha + \lambda\gamma}{\varepsilon Z\alpha - \lambda\gamma}\right)^{\gamma/2}. \qquad (41)$$

The results (39) and (41) differ within the error between $n!$ and the Stirling formula.

Now we calculate position of the quasistationary level with an accuracy of $O(F^2)$. Therefore it is necessary to evaluate the turning point with more accuracy and we must also find the other two solutions of the equation $p^2(r) = 0$. Then

$$r_{1,2} = r_{1,2}^0 \left[1 + \frac{r_{1,2}^0 F}{\lambda^2}\left(\varepsilon \mp \frac{Z\alpha}{\nu}\right)\right], \qquad r_{3,4} = \frac{\mp 1 - \varepsilon}{F} + \frac{Z\alpha}{\varepsilon \pm 1}, \qquad (42)$$

where $r_0 = r_1$, $r_- = r_2$, $r_+ = r_4$. The position of close ("atomic") turning points $r_{1,2}$ depends on F weakly and is determined principally by the Coulomb field of a nucleus. The turning points $r_{3,4}$ distances from the atom depend principally on the field but their value is "controlled" by quantity $Z\alpha/(\varepsilon \pm 1) \approx 1$ that is determined by long distance Coulomb interaction. If $F \ll 1$ (weak field) then $\{|r_3|, r_4\} \gg \{r_1, r_2\}$. It allows to expand quasi-momentum in positive powers of F in the region $r_1 \leq r \leq r_2$.

$$p(r) = F\frac{\sqrt{(r - r_1)(r - r_2)(r - r_3)(r - r_4)}}{r}$$
$$\approx F\sqrt{-r_3 r_4}\frac{(r - r_1)(r - r_2)}{r}\left[1 - \frac{r_3 + r_4}{2r_3 r_4}\right]. \qquad (43)$$

Taking into account (42), (43), and calculating the integral in (31), we obtain

$$\varepsilon = \varepsilon_0 + \frac{F}{2Z\alpha}\left[k - \varepsilon_0\left(\frac{3Z^2\alpha^2}{\lambda_0^2} - k^2\right)\right], \qquad (44)$$

where

$$\lambda_0 = \sqrt{1 - \varepsilon_0^2}, \qquad \varepsilon_0 = \left[1 + \frac{(Z\alpha)^2}{n + \sqrt{k^2 - (Z\alpha)^2}}\right]^{-3/2}, \qquad n = n_r + \frac{1 + \operatorname{sgn} k}{2}.$$

To determine ε we can also use the perturbation theory (at $F \ll 1$) in which the restriction to quantum numbers does not exist unlike the quasi-classical method. It is surprising but calculations lead to the same expression (44). So the quasiclassical condition (31) gives excellent results even for low-energy levels of discrete spectrum.

Now, we evaluate the width by (33). We rewrite q in the form

$$q(r) = \frac{F\sqrt{(r - r_1)(r - r_2)(r - r_3)(r_4 - r)}}{r}. \qquad (45)$$

As in the case of quasi-momentum we expand q in power series of F. Practical expansion is realized in following form. We divide the integration region $r_2 \ll$

$r \ll r_4$ into two regions introducing the point $r_0' \approx F^{-1/2}$ which satisfies the condition $r_2 \ll r_0' \ll r_4$. In the first region ($r_2 \ll r \ll r_0'$) the Coulomb interaction predominates but the interaction between the electron and an electric field can be regarded as the perturbation.

Expanding $q(r)$ in this region in powers of a small parameter (perturbation) yields the asymptotic representation

$$q(r) \approx F\sqrt{-r_3 r_4}\, \frac{\sqrt{(r - r_1)(r - r_2)}}{r} \left(1 - \frac{r_3 + r_4}{2r_3 r_4} r \right). \tag{46}$$

On the contrary, the potential of an electric field dominates in the region $r_0' \leq r \leq r_4$ but the Coulomb field of the nucleus can be regarded as a perturbation

$$q(r) \approx F\sqrt{(r - r_3)(r_4 - r)} \left(1 - \frac{r_1 + r_2}{2r} \right). \tag{47}$$

Calculating integrals in (33) and taking (39), we obtain

$$\Gamma = 2\lambda |A|^2 \left(\frac{2\lambda^2}{F} \right)^{2\varepsilon Z\alpha/\lambda} \exp\left(-\frac{\Phi(\varepsilon)}{\alpha^3 F} + 2Z\alpha \arccos\varepsilon \right), \tag{48}$$

where

$$\Phi(\varepsilon) = \arccos\varepsilon - \varepsilon\sqrt{1 - \varepsilon^2}. \tag{49}$$

It is seen from (48) the width of an atomic level is proportional to $|A|^2$. It is not surprising because for $F \ll 1$ the ionization comes from "a tail" of atomic wavefunction and the barrier is broad.

Next let us investigate some limiting cases of the obtained expression (48).

1. We begin from the ionization of the s-level, which is bound by short range forces ($Z = 0$). In this case (48) gives

$$\Gamma = 2\lambda |A|^2 \exp\left(-\frac{\Phi(\varepsilon)}{\alpha^3 F} \right). \tag{50}$$

   This is the same result as in [28, 29] for the Stark ionization of the s-level of negative ions (H^-, Na^- etc.).

2. At present Coulomb field it is appropriate to consider different limiting cases for the functions in the square brackets of the formula (48):

$$\arccos\varepsilon = \begin{cases} (1 - \varepsilon^2)^{1/2} + \frac{1}{6}(1 - \varepsilon^2)^{3/2} + \dots, & \varepsilon \to 1, \\ \frac{1}{2}\pi - \varepsilon - \frac{1}{6}\varepsilon^3 - \dots, & \varepsilon \to 0, \\ \pi - (1 - \varepsilon^2)^{1/2} - \frac{1}{6}(1 - \varepsilon^2)^{3/2} - \dots, & \varepsilon \to -1. \end{cases} \tag{51}$$

$$\Phi(\varepsilon) = \begin{cases} \frac{1}{3}2^{5/2}(1 - \varepsilon)^{3/2} \left[1 - \frac{3}{20}(1 - \varepsilon) + \dots \right], & \varepsilon \to 1, \\ \frac{1}{2}\pi - 2\varepsilon + \frac{1}{3}\varepsilon^3 - \dots, & \varepsilon \to 0, \\ \pi - \frac{1}{3}2^{5/2}(1 + \varepsilon)^{3/2} + \dots, & \varepsilon \to -1. \end{cases} \tag{52}$$

In the nonrelativistic limit ($\alpha \to 0$, $\varepsilon \to 1$) the formula (48) passes into the expression obtained in [30]. At $\varepsilon \to -1$ (when the level enters the negative-energy continuum) the exponential factor equals $\exp\left[-\pi/(\alpha^3 F)\right]$ that coincides with the corresponding factor in the Schwinger formula [31] for the probability of electron-positron pairs creation from vacuum in a constant electric field obtained within the framework of quantum field theory.

6 Description of an energetic spectrum of the quark system with the Cornell potential

As it is known, the formulation of the two-body problem within the framework of the relativistic quantum theory strikes on principal difficulties which have mathematical and logical character. Until now there is no satisfactory relativistic theory of two-particle systems. But the great need of the logically uncontradictory two-body equations exists since we know that the wide class of really existing fundamental particles (mesons) are considered as the bound states of the two-quark systems.

It seems also by now to be proven that the quantum chromodynamics (QCD) is able to correctly describe the most pronounced features of the quark-antiquark interaction. However, the standard perturbative QCD gives rather reliable recipes for the calculation of various characteristics only for description so-called "hard" processes characterized by the large transmitted momentum, and not applicable for calculation of the characteristics defined by "soft" processes (the mass spectrum, confinement of quarks, decay widths of hadrons). At the same time the nature of the formation of the bound states of interacting particles must be determined undoubtedly by nonperturbative effects. The confinement is the result of circumstance that, unlike quantum electrodynamics in which interaction mediators — photons — are electroneutral, exchange particles in quantum chromodynamics — gluons — possess non-zero colour charge and therefore can interact one with other. Thus confinement is not nested in the framework of the perturbation theory. Because of this, at the present time the structure of interquark forces cannot be completely defined from the principles of quantum chromodynamics.

The utilization of dynamical equation represents the most effective way of the construction of the theory for bound states. The point is that even if we are able to construct the kernels (potentials) of dynamic equations only for the lower orders of the perturbation theory, then elaboration of the methods for exact or approximative solving of these equations allows to take into account contributions of nonperturbative effects of the interaction for calculation of observable characteristics of bound states. In the nonrelativistic constituent quark model the meson is basically described by the Schrödinger equation with appropriate potential which describes the quark-antiquark interaction. This interaction is a sum of the linearly rising potential responsible for confinement in long range and of the quasi-Coulomb short range interaction caused by the one-gluon-exchange. Overview of the results and successes of nonrelativistic approach is given for example in [32].

However, the consideration of meson containing the light quarks is a complicated task, and it demands the relativistic effects to be taken into account. A more ambitious approach is based on the generalization of the Schrödinger equation to the relativistic Bethe–Salpeter equation or the Salpeter equation (see e.g. [33]). Another form used to describe the two-quark system is the two-body Dirac equation or the one body Dirac equation (see e.g. [34]).

In the paper [18] attempts were made to give the theoretical reasons (though it is not full consistent) to the relativistic potential model of the quark-antiquark systems consisting of one heavy quark (antiquark) Q (b and c quarks) with the mass m_2 and one light antiquark (quark) q (u, d, s - quarks) with mass m_1 much less. Such heavy-light quark systems are QCD analogues of the hydrogen atom and thus of fundamental importance. Hydrogen-like quark systems are e.g. $B^+ = \bar{b}u$, $B^0 = \bar{b}d$, $B^0_s = \bar{b}s$, $D^0 = \bar{c}u$, $D^- = \bar{c}d$, $D^-_s = \bar{c}s$. Using the Dirac equation we implement explicitly relativistic dynamics and can study relativistic properties of the spectrum, and also spin splittings of energy levels. We use the Dirac equation with a static QCD-motivated potential to describe the light quark or antiquark motion in the field of the heavy quark.

In the present paper, the color Coulomb interaction between quarks is of a vector nature and the confining interaction is scalar. It was shown in [18] that:

1. When we have superposition of the scalar potential $S(r) = \mu r$ and the vector potential $V(r) = -\alpha/r$ then the Dirac equation has discrete spectrum.

2. When $S(r) = 0$ and $V(r) = -\alpha/r + \mu r$ the discrete spectrum is absent, and all solutions of the Dirac equation, if any, are quasistationary.

3. For purely scalar interaction $S(r) = -\alpha/r + \mu r$, $V = 0$ the spectrum is discrete. In this case the scalar potential explicitly breaks the chiral symmetry and states with opposite parities are not degenerate.

It is widely accepted that interaction between the two quarks is very well described by the Cornell potential [33].

It is interesting to note that some authors assumed that the confinement part of the potential is a pure vector [35], but a further search showed that confining potential which, in principle, is scalar can contain a small fraction of the vector potential [36].

We are also inclined to the Cornell potential. In previous paper [37] a very good spin average mass-spectrum of the light and heavy quark-system was also obtained by this potential.

Let's assume

$$V(r) = -\frac{\alpha}{r}, \qquad S(r) = \mu r, \tag{53}$$

where $\alpha = 4/3\,\alpha_s$, $\mu = 0.18\,\text{GeV}^2$, α_s is the running coupling constant.

Both the Schrödinger equation and the Dirac equation with Cornell potential has no analytical solutions. The ordinary WKB method was used in the paper

[38] for solving the first equation. For finding asymptotic solutions of the Dirac equation (3) with potential (53), we shall use the version of the WKB approximation elaborated by us recently.

Due to the confinement of quarks we are interested in only classically allowed region ($r_0 < r < r_-$, $q^2 < 0$, a potential well) that corresponds to only the discrete energy spectrum of the quark-antiquark system. Then on the basis of (23)

$$p(r) = \left[\left(E + \frac{\alpha}{r} \right)^2 - (m + \mu r)^2 - \left(\frac{k}{r} \right)^2 \right]^{1/2}. \tag{54}$$

We represent the left-hand side of the quantization condition (31) in the form of a sum of two integrals I_1 and I_2:

$$I_1 = \int_b^a p(r)\mathrm{d}r, \qquad I_2 = \int_b^a \frac{k\,w}{p(r)r}\mathrm{d}r. \tag{55}$$

The integration is between the two turning points $r_0 = b$ and $r_- = a$, which are real and positive ($a > b > 0$) roots of the equation $p^2(r) = 0$. Two roots of this equation are real and negatives ($d < c < 0$):

$$a, b = -\tfrac{1}{4}h + \tfrac{1}{2}(\Omega \pm \Delta_+), \qquad c, d = -\tfrac{1}{4}h - \tfrac{1}{2}(\Omega \mp \Delta_-), \tag{56}$$

where

$$\Omega = \left[\frac{h^2}{4} - \frac{2f}{3} + \frac{u}{3}\left(\frac{2}{Z} \right)^{1/3} + \frac{1}{3}\left(\frac{Z}{2} \right)^{1/3} \right]^{1/2}, \quad \Delta_\pm = \sqrt{F \pm \frac{D}{4\Omega}},$$

$$F = \frac{h^2}{2} - \frac{4f}{3} - \frac{u}{3}\left(\frac{2}{Z} \right)^{1/3} - \frac{1}{3}\left(\frac{Z}{2} \right)^{1/3}, \quad Z = v + \sqrt{-4u^3 + v^2}, \tag{57}$$

$$D = -h^3 + 4hf - 8g, \qquad v = 2f^3 - 9hfg + 27g^2 + 27h^2l - 72fl,$$

$$u = f^2 - 3hg + 12l$$

and

$$h = \frac{2m}{\mu}, \quad f = -\frac{E^2 - m^2}{\mu^2}, \quad g = -\frac{2E\alpha}{\mu^2}, \quad l = \frac{k^2 - \alpha^2}{\mu^2}. \tag{58}$$

Formulae (55) can now be re-expressed in terms of a, b, c and d as

$$I_1 = -\mu \int_b^a \frac{r^3 + hr^2 + fr + g + lr^{-1}}{R}\mathrm{d}r, \tag{59}$$

$$I_2 = -\frac{k}{2\mu}\left[\int_b^a \frac{\mathrm{d}r}{(r - \lambda_+)R} + \int_b^a \frac{\mathrm{d}r}{(r - \lambda_-)R} \right], \tag{60}$$

where

$$R = \left[(a-r)(r-b)(r-c)(r-d) \right]^{1/2},$$

$$\lambda = \lambda_+ - \lambda_-, \quad \lambda_\pm = -\frac{E + m \mp \sqrt{(E+m)^2 - 4\mu\alpha}}{2\mu}.$$

By replacement

$$r = \frac{b(a-c) - c(a-b)\sin^2\phi}{a - c - (a-b)\sin^2\phi} \tag{61}$$

the integrals (59) and (60) can be expressed through elliptic integrals. Then the quantization condition (31) gives the transcendental equation for the energy $E_{n_r,k}$:

$$-\frac{\mu(b-c)^2}{\Re_1} \left[N_1 F(\xi) + N_2 E(\xi) + N_3 \Pi(\nu,\xi) + N_4 \Pi\left(\frac{c}{b}\nu,\xi\right) \right]$$

$$+ \frac{k}{\mu} \left[\frac{b-c}{2} \left(N_5 \Pi(\nu_1,\xi) + N_6 \Pi(\nu_2,\xi) \right) + N_7 F(\xi) \right]$$

$$= \tfrac{1}{2}\sqrt{(a-c)(b-d)} \left(n_r + \tfrac{1}{2} \right) \pi, \tag{62}$$

where

$$\Re_1 = (1-\nu)\left(\xi^2 - \nu\right), \quad \aleph_1 = \xi^2(3 - 2\nu) + \nu(\nu - 2),$$

$$\xi = \sqrt{\frac{(a-b)(c-d)}{(a-c)(b-d)}}, \qquad \nu = \frac{a-b}{a-c}, \qquad \nu_1 = \frac{\lambda_+ - c}{\lambda_+ - b}\nu, \qquad \nu_2 = \frac{\lambda_- - c}{\lambda_- - b}\nu,$$

$$N_1 = \frac{\xi^2(b-c)}{4} - \frac{3\aleph_1(b-c)}{8(1-\nu)} - \frac{\xi^2 - \nu}{2}(h + 3c)$$

$$+ \frac{\Re_1}{(b-c)^2}\left(c^3 + c^2 h + cf + g + \frac{l}{c} \right),$$

$$N_2 = -\frac{\nu}{2}\left[h + 3c + \frac{3}{4}\frac{(b-c)\aleph_1}{\Re_1} \right],$$

$$N_3 = \frac{1}{2}\left[\frac{3}{4}\frac{(b-c)\aleph_1^2}{\Re_1} + \frac{2\Re_1}{(b-c)}(3c^2 + 2ch + f) + (b-c)\left((1 + \xi^2)\nu - 3\xi^2\right) \right.$$

$$\left. + \xi^2\aleph_1(h + 3c) \right],$$

$$N_4 = -\frac{\Re_1}{(b-c)}\frac{l}{bc}, \quad N_5 = \left[(b-\lambda_+)(\lambda_+ - c) \right]^{-1},$$

$$N_6 = \left[(b-\lambda_-)(\lambda_- - c) \right]^{-1}, \quad N_7 = \left(c + \frac{E+m}{2\mu} \right)\left[(\lambda_+ - c)(\lambda_- - c) \right]^{-1}$$

and $F(\xi)$, $E(\xi)$, and $\Pi(\nu,\xi)$ are the complete elliptic integrals of the first, second, and third kinds, respectively.

Table 1. The eigenvalues $\varepsilon_{n_r,k}$ of the Dirac equation with $\alpha = 0$.

$^{2s+1}N_j$	k, n_r	$m = 0$			$m = 0.3$	
		[18]	(62)	(63)	[18]	(62)
$^2S_{1/2}$	$-1, 0$	1.6194	1.6229	1.4142	1.8444	1.8484
	$-1, 1$	2.6026	2.6038	2.5887	2.8068	2.8082
	$-1, 2$	3.2912	3.2918	3.2886	3.4908	3.4915
	$-1, 3$	3.8554	3.8558	3.8555	4.0530	4.0534
$^2P_{3/2}$	$-2, 0$	2.1465	2.1473	2.0000	2.3676	2.3686
	$-2, 1$	2.9520	2.9524	2.9208	3.1585	3.1590
	$-2, 2$	3.5735	3.5738	3.5616	3.7751	3.7754
	$-2, 3$	4.0995	4.0996	4.0941	4.2985	4.2987
$^2D_{5/2}$	$-3, 0$	2.5693	2.5696	2.4495	2.7885	2.7889
	$-3, 1$	3.2685	3.2687	3.2287	3.4765	3.4767
$^2F_{7/2}$	$-4, 0$	2.9322	2.9323	2.8284	3.1503	3.1505
$^2P_{1/2}$	$1, 0$	2.2940	2.2925	2.3178	2.4921	2.4911
	$1, 1$	3.0310	3.0303	3.0359	3.2275	3.2269
	$1, 2$	3.6260	3.6255	3.6265	3.8216	3.8213
$^2D_{3/2}$	$2, 0$	2.7044	2.7040	2.7443	2.9065	2.9057
	$2, 1$	3.3538	3.3535	3.3693	3.5522	3.5520
$^2F_{5/2}$	$3, 0$	3.0597	3.0558	3.1021	3.2618	3.2598
$^2G_{7/2}$	$4, 0$	3.4087	3.3695	3.4183	3.6000	3.5750

In the case $m = 0$, $\alpha = 0$, by asymptotics of elliptic integrals [26], the formula (62) leads to expression

$$\varepsilon_{n_r,k} = \frac{E_{n_r,k}}{\sqrt{\sigma}} = \pm\sqrt{N + (N^2 + 2k)^{1/2}}, \quad N = 2n_r + 1 + |k| + \frac{1}{2}\operatorname{sgn} k, \quad (63)$$

which coincides with equation (36), where one takes $\alpha' = 0$.

We show in Tables 1 and 2 the values of the energy obtained with the aid of (62), (63), and the values calculated by numerical solution of the system (3) in [18].

Analysis of results in Tables 1 and 2 shows that for $\alpha = 0$ the general formula (62) gives good coincidence with [18] (for $k = \pm 1$ states the relative error is less than 2%). Formula (63) has worse accuracy, but precision of both (62) and (63) rapidly increases with increasing quantum numbers. In the case $\alpha = |k|$ the relative error of formula (62) does not exceed 2.5% (besides the ground state).

Corresponding wavefunctions for light quarks in the quasi-classical allowed region (a potential well) are determined by formulae (22), where the normalized constants have the following form:

$$|C_1^{\pm}| = \left[\frac{\mu\sqrt{(a-c)(b-d)}}{2((Ec+\alpha)F(\xi) + E(b-c)\Pi(\nu,\xi))}\right]^{1/2}, \quad (64)$$

Table 2. The eigenvalues $\varepsilon_{n_r,k}$ of the Dirac equation with $\alpha = |k|$.

$^{2s+1}N_j$	k, n_r	$m = 0$		$m = 0.3$	
		[18]	(62)	[18]	(62)
$^2S_{1/2}$	$-1, 0$	0.0000	0.0769	0.0000	0.0000
	$-1, 1$	1.0690	1.0922	1.1749	1.2008
	$-1, 2$	1.9685	1.9887	2.1171	2.1379
	$-1, 3$	2.6786	2.6954	2.8428	2.8600
$^2P_{3/2}$	$-2, 0$	0.0000	0.0385	0.0000	0.0000
	$-2, 1$	0.6444	0.6549	0.7095	0.7216
	$-2, 2$	1.3977	1.4079	1.5108	1.5213
	$-2, 3$	2.0853	2.0940	2.2234	2.2323
$^2D_{5/2}$	$-3, 0$	0.0000	0.0257	0.0000	0.0000
	$-3, 1$	0.4500	0.4568	0.4957	0.5036
$^2F_{7/2}$	$-4, 0$	0.0000	0.0193	0.0000	0.0000
$^2P_{1/2}$	$1, 0$	0.6402	0.7090	0.7901	0.8502
	$1, 1$	1.6259	1.6561	1.7974	1.8268
	$1, 2$	2.3902	2.4116	2.5702	2.5914
$^2D_{3/2}$	$2, 0$	0.3692	0.4059	0.4615	0.4929
	$2, 1$	1.1200	1.1362	1.2506	1.2662
$^2F_{5/2}$	$3, 0$	0.2545	0.2793	0.3193	0.3404
$^2G_{7/2}$	$4, 0$	0.1933	0.2120	0.2428	0.2587

the quasi-momentum is defined by (23) and phases Θ_1, Θ_2 are

$$\Theta_1 = -\frac{2}{\sqrt{(a-c)(b-d)}} \left[\frac{\mu(b-c)^2}{\Re_1} \Big[N_1 F(\Phi, \xi) + N_2 E(\Phi, \xi) + \right.$$
$$+ N_3 \Pi(\Phi, \nu, \xi) + N_4 \Pi(\Phi, \frac{c}{b}\nu, \xi) + L \Big]$$
$$\left. + \frac{k}{\mu} \left[\frac{b-c}{2} (N_5 \Pi(\Phi, \nu_1, \xi) + N_6 \Pi(\Phi, \nu_2, \xi)) + N_7 F(\Phi, \xi) \right] \right] + \frac{\pi}{4}, \quad (65)$$

$$\Theta_2 = -\frac{2}{\sqrt{(a-c)(b-d)}} \left\{ \frac{\mu(b-c)^2}{\Re_1} [N_1 F(\Phi, \xi) + N_2 E(\Phi, \xi) \right.$$
$$+ N_3 \Pi(\Phi, \nu, \xi) + N_4 \Pi(\Phi, \frac{c}{b}\nu, \xi) + L$$
$$\left. - \frac{k}{\mu} \left[\frac{b-c}{2} (N_8 \Pi(\Phi, \nu_1^*, \xi) + N_9 \Pi(\Phi, \nu_2^*, \xi)) + N_{10} F(\Phi, \xi) \right] \right\} + \frac{\pi}{4}. \quad (66)$$

Here

$$\Phi = \arcsin \sqrt{\frac{(a-c)(r-b)}{(a-b)(r-c)}}, \quad \Delta = \sqrt{1 - \xi^2 \sin^2 \Phi},$$

$$L = \frac{\nu^2(b-c)^3}{\Re_1} \frac{\Delta \sin \Phi \cos \Phi}{1 - \nu \sin^2 \Phi} \left[\frac{3(b-c)\aleph_1^2}{8\Re_1} + \frac{1}{4\left(1 - \nu \sin^2 \Phi\right)} + \frac{h}{2\left(b-c\right)} \right],$$

$$\lambda^* = \lambda_+^* - \lambda_-^*, \quad \lambda_\pm^* = \frac{E - m \pm \sqrt{(E-m)^2 + 4\mu\alpha}}{2\mu},$$

$$N_8 = [(b - \lambda_+^*)(\lambda_+^* - c)]^{-1}, \quad N_9 = [(b - \lambda_-^*)(\lambda_-^* - c)]^{-1}, \quad \nu_1^* = \frac{\lambda_+^* - c}{\lambda_+^* - b} \nu,$$

$$N_{10} = (c - (E-m)/(2\mu)) \left[(\lambda_+^* - c)(\lambda_-^* - c)\right]^{-1}, \quad \nu_2^* = \frac{\lambda_-^* - c}{\lambda_-^* - b} \nu;$$

$F(\Phi, \xi)$, $E(\Phi, \xi)$ and $\Pi(\Phi, \nu, \xi)$ are the elliptic integrals of the first, second, and third kind, respectively.

For determination of various characteristics of mesons (size of $Q\bar{q}$ system, polarization coefficient of meson in an external field, etc.) it is necessary to know the mean value of r^i ($i \in \mathbf{Z}$). In the WKB approximation it is of the form

$$\langle r^i \rangle = \int_{r_0}^{r_-} \Psi^+ r^i \Psi \, dr = \int_{r_0}^{r_-} \left(|F(r)|^2 + |G(r)|^2 \right) r^i dr$$

$$= \frac{2}{T} \int_{r_0}^{r_-} \frac{E - V(r)}{p(r)} r^i dr. \tag{67}$$

Specifically the mean radius of heavy-light quark system is given by the expression

$$\langle r \rangle = \frac{\lambda_1 F(\xi) + \lambda_2 E(\xi) + \lambda_3 \Pi(\nu, \xi)}{\lambda_4 F(\xi) + \lambda_5 \Pi(\nu, \xi)}, \tag{68}$$

where

$$\lambda_1 = \alpha c + E\left(c^2 - \frac{(b-c)^2}{2(1-\nu)}\right), \quad \lambda_2 = -\frac{\nu(b-c)^2}{2\Re_1} E,$$

$$\lambda_3 = (b-c)\left[\alpha + E\left(2c + \frac{(b-c)\aleph_1}{2\Re_1}\right)\right], \quad \lambda_4 = \alpha + Ec, \quad \lambda_5 = (b-c)E.$$

7 Summary

We have shown in this paper that the relativistic version of the WKB approximation is an accurate and powerful method. It makes possible the derivation of reliable analytical formulas, from which one can easily study, e.g., the dependence of some physical quantities (the relativistic mass spectrum, confinement of quarks, decay widths, wavefunction of Dirac particle, etc.) on the parameters of potentials, quarks masses, quantum numbers. Though sometimes the WKB method yields complicated formulas, it is often possible to extract interesting information from approximate analytical relations and in this way guide full quantum calculations.

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