Studies of the University of Žilina
Mathematical Series
Vol. 23/2009, 67–76

INVESTIGATING OF BOUNDARY VALUE PROBLEMS FOR ORDINARY DIFFERENTIAL EQUATIONS IN A CRITICAL CASE

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ABSTRACT. The numerical-analytic method for investigating and approximate constructing of the solutions of boundary value problems for nonlinear differential systems in a critical case is suggested.

1. Introduction

The theory of boundary value problems (BVP) is an important branch of the general theory of differential equations due to their strong connection with the practical applications. Various types of methods (numerical, analytic, numerical-analytic, functional etc) [1–7] for investigating the problems of existing and approximate constructing of the solutions are studied.

The modification of the numerical-analytic method, which is developed in this paper allow us to study BVP

$$\frac{dx}{dt} = A(t)x + f(t,x), \qquad B_1x(a) + B_2x(b) + \int_a^b [dB(t)]\,x(t) = d$$

in a critical [2] case – when there exist a nontrivial solutions of the correspondence linear homogeneous problem. We can use this modification when the boundary condition is "degenerate" [7]. Also we can note that due to using this modification the restriction for Lipschitz matrix concerns not the whole right part of the differential equation, but only the nonlinearity $f(t,x)$, and this is less difficult to deal with.

2. Linear BVP

In the beginning let us consider a linear inhomogeneous system of differential equations

$$\frac{dx}{dt} = A(t)x + h(t), \tag{1}$$

Key words and phrases. ordinary differential equations, boundary value problems, numerical-analytic method, critical case, successive approximations.

with the additional linear condition, which can be represent by means of Riemann-Stieltjes integral

$$B_1x(a) + B_2x(b) + \int\limits_a^b [dB(t)]\,x(t) = d. \tag{2}$$

Let us assume that matrix $A(t) : [a,b] \to \mathbb{R}^{n\times n}$ and function $h(t) : [a,b] \to \mathbb{R}^n$ are continuous, B_1, $B_2 \in \mathbb{R}^{n\times n}$ are the constant matrices, matrix $B(t) : [a,b] \to \mathbb{R}^{n\times n}$ is of bounded variation, $d \in \mathbb{R}^n$.

A continuously differentiable n-dimensional function $x \in C^1([a,b],\mathbb{R}^n)$, $t \in [a,b]$ is said to be a solution of the problem (1), (2) if it satisfies the equation (1) and verifies also the boundary condition (2).

It is known that the solution $x(t,x_0)$ of the differential system (1) with the initial value $x(a) = x_0$ is of the form

$$x(t,x_0) = \Omega_a^t x_0 + \int\limits_a^t \Omega_s^t h(s)ds, \tag{3}$$

where Ω_a^t, $\Omega_a^a = I_n$ is a matriciant of the corresponding to (1) homogeneous system of differential equations

$$\frac{dx}{dt} = A(t)x, \tag{4}$$

I_n is identity $(n\times n)$-matrix.

While substituting (3) into the boundary condition (2) one can see that $x(t,x_0)$ satisfies the boundary condition (2) if and only if x_0 is a solution of the algebraic system

$$Gx_0 = d - \int\limits_a^b W(s)h(s)ds, \tag{5}$$

where

$$W(s) = B_2\Omega_s^b + \int\limits_s^b [dB(t)]\Omega_s^t, \quad G = B_1 + W(a) = B_1 + B_2\Omega_a^b + \int\limits_a^b [dB(t)]\,\Omega_a^t. \tag{6}$$

Obviously that in a noncritical case [2] – when a correspondent to (1), (2) homogeneous BVP

$$\frac{dx}{dt} = A\,(t)\,x, \qquad B_1x(a) + B_2x(b) + \int\limits_a^b [dB(t)]\,x(t) = 0 \tag{7}$$

does not have the nontrivial solutions, the algebraic system (5) has a unique solution

$$x_0 = G^{-1}\left(d - \int\limits_a^b W(s)h(s)ds\right),$$

which is the initial value of a unique solution of the BVP (1), (2)

$$x(t) = \Omega_a^t G^{-1} d - \Omega_a^t G^{-1} \int_a^b W(s)h(s)ds + \int_a^t \Omega_s^t h(s)ds.$$

Let us consider the problem of existence of the solutions of the BVP (1), (2) in a critical case [2] i.e. when

A) the corresponding homogeneous BVP (7) has k nontrivial linearly independent solutions, $k = n - rank(G)$, $1 \leq k \leq n$.

Lemma 2.1. *Assume that the linear homogeneous BVP* (2), (4) *has k linearly independent solutions. Then for an arbitrary function $h(t)$ there exist a function $H(t)$ such that inhomogeneous differential system*

$$x' = A(t)x + h(t) + H(t), \tag{8}$$

possesses a k-parametric family of solutions, which satisfies the boundary condition (2).

Proof. From (5) we have [2, 4] that the BVP (8), (2) has a solution if and only if the condition

$$P_{G^*}\left(d - \int_a^b W(s)\Big(h(s) + H(s)\Big)ds\right) = 0 \tag{9}$$

is fulfilled, where P_{G^*} is an orthoprojector from the space $\mathbb{R}^n$ to the null space $\ker G^*$. Respectively P_G is an orthoprojector from the space $\mathbb{R}^n$ to the null space $\ker G$:

$$P_G\colon \mathbb{R}^n \to \ker G, \qquad \ker G = \{y : y \in \mathbb{R}^n,\ Gy = 0\},$$
$$P_{G^*}\colon \mathbb{R}^n \to \ker G^*, \qquad \ker G^* = \{z : z \in \mathbb{R}^n,\ zG^* = 0\},$$
$$rankP_G = rankP_{G_k} = rankP_{G^*} = rankP_{G_k^*} = k.$$

We will denote by P_{G_k} the $(n \times k)$-matrix, which columns constituting a basis of the kernel $\ker(G)$ and they are the linearly independent columns of P_G. Respectively by $P_{G_k^*}$ we denote $(k \times n)$-matrix, which rows constituting a basis of the $\ker(G^*)$ and are the linearly independent rows of matrix P_{G^*}.

Let us denote

$$H(t) = W^*(t)(P_{G_k^*})^* R_1^{-1} P_{G_k^*}\left(d - \int_a^b W(s)h(s)ds\right), \tag{10}$$

where

$$R_1 = P_{G_k^*} R_2 (P_{G_k^*})^*,$$
$$R_2 = \int_a^b W(s)W^*(s)ds = B_2 \int_a^b \Omega_s^b W^*(s)ds + \int_a^b [dB(t)] \int_a^t \Omega_s^t W^*(s)ds. \tag{11}$$

Substituting (10) into (9) we obtain

$$P_{G_k^*}\left(d-\int_a^b W(s)\Big(h(s)+H(s)\Big)ds\right)=P_{G_k^*}\left(d-\int_a^b W(s)h(s)ds\right.$$
$$\left.-\int_a^b W(s)W^*(s)ds(P_{G_k^*})^*R_1^{-1}P_{G_k^*}\left(d-\int_a^b W(s)h(s)ds\right)\right)$$
$$=\left(I_k-P_{G_k^*}\int_a^b W(s)W^*(s)ds(P_{G_k^*})^*R_1^{-1}\right)P_{G_k^*}\left(d-\int_a^b W(s)h(s)ds\right)=0.$$

Thus condition (9) is fulfilled if $H(t)$ is of the form (10). The solution $x(t,x_0)$ of the system (8), (10) with the initial value $x(a,x_0)=x_0$ has the form

$$x(t,x_0)=\Omega_a^t x_0+\int_a^t \Omega_s^t\left\{h(s)+W^*(s)(P_{G_k^*})^*R_1^{-1}P_{G_k^*}\left(d-\int_a^b W(s)h(s)ds\right)\right\}ds. \quad (12)$$

Substituting $x(t,x_0)$ of the form (12) into the boundary condition (2) we can see that $x(t,x_0)$ satisfies the boundary condition (2) if and only if the initial value x_0 is a solution of the algebraic system

$$Gx_0=d-\int_a^b W(s)\left\{h(s)+W^*(s)(P_{G_k^*})^*R_1^{-1}P_{G_k^*}\left(d-\int_a^b W(s)h(s)ds\right)\right\},$$

which we can rewrite in the form

$$Gx_0=\left(I_n-R_2(P_{G_k^*})^*R_1^{-1}P_{G_k^*}\right)\left(d-\int_a^b W(s)h(s)ds\right). \quad (13)$$

Multiplying the right side of this equation from the left by matrix $P_{G_k^*}$ we will obtain the zero vector. It means that system (13) is solvable and (because $rankG=n-k$) it has k-parametric solution of the form [2, 4]

$$x_0=P_{G_k}\xi+G^+\left(I_n-R_2(P_{G_k^*})^*R_1^{-1}P_{G_k^*}\right)\left(d-\int_a^b W(s)h(s)ds\right), \quad (14)$$

where G^+ is a unique Moore-Penrose generalized inverse $(n\times n)$-matrix [2, 4, 8, 9], $\xi\in R^k$ is an arbitrary vector. Substituting x_0 of the form (14) into (12) we obtain the general k-parametric solution of the BVP (2), (8):

$$x(t,x_0)=x(t,\xi)$$
$$=\Omega_a^t P_{G_k}\xi+\Omega_a^t G^+\left(I_n-R_2(P_{G_k^*})^*R_1^{-1}P_{G_k^*}\right)\left(d-\int_a^b W(s)h(s)ds\right)$$
$$+\int_a^t \Omega_s^t\left\{h(s)+W^*(s)(P_{G_k^*})^*R_1^{-1}P_{G_k^*}\left(d-\int_a^b W(s)h(s)ds\right)\right\}ds.$$

Finally we can rewrite it in the form

$$x(t,\xi) = \Omega_a^t P_{G_k}\xi + \Omega_a^t \Big(G^+ + (R(t) - G^+ R_2)(P_{G_k^*})^* R_1^{-1} P_{G_k^*}\Big) d + \int_a^b L(t,s)h(s)ds,$$

where

$$R(t) = \int_a^t \Omega_s^a W^*(s)ds,$$

and

$$L(t,s) = \begin{cases} \Omega_s^t - \Omega_a^t \big(G^+ + \big(R(t) - G^+ R_2\big)(P_{G_k^*})^* R_1^{-1} P_{G_k^*}\big) W(s), & 0 \le s \le t \le b, \\ -\Omega_a^t \big(G^+ + \big(R(t) - G^+ R_2\big)(P_{G_k^*})^* R_1^{-1} P_{G_k^*}\big) W(s), & 0 \le t < s \le b. \end{cases}$$

The lemma is proved. □

3. The numerical-analytic method for investigating nonlinear differential systems with linear boundary condition

Now let us discuss the problem of existence and approximate constructing of the solutions of nonlinear differential systems

$$\frac{dx}{dt} = A(t)x + f(t,x), \quad x, f \in R^n, \tag{15}$$

which fulfills the additional condition (2). We will consider a critical case i.e. the case when the condition **A** is fulfilled.

We suppose that on $\Omega = [a,b] \times D$, where $D \subset \mathbb{R}^n$ is a closed and bounded domain, the following conditions are hold:

B) matrix $A(t) : [a,b] \to \mathbb{R}^{n \times n}$ and function $f : \Omega \to \mathbb{R}^n$ are continuous for t and the following inequalities are hold:

$$|f(t,x)| \le m(t), \qquad |f(t,x') - f(t,x'')| \le K(t)|x' - x''|,$$

where vector $m(t)$ and matrix $K(t)$ are continuous and their components are nonnegative;

C) the domain $D_0 \equiv \big\{\xi \in R^k \mid B(x_0(t,\xi), \beta) \subseteq D\big\}$ is not empty:

$$D_0 \ne \emptyset,$$

where

$$\beta = \max_{t \in [a,b]} \left(\left| \Omega_a^t \Big(G^+ + (R(t) - G^+ R_2)(P_{G_k^*})^* R_1^{-1} P_{G_k^*}\Big) d \right| + \int_a^b \big|L(t,s)\big| m(s) ds \right)$$

and $B(y,\varrho) = \{x \in \mathbb{R}^n : |x - y| \le \varrho\}$ for fixed $y, \varrho \in \mathbb{R}^n$;

D) maximum eigenvalue of the following matrix Q is less then one:

$$Q = \max_{t \in [a,b]} \int_a^b \big|L(t,s)\big| K(s) ds.$$

The notation $|\cdot|$ means the absolute value: $|x| = col(|x_1|, \ldots, |x_n|)$, where $x \in R^n$ and $|A| = (|A_{ij}|)_{i,j=1}^n$. The inequalities are meant componentwise.

Next lemmas give us necessary and sufficient conditions for existing solutions of the BVP (15), (2).

Lemma 3.1 ([10]). *Let the linear homogeneous BVP* (7) *has* k, $1 \le k \le n$ *linearly independent solutions and the vector-function* $\varphi(t) \in C^1([a,b], \mathbb{R}^n)$ *is a solution of the BVP* (15), (2) *with the initial value*

$$\varphi(a) = \varphi_0 := P_{G_k}\xi + G^+\left(d - \int_a^b W(s)f(s,\varphi(s))ds\right). \tag{16}$$

Then φ *is a solution of the system of equations*

$$x(t) = \Omega_a^t P_{G_k}\xi$$
$$+\Omega_a^t\left(G^+ + (R(t) - G^+R_2)(P_{G_k^*})^* R_1^{-1} P_{G_k^*}\right)d + \int_a^b L(t,s)f(s,x(s))ds, \tag{17}$$

$$P_{G_k^*}\left(d - \int_a^b W(s)f(s,x(s))ds\right) = 0. \tag{18}$$

Lemma 3.2. *Let the linear homogeneous BVP* (7) *has* k, $1 \le k \le n$ *linearly independent solutions. Then:*

1) *if the vector-function* $\varphi(t)$ *satisfies the equation* (17) *then* $\varphi(t) \in C^1([a,b], \mathbb{R}^n)$ *and* $\varphi(t)$ *satisfies the boundary condition* (2);
2) *if furthermore function* $\varphi(t) = \varphi(t, \xi^*)$ *satisfies the equation* (18), *then* $\varphi(t) = \varphi(t, \xi^*)$ *is a solution of the BVP* (15), (2) *with the initial value* (16).

Proof. Let us assume that $\varphi(t)$ satisfies the equation (17).
Then $\varphi(t) \in C^1([a,b], \mathbb{R}^n)$ and the identity

$$\varphi(t) \equiv \Omega_a^t P_{G_k}\xi + \Omega_a^t G^+\left(I_n - R_2(P_{G_k^*})^* R_1^{-1} P_{G_k^*}\right)\left(d - \int_a^b W(s)f(s,\varphi(s))ds\right)$$
$$+ \int_a^t \Omega_s^t f(s,\varphi(s))ds$$
$$+ \int_a^t \Omega_s^t W^*(s)ds(P_{G_k^*})^* R_1^{-1} P_{G_k^*}\left(d - \int_a^b W(s)f(s,\varphi(s))ds\right) \tag{19}$$

is fulfilled. Substituting $\varphi(t)$ of the form (19) into the boundary condition (2) and taking into the consideration (6), (11) we obtain

$$
\begin{aligned}
&d - B_1\varphi(0) - B_2\varphi(b) - \int_a^b [dB(t)]\varphi(t) \\
&= d - \left(B_1 + B_2\Omega_a^b - \int_a^b [dB(t)]\Omega_a^t\right) \\
&\qquad \times \left(P_{G_k}\xi + G^+\left(I_n - R_2(P_{G_k^*})^* R_1^{-1} P_{G_k^*}\right)\right)\left(d - \int_a^b W(s)f(s,\varphi(s))ds\right) \\
&\quad -\left(B_2\int_a^b \Omega_s^b f(s,\varphi(s))ds + \int_a^b [dB(t)]\int_a^t \Omega_s^t f(s,\varphi(s))ds\right) \\
&\quad -\left(B_2\int_a^b \Omega_s^b W^*(s)ds + \int_a^b [dB(t)]\int_a^t \Omega_s^t W^*(s)ds\right) \\
&\qquad\qquad \times (P_{G_k^*})^* R_1^{-1} P_{G_k^*}\left(d - \int_a^b W(s)f(s,\varphi(s))ds\right) \\
&= d - GG^+\left(I_n - R_2(P_{G_k^*})^* R_1^{-1} P_{G_k^*}\right)\left(d - \int_a^b W(s)f(s,\varphi(s))ds\right) \\
&\quad - \int_a^b W(s)f(s,\varphi(s))ds - R_2(P_{G_k^*})^* R_1^{-1} P_{G_k^*}\left(d - \int_a^b W(s)f(s,\varphi(s))ds\right) \\
&= (I_n - GG^+)\left(I_n - R_2(P_{G_k^*})^* R_1^{-1} P_{G_k^*}\right)\left(d - \int_a^b W(s)f(s,\varphi(s))ds\right) \\
&= P_{G_k^*}\left(I_n - R_2(P_{G_k^*})^* R_1^{-1} P_{G_k^*}\right)\left(d - \int_a^b W(s)f(s,\varphi(s))ds\right) \\
&= \left(P_{G_k^*} - P_{G_k^*}R_2(P_{G_k^*})^* R_1^{-1} P_{G_k^*}\right)\left(d - \int_a^b W(s)f(s,\varphi(s))ds\right) = 0.
\end{aligned}
$$

Therefore $\varphi(t)$ satisfies the boundary condition (2). If furthermore (18) holds, then from (19) it follows that $\varphi(t)$ satisfies the identity

$$\varphi(t) \equiv \Omega_a^t P_{G_k}\xi + \Omega_a^t G^+\left(d - \int_a^b W(s)f(s,\varphi(s))ds\right) + \int_a^t \Omega_s^t f(s,\varphi(s))ds, \quad (20)$$

i.e. $\varphi(t)$ is a solution of the system (15) with the initial value (16).

Thus the lemma is proved. □

For investigating the problem of existence and approximate constructing of the solutions of the BVP (15), (2) we consider the k-parametric sequence of functions given by the formula

$$x_m(t,\xi)=x_0(t,\xi)+\Omega_a^t\Big(G^+ +\big(R(t)-G^+R_2\big)(P_{G_k^*})^*R_1^{-1}P_{G_k^*}\Big)d$$
$$+\int_a^b L(t,s)f(s,x_{m-1}(s,\xi))ds, \quad x_0(t,\xi)=\Omega_a^t P_{G_k}\xi, \quad \xi\in\mathbb{R}^k,\ m=1,2,\dots . \quad (21)$$

It can be easily verified that each of this functions satisfies the boundary condition (2). Let us now establish the main result of this paper.

Theorem 3.3. *Assume that for the BVP* (2), (15) *the conditions A—D are hold. Then:*

1) *the sequence of functions* (21) *converges as* $m\to\infty$, *uniformly in* $(t,\xi)\in[a,b]\times D_0$ *to the limit function* $x^*(t,\xi)$ *and the following error estimate holds*
$$|x^*(t,\xi)-x_m(t,\xi)|\le(E-Q)^{-1}Q^m\beta; \quad (22)$$
2) *the limit function* $x^*(t)=x^*(t,\xi^*)$ *is a solution of BVP* (2), (15) *if and only if* ξ^* *is a solution of the determining equation* $\Delta(\xi)=0$, *where*
$$\Delta(\xi)\overset{def}{=}P_{G_k^*}\left(d-\int_a^b W(s)f(s,x^*(s,\xi))ds\right) \quad (23)$$
and its initial value is
$$x^*(a)=P_{G_k}\xi^*+G^+\left(d-\int_a^b W(s)f(s,x^*(s))ds\right). \quad (24)$$

Proof. Considering the difference

$$|x_1(t,\xi)-x_0(t,\xi)|\le\left|\Omega_a^t\Big(G^+ +\big(R(t)-G^+R_2\big)(P_{G_k^*})^*R_1^{-1}P_{G_k^*}\Big)d\right|$$
$$+\int_a^b |L(t,s)f(s,x_0(s,\xi))|\,ds\le\beta,$$

we see that $x_1(t,\xi)\in D$. It can be shown by induction that $x_m(t,\xi)\in D$ for all $\xi\in D_0$, $m\in\mathbb{N}$. Furthermore, from the Lipschitz condition we obtain the estimates:

$$|x_{m+1}(t,\xi)-x_m(t,\xi)|\le\int_a^b |L(t,s)|\cdot|f(s,x_m(s,\xi))-f(s,x_{m-1}(s,\xi))|\,ds$$
$$\le Q|x_m(t,\xi)-x_{m-1}(t,\xi)|\le Q^2|x_{m-1}(t,\xi)-x_{m-2}(t,\xi)|\le\dots\le Q^m\beta,$$

From (22), (23), (26) and the Lipschitz condition we have

$$|\Delta(\xi) - \Delta_m(\xi)| \le \int_a^b \left|P_{G_k^*} W(s)\right| \cdot |f(s, x^*(s,\xi)) - f(s, x_m(s,\xi))\, ds| \le$$

$$\le \int_a^b \left|P_{G_k^*} W(s)\right| K(s) |x^*(s,\xi) - x_m(s,\xi)| ds \le Q_1 (I_n - Q)^{-1} Q^m \beta.$$

But in this case from (28) we obtain the inequality

$$|\Delta_m(\xi)| \le |\Delta(\xi) - \Delta_m(\xi)| \le Q_1 (I_n - Q)^{-1} Q^m \beta,$$

which contradict the condition (27). It means that the field family $\Delta(\theta, \xi)$ does not assume the value zero on ∂D_1, therefore vector fields $\Delta(\xi)$ and $\Delta_m(\xi)$ are homotopic. It means that the rotation of vector field $\Delta(\xi)$ on the boundary ∂D_1 is also non-zero and consequently $\Delta(\xi)$ assumes the value zero at least in one point $\xi = \xi^* \in D_1$. Thus the theorem is proved. □

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