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FURTHER RESULTS ON THE INVESTIGATION OF SOLUTIONS OF INTEGRAL BOUNDARY VALUE PROBLEMS

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ABSTRACT. We give a new approach for the investigation of existence and construction of an approximate solutions of nonlinear non-autonomous systems of ordinary differential equations under nonlinear integral boundary conditions depending on the derivative. The constructivity of a suggested technique is shown on the example of non-linear integral boundary value problem with two solutions.

1. Introduction

In this paper we use the technique suggested in [1] for the investigation of existence and approximate construction of solutions of a new class of non-linear boundary value problems with nonlinear integral boundary conditions involving the derivative. At first, we reduce the given problem to a simpler model problem with two-point separated linear parametrized boundary conditions. Then, the transformed problem is replaced by the Cauchy problem for a suitably perturbed system containing some artificially introduced vector parameters. The solution of the Cauchy problem for the perturbed system is sought out by successive approximations. We give conditions sufficient for the uniform convergence of the successive approximations. The functional perturbation term, by which the modified equation differs from the original one, essentially depends on the parameters and generates finitely many *determining* equations from which the numerical values of the parameters should be found. The solvability of the determining equations, in turn, may be checked by studying some approximations that can be constructed explicitly.

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Such an approach belongs to the few of them that offer constructive possibilities both for the investigation of the existence of solution and it approximate construction, see, e.g, [3], [5], [6], [9]–[13], [15], [17].

2. Notation and symbols

In the sequel, for any vector $x = \operatorname{col}(x_1, \dots, x_n) \in \mathbb{R}^n$ the obvious notation $|x| = \operatorname{col}(|x_1|, \dots, |x_n|)$ is used and the inequalities between vectors are understood componentwise.

The same convention is adopted implicitly for operations 'max', 'min', 'sup', 'inf'. The symbols 1_n and 0_n stand respectively for the unit and zero matrix of dimension n, and $r(K)$ denotes the maximal, in modulus, eigenvalue of a square matrix K.

Definition 1. For any non-negative vector $\rho \in \mathbb{R}^n$ under the componentwise ρ-neighbourhood of a point $z \in \mathbb{R}^n$ we understand

$$B(z, \rho) := \left\{ \xi \in \mathbb{R}^n : |\xi - z| \leq \rho \right\}. \tag{2.1}$$

Similarly, for the given bounded connected set $\Omega \subset \mathbb{R}^n$, we define its componentwise ρ-neighbourhood by putting

$$B(\Omega, \rho) := \underset{\xi \in \Omega}{\cup} B(\xi, \rho). \tag{2.2}$$

Definition 2. For given two bounded connected sets $D_a \subset \mathbb{R}^n$ and $D_b \subset \mathbb{R}^n$, introduce the set

$$D_{a,b} := (1 - \theta)z + \theta\eta, \qquad z \in D_a, \ \ \eta \in D_b, \ \ \theta \in [0, 1] \tag{2.3}$$

and its componentwise ρ-neighbourhood

$$D := B(D_{a,b}, \rho). \tag{2.4}$$

For a set $D \subset \mathbb{R}^n$, closed interval $[a, b] \subset \mathbb{R}$, Caratheodory function $f : [a, b] \times D \to \mathbb{R}^n$, $n \times n$ matrix K with non-negative entires, we write

$$f \in Lip(K, D) \tag{2.5}$$

if the inequality

$$|f(t, u) - f(t, v)| \leq K \, |u - v| \tag{2.6}$$

holds for all $\{u, v\} \subset D$ and a.e. $t \in [a, b]$.

Finally, on the base of function $f : [a, b] \times D \to \mathbb{R}^n$ we introduce the vector

$$\delta_{[a,b],D}(f) := \frac{1}{2} \left[\underset{(t,x) \in [a,b] \times D}{ess \sup} f(t, x) \ - \underset{(t,x) \in [a,b] \times D}{ess \inf} f(t, x) \right]. \tag{2.7}$$

3. Problem setting and reduction to a model-type, some subsidiary statements

Let us consider the nonlinear integral boundary value problem

$$\frac{dx}{dt} = f(t,x), \qquad t \in [a,b], \tag{3.1}$$

$$\int_a^b \Big[g\big(s, x(s)\big) + h\big(s, x'(s)\big)\Big] ds = d. \tag{3.2}$$

Let D_a and D_b be convex subsets of $\mathbb{R}^n$, where one looks for the values of the solution of the boundary value problem (3.1), (3.2) at a and b, respectively. Based on the sets D_a and D_b we introduce the set $D_{a,b}$ according to (2.3) and its componentwise ρ-neighbourhood D as in (2.4). So, the domain of the space variables in the given problem is D defined according to (2.4).

From now on, we suppose that the functions

$$f : [a,b] \times D \to \mathbb{R}^n \quad \text{and} \quad g : [a,b] \times D \to \mathbb{R}^n, \quad h : [a,b] \times D \to \mathbb{R}^n$$

satisfy the Caratheodory and the Lipschitz condition in the domain D with ρ satisfying the inequality

$$\rho \geq \frac{b-a}{2}\delta_{[a,b],D}(f). \tag{3.3}$$

Here $\delta_{[a,b],D}(f)$ is given in (2.7) and for the maximal in modulus eigenvalue of the matrix

$$Q = \frac{3(b-a)}{10}K \tag{3.4}$$

holds

$$r(Q) < 1. \tag{3.5}$$

It is important to emphasize that $D \subset \mathbb{R}^n$ is bounded and, thus, the Lipschitz condition is not assumed globally.

The problem is to find an absolutely continuous solution $x\colon [a,b] \to D$ of the problem (3.1), (3.2) with initial value $x(a) \in D_a$.

At first we simplify the boundary conditions (3.2) and reduce them to some two-point separated conditions. To replace the boundary conditions (3.2) by certain linear two-point linear separated ones, similarly to [8]–[12], [14], [16], we apply a certain "freezing" technique. Namely, we introduce the vectors of parameters

$$z = \operatorname{col}(z_1, z_2, \ldots, z_n), \quad \eta = \operatorname{col}(\eta_1, \eta_2, \ldots, \eta_n) \tag{3.6}$$

by formally putting

$$z := x(a), \quad \eta = x(b). \tag{3.7}$$

Now, instead of the integral boundary value problem (3.1), (3.2) we will consider the following "model-type", two-point BVP with separated parameterized conditions

$$\frac{dx}{dt} = f\left(t, x\right), \quad t \in [a, b]\,, \tag{3.8}$$

$$x(a) = z, \quad x(b) = \eta. \tag{3.9}$$

The parametrization technique that we are going to use suggest that, instead of the original boundary value problem with nonlinear integral boundary conditions (3.2), we study the family of parametrized boundary value problems (3.8), (3.9), where the boundary restrictions are linear and separated. We then go back to the original problem by choosing the values of the introduced parameters appropriately.

Remark 1. The set of solutions of the non-linear integral boundary value problem (3.1), (3.2) coincides with the set of the solutions of the parametrized problem (3.8), (3.9) with separated restrictions, satisfying the additional conditions (3.9).

We recall some subsidiary statements which are needed below in the following form.

Lemma 1 ([4, Lemma 3.13])**.** *Let $f\colon [\tau, \tau+I] \to \mathbb{R}^n$ be a continuous function. Then, for an arbitrary $t \in [\tau, \tau+I]$, the inequality*

$$\left| \int_{\tau}^{t} \left[f(\tau) - \frac{1}{I} \int_{\tau}^{\tau+I} f(s)\,ds \right] d\tau \right| \le \alpha_1(t, \tau, I)\, \delta_{[\tau,\tau+I]}(f) \tag{3.10}$$

holds, where

$$\alpha_1(t, \tau, I) = 2\,(t-\tau)\left(1 - \frac{t-\tau}{I}\right), \qquad |\alpha_1(t, \tau, I)| \le \frac{I}{2}, \quad t \in [\tau, \tau+I]\,, \tag{3.11}$$

and

$$\delta_{[\tau,\tau+I]}(f) = \frac{\max\limits_{t\in[\tau,\tau+I]} f(t) - \min\limits_{t\in[\tau,\tau+I]} f(t)}{2}.$$

Lemma 2 ([4, Lemma 3.16])**.** *Let the sequence of continuous functions*

$$\left\{\alpha_m(t, \tau, I)\right\}_{m=0}^{\infty}, \qquad \textit{for} \quad t \in [\tau, \tau+I]$$

be defined by the recurrence relation

$$\alpha_{m+1}(t, \tau, I) = \left(1 - \frac{t-\tau}{I}\right)\int_{\tau}^{t} \alpha_m(s, \tau, I)\,ds + \frac{t-\tau}{I}\int_{t}^{\tau+I} \alpha_m(s, \tau, I)\,ds,$$
$$m = 0, 1, 2, \dots, \quad \textit{where} \quad \alpha_0(t, \tau, I) = 1. \tag{3.12}$$

Then the following estimates hold for $t \in [\tau, \tau + I]$:

$$\alpha_{m+1}(t,\tau,I) \leq \frac{10}{9}\left(\frac{3I}{10}\right)^m \alpha_1(t,\tau,I), \qquad m \geqslant 0, \tag{3.13}$$

$$\alpha_{m+1}(t,\tau,I) \leq \frac{3I}{10}\,\alpha_m(t,\tau,I), \qquad m \geqslant 2,$$

where $\alpha_1(t,\tau,I)$ *is given in* (3.11).

4. Investigation of the model type BVP

Let us connect with the two-point model type BVP (3.8), (3.9) the sequence of functions

$$x_{m+1}(t,z,\eta) = z + \int_a^t f\big(s, x_m(s,z,\eta)\big)\,ds - \frac{t-a}{b-a}\int_a^b f\big(\tau, x_m(\tau,z,\eta)\big)\,d\tau \tag{4.1}$$
$$+ \frac{t-a}{b-a}\,[\eta - z]\,, \qquad t \in [a,b]\,,\ \ m = 1,2,\ldots,$$

satisfying (3.9) for arbitrary $z, \eta \in \mathbb{R}^n$, where

$$x_0(t,z,\eta) = z + \frac{t-a}{b-a}\,[\eta - z] = \left(1 - \frac{t-a}{b-a}\right) z + \frac{t-a}{b-a}\,\eta, \qquad t \in [a,b] \tag{4.2}$$

and $z \in D_a,\ \eta \in D_b$ are considered as parameters.

It is easy to see from (4.2) that $x_0(t,z,\eta) \in D$ as a convex combination of vectors z and η, when $z \in D_a$, $\eta \in D_b$.

The following statement establishes the uniformly convergence of the sequence (4.1) to some parametrized limit function.

Theorem 1. *Let there exist a non negative vector* ρ *satisfying the inequality* (3.3) *and* $f : [a,b] \times D \to \mathbb{R}^n$ *be a function satisfying the Caratheodory and the Lipschitz condition* $f \in Lip(K,D)$ *in the domain* D *of form* (2.4) *with a matrix* K *for which*

$$r\left(Q = \frac{3(b-a)}{10} K\right) < 1. \tag{4.3}$$

Then, for all fixed $(z,\eta) \in D_a \times D_b$:

1. *The functions of the sequence* (4.1) *are absolutely continuous functions on the interval* $t \in [a,b]$, *have values in the domain* D *and satisfy the two-point separated boundary conditions* (3.9).

2. *The sequence of functions* (4.1) *in* $t \in [a, b]$ *converges uniformly as* $m \to \infty$ *to the limit function*
$$x_\infty(t, z, \eta) = \lim_{m\to\infty} x_m(t, z, \eta).$$
3. *The limit function satisfies the initial condition*
$$x_\infty(a, z, \eta) = z \tag{4.4}$$
and the two-point separated boundary conditions (3.9)*.*
4. *The function* $x_\infty(t, z, \eta)$ *is a unique absolutely continuous solution of the integral equation*
$$x(t) = z + \int_a^t f\big(s, x(s)\big)\, ds - \frac{t-a}{b-a}\int_a^b f\big(s, x(s)\big)\, ds + \frac{t-a}{b-a}\,[\eta - z]\,. \tag{4.5}$$
In other words, $x_\infty(t, z, \eta)$ *satisfies the Cauchy problem for the modified system of integro-differential equations:*
$$\frac{dx}{dt} = f(t, x) + \frac{1}{b-a}\Delta(z, \eta),$$
$$x(a) = z \tag{4.6}$$
where $\Delta(z, \eta)\colon D_a \times D_b \to \mathbb{R}^n$ *is a mapping given by formula*
$$\Delta(z, \eta) := \eta - z - \int_a^b f\big(s, x_\infty(s, z, \eta)\big)\, ds\,. \tag{4.7}$$
5. *The following error estimate holds:*
$$|x_\infty(t, z, \eta) - x_m(t, z, \eta)| \leqslant \frac{10}{9}\alpha_1(t, a, b) Q^m (1_n - Q)^{-1}\, \delta_{[a,b],D}(f), \tag{4.8}$$
for any $t \in [a, b]$ *and* $m \geq 0$*, where* $\delta_{[a,b],D}(f)$ *is given in* (2.7) *and*
$$\alpha_1(t, a, b) = 2(t-a)\left(1 - \frac{t-a}{b-a}\right) \tag{4.9}$$
is defined by (3.11) *for which*
$$\alpha_1(t, a, b) \leq \frac{b-a}{2}\,.$$
holds.

P r o o f. The validity of assertion 1 is verified by direct computation. To obtain the other required properties, similarly to [1] we will prove that under the condition assumed for fixed $z \in D_a$, $\eta \in D_b$ and $t \in [a, b]$ the functions of the sequence (4.1) are contained in the domain D and (4.1) is a Cauchy sequence in the Banach space $C\big([a, b], \mathbb{R}^n\big)$ equipped with the standard uniform norm.

Indeed, using the estimate (3.10) of Lemma1 for $\tau = a$, $I = b - a$, relation (4.1) for $m = 0$, $t \in [a, b]$ implies that

$$\begin{aligned} &|x_1(t,z,\eta) - x_0(t,z,\eta)| \\ &\le \frac{1}{2}\alpha_1(t,a,b)\left[ess\sup_{t\in[a,b]} f\big(t, x_0(t,z,\eta)\big) - ess\inf_{t\in[a,b]} f\big(t, x_0(t,z,\eta)\big)\right] \\ &\le \alpha_1(t,a,b)\delta_{[a,b],D}(f) \le \frac{b-a}{2}\delta_{[a,b],D}(f), \end{aligned} \tag{4.10}$$

which means, that $x_1(t,z,\lambda,\eta) \in D$, whenever $(t,z,\eta) \in [a,b] \times Da \times D_b$.

Using this and arguing by induction according to Lemma 1 we can easily establish that

$$\begin{aligned} |x_m(t,z,\eta) - x_0(t,z,\eta)| &\le \alpha_1(t,a,b)\delta_{[a,b],D}(f) \\ &\le \frac{b-a}{2}\delta_{[a,b],D}(f), \qquad m = 2,3,\ldots, \end{aligned} \tag{4.11}$$

which means that all the functions (4.1) are also contained in the domain D, for all $m = 1,2,3,\ldots$ and $(t,z,\eta) \in [a,b] \times D_a \times D_b$.

Now, consider the difference of functions

$$\begin{aligned} &x_{m+1}(t,z,\eta) - x_m(t,z,\eta) \\ &= \int_a^t \Big[f\big(s, x_m(s,z,\eta)\big) - f\big(s, x_{m-1}(s,z,\eta)\big)\Big] ds \\ &\quad - \frac{t-a}{b-a}\int_a^b \Big[f\big(s, x_m(s,z,\eta)\big) - f\big(s, x_{m-1}(s,z,\eta)\big)\Big] ds, \qquad m = 1,2,\ldots \end{aligned} \tag{4.12}$$

and introduce the notation

$$r_m(t,z,\eta) = |x_m(t,z,\eta) - x_{m-1}(t,z,\eta)|, \qquad m = 1,2,\ldots \tag{4.13}$$

According to the recurrence relation (3.12) of Lemma 2, using the Lipschitz condition (2.6) and the estimation (3.13), for $m = 1$ from (4.12) and (4.10) it follows that

$$\begin{aligned} r_2(t,z,\eta) &\le K\left[\left(1 - \frac{t-a}{b-a}\right)\int_0^t \alpha_1(s,a,b)\,ds + \frac{t-a}{b-a}\int_0^T \alpha_1(s,a,b)\,ds\right]\delta_{[a,b],D}(f) \\ &\le K\alpha_2(t,a,b)\delta_{[a,b],D}(f) \le \frac{10}{9}Q\alpha_1(t,a,b)\delta_{[a,b],D}(f), \end{aligned} \tag{4.14}$$

where the matrix Q has the form (3.4). By induction we can easily establish that

$$r_{m+1}(t,z,\eta) \le K^m\alpha_{m+1}(t,a,b)\delta_{[a,b],D}(f) \le \frac{10}{9}Q^m\alpha_1(t,a,b)\delta_{[a,b],D}(f). \tag{4.15}$$

Therefore, in view of (4.15)

$$\begin{aligned}
&|x_{m+j}(t,z,\eta)-x_m(t,z,\eta)| \leq |x_{m+j}(t,z,\eta)-x_{m+j-1}(t,z,\eta)| \\
&\quad + |x_{m+j-1}(t,z,\eta)-x_{m+j-2}(t,z,\eta)| + |x_{m+1}(t,z,\eta)-x_m(t,z,\eta)| \\
&= \sum_{i=1}^{j} r_{m+i}(t,z,\eta) \leq \frac{10}{9}\alpha_1(t,a,b)\sum_{i=1}^{j} Q^{m+i-1}\delta_{[a,b],D}(f) \\
&= \frac{10}{9}\alpha_1(t,a,b)Q^m\sum_{i=0}^{j-1} Q^i\delta_{[a,b],D}(f), \qquad (4.16)
\end{aligned}$$

where $\delta_{[a,b],D}(f)$ is given by (2.7). Since, due to (3.5), the maximum eigenvalue of the matrix Q does not exceed the unity, we have

$$\sum_{i=0}^{j-1} Q^i \leq (1_n - Q)^{-1}, \qquad \lim_{m\to\infty} Q^m = 0_n. \tag{4.17}$$

Therefore, we conclude from (4.16) that, according to Cauchy criterium, the sequence $\{x_m(t,z,\eta)\}_{m=0}^{\infty}$ of the form (4.1) uniformly converges in the domain $(t,z,\eta) \in [a,b]\times D_a \times D_b$ to the limit function $x_\infty(t,z,\eta)$. Since all functions of the sequence (4.1) satisfy the boundary conditions (3.9) for all values of the introduced parameter $z \in D_a$, $\eta \in D_b$ the limit function $x_\infty(t,z,\eta)$ also satisfies these conditions. Passing to the limit as $m\to\infty$ in the equality (4.1) we show that the limit function satisfies both the integral equation (4.5) and the Cauchy problem (4.6), where $\Delta(z,\eta)$ is given by (4.7). Passing to the limit as $j\to\infty$ in (4.16) we get the estimation (4.8). □

5. Connection of the limit function $x_\infty(\cdot,z,\eta)$ to the solution of the original integral BVP

Along with (3.1), consider the system of differential equations with the additive perturbation of the right-hand side

$$\frac{dx}{dt} = f(t,x) + \frac{1}{b-a}\mu, \qquad t\in[a,b], \tag{5.1}$$

with the initial condition

$$x(a) = z, \tag{5.2}$$

where

$$\mu = \operatorname{col}(\mu_1,\ldots,\mu_n) \in \mathbb{R}^n \tag{5.3}$$

is some control parameter.

Theorem 2. *Let $z \in D_a$ and $\eta \in D_b$ be fixed. Suppose that all conditions of Theorem 1 hold.*

Then, for the solution $x(\cdot, a, z)$ of the Cauchy problem (5.1), (5.2) we have the property

$$x(b, a, z) = \eta,$$

by other words, to satisfy the parametrized separated two-point boundary conditions (3.9), it is necessary and sufficient that the control parameter μ is given by the formula

$$\mu := \eta - z - \int_a^b f\big(s, x_\infty(s, z, \eta)\big)\, ds, \tag{5.4}$$

where $x_\infty(\cdot, z, \eta)$ is the limit function of the sequence (4.1). Moreover, in that case

$$x(\cdot, a, z) = x_\infty(\cdot, z, \eta). \tag{5.5}$$

Proof. Sufficiency. Following to [14] let us suppose that μ in (5.1) is given according to (5.4). By virtue of Theorem1 the limit function $x_\infty(\cdot, z, \eta)$ of the sequence (4.1) satisfying the two-point boundary conditions (3.12) is a unique solution of the initial value problem (4.6), where $\Delta(z, \eta) = \mu$ is given by (5.4), i.e., it is a solution of the Cauchy problem (5.1), (5.2) when μ is given by (5.4). Thus, we have found the value (5.4) of the parameter μ for which (5.5) holds.

Necessity. Now we show that the control parameter value (5.4) is unique because for any

$$\mu = \widetilde{\mu} \neq \eta - z - \int_a^b f\big(s, x_\infty(s, z, \eta)\big)\, ds \tag{5.6}$$

the corresponding solution $\widetilde{x}(\cdot, a, z)$ of the Cauchy problem

$$\frac{dx}{dt} = f(t, x) + \frac{1}{b-a}\widetilde{\mu}, \qquad t \in [a, b], \tag{5.7}$$

with the initial condition

$$x(a) = z, \tag{5.8}$$

does not satisfy the boundary conditions (3.9).

Let us suppose the opposite, that the solution $\widetilde{x}(\cdot, a, z)$ of the initial value problem (5.7), (5.8) satisfy the boundary conditions (3.9). It is obvious from (5.1), (5.2) with μ equal to (5.4) and from (5.7), (5.8) that the functions $x(\cdot, a, z)$ and $\widetilde{x}(\cdot, a, z)$ satisfies the Volterra integral equations

$$x(t, a, z) = z + \int_a^t f\big(s, x(s, a, z)\big)\, ds + \mu\frac{t-a}{b-a}, \qquad t \in [a, b] \tag{5.9}$$

and

$$\widetilde{x}(t,a,z) = z + \int_a^t f\big(s, \widetilde{x}(s,a,z)\big)\, ds + \widetilde{\mu}\frac{t-a}{b-a}, \qquad t \in [a,b]\,. \tag{5.10}$$

By assumption, the functions $x(\cdot,a,z)$ and $\widetilde{x}(\cdot,a,z)$ satisfy the boundary conditions (3.9) and the initial conditions (5.8)

$$x(a,a,z) = z, \quad \widetilde{x}(a,a,z) = z,$$

$$x(b,a,z) = \eta, \quad \widetilde{x}(b,a,z) = \eta. \tag{5.11}$$

Relations (5.9)–(5.11) for $t = b$ give

$$\mu = \eta - z - \int_a^b f\big(s, x(s,a,z)\big)\, ds, \tag{5.12}$$

$$\widetilde{\mu} = \eta - z - \int_a^b f\big(s, \widetilde{x}(s,a,z)\big)\, ds. \tag{5.13}$$

Substituting (5.12), (5.13) into the integral equations (5.9) and (5.10), we get that for all $t \in [a,b]$

$$\begin{aligned} x(t,a,z) = z &+ \int_a^t f\big(s, x(s,a,z)\big)\, ds \\ &- \frac{t-a}{b-a}\int_a^b f\big(s, x(s,a,z)\big)\, ds + \frac{t-a}{b-a}\,[\eta - z] \end{aligned} \tag{5.14}$$

and

$$\begin{aligned} \widetilde{x}(t,a,z) = z &+ \int_a^t f\big(s, \widetilde{x}(s,a,z)\big)\, ds \\ &- \frac{t-a}{b-a}\int_a^b f\big(s, \widetilde{x}\big(s,a,z\big)\big)\, ds + \frac{t-a}{b-a}\,[\eta - z]\,. \end{aligned} \tag{5.15}$$

As $z \in D_a$ and $\eta \in D_b$, by analogy to the proof of Theorem 1, according to the form of equations (5.14), (5.15) and the definition of the set D and vector $\delta_{[a,b],D}(f)$, respectively in (2.4) and (2.7), it can be shown that the values of the functions $x(\cdot,a,z)$ and $\widetilde{x}(\cdot,a,z)$ are contained in D.

It is clear from (5.14) and (5.15) that

$$x(t,a,z)-\widetilde{x}(t,a,z)=\int_a^t\Big[f\big(s,x(s,a,z)\big)-f\big(s,\widetilde{x}(s,a,z)\big)\Big]ds \tag{5.16}$$
$$-\frac{t-a}{b-a}\int_a^b\Big[f\big(s,x(s,a,z)\big)-f\big(s,\widetilde{x}(s,a,z)\big)\Big]ds,\quad t\in[a,b]\,.$$

By virtue of the Lipschitz condition (2.6), from the relation (5.16) we get that the function

$$\omega(t):=|x(t,a,z)-\widetilde{x}(t,a,z)| \tag{5.17}$$

satisfies the integral inequalities

$$\omega(t)\le K\left[\int_a^t\omega(s)\,ds+\frac{t-a}{b-a}\int_a^b\omega(s)\,ds\right]$$
$$\le K\alpha_1(t,a,b)\max_{s\in[a,b]}\omega\,(s)\,,\qquad t\in[a,b]\,, \tag{5.18}$$

where $\alpha_1(t,a,b)$ is given by (4.9). Using (5.18) recursively, we arrive at the inequality

$$\omega(t)\le K^{m+1}\alpha_{m+1}(t,a,b)\max_{s\in[a,b]}\omega\,(s)\,,\qquad t\in[a,b]\,, \tag{5.19}$$

where $m\in\mathbb{N}$ is arbitrary and the functions $\alpha_{m+1}(t,a,b),\ m\ge1$ are given by the formula (3.12), where $\tau=a,\ I=b-a$.

Taking (3.13) into account from (5.19) we get the following estimate for every $m\in\mathbb{N}$,

$$\omega(t)\le\frac{10}{9}\alpha_1(t,a,b)K\left(\frac{3(b-a)}{10}K\right)^m\max_{s\in[a,b]}\omega\,(s)\,,\qquad t\in[a,b]\,. \tag{5.20}$$

By passing to the limit as $m\to\infty$ in the last inequality and by virtue of (3.4), (3.5), we come to the conclusion that

$$\omega(t)=0,\qquad t\in[a,b]\,. \tag{5.21}$$

According to (5.17), this means that the function $x(t,a,z)$ coincides with the function $\widetilde{x}(t,a,z)$. Using (5.12) and (5.13), we get that

$$\mu=\widetilde{\mu}=\eta-z-\int_a^b f\big(s,x_\infty\,(s,z,\eta)\big)\,ds.$$

This contradiction proves the theorem. □

Let us find the relation of the limit function $x_\infty\,(\cdot,z,\eta)$ of the sequence (4.1) to the solution of the original integral boundary value problem (3.1), (3.2).

Theorem 3. *Under the assumptions of Theorem 1, the limit function*

$$x_\infty(t,z^*,\eta^*) = \lim_{m\to\infty} x_m(t,z^*,\eta^*) \tag{5.22}$$

of the sequence (4.1) *is an absolutely continuous solution of the integral boundary value problem* (3.1), (3.2) *if and only if the pair of parameters* (z^*,η^*) *satisfies the system of* $2n$ *algebraic "determining" equations*

$$\begin{aligned}\Delta(z,\eta) &:= \eta - z - \int_a^b f\big(s, x_\infty(s,z,\eta)\big)\,ds = 0,\\ \Lambda(z,\eta) &:= \int_a^b \Big[g\big(s, x_\infty(s,z,\eta)\big) + h\Big(s, f\big(s, x_\infty(s,z,\eta)\big)\Big)\Big]ds - d = 0.\end{aligned} \tag{5.23}$$

P r o o f. It suffices to apply Theorem 2 and notice that the differential equation (4.6) coincides with (3.1) if and only if (z^*,η^*) satisfies the equation

$$\Delta(z^*,\eta^*) := \eta^* - z^* - \int_a^b f\big(s, x_\infty(z^*,\eta^*)\big)\,ds = 0.$$

Moreover, it is clear that the limit function $x_\infty(\cdot,z^*,\eta^*)$ coincides with the solution of the integral boundary value problem (3.1), (3.2) if and only if $x_\infty(\cdot,z^*,\eta^*)$ satisfies the equation

$$\int_a^b \Big[g\big(s, x_\infty(\cdot,z^*,\eta^*)\big) + h\Big(s, f\big(s, x_\infty(\cdot,z^*,\eta^*)\big)\Big]ds = d. \tag{5.24}$$

This means that the limit function $x_\infty(z^*,\eta^*)$ is the solution the integral boundary value problem (3.1), (3.2) if and only if the equations (5.23) hold. □

The next statement claims that the system of determining equations (5.23), in fact, determines all possible solutions of the original non-linear integral boundary value problem (3.1), (3.2).

Theorem 4. *Assume that conditions of Theorem 1 are satisfied.*

If there exists some pair of vectors $\big(z^0,\eta^0\big) \in D_a \times D_b$ *that satisfy the system of determining equations* (5.23)*, then the integral boundary value problem* (3.1), (3.2) *has a solution* $x^0(\cdot)$ *such that*

$$x^0(a) = z^0, \quad x^0(b) = \eta^0$$

and

$$\int_a^b \Big[g\big(s, x^0(s)\big) + h\Big(s, f(s, x^0(s))\Big)\Big]ds = d\,, \tag{5.25}$$

Moreover, this solution is given by the limit function of the sequence (4.1)

$$x^0(t) = x_\infty\left(t, z^0, \eta^0\right) = \lim_{m\to\infty} x_m(\cdot, z^0, \eta^0), \qquad t \in [a,b]\,. \tag{5.26}$$

Conversely, if the integral boundary value problem (3.1), (3.2) *has a solution* $x^0(\cdot)$*, then* $x^0(\cdot)$ *necessarily has the form* (5.26) *and the system of determining equations* (5.23) *is satisfied with*

$$z = x^0(a), \quad \eta = x^0\,(b)\,. \tag{5.27}$$

P r o o f. If there exists a pair $\left(z^0, \eta^0\right) \in D_a \times D_b$ that satisfies the system of determining equations (5.23), then according to Theorem 3 the function (5.26) is a solution of the given integral boundary value problem (3.1), (3.2).

On the other hand, if $x^0(\cdot)$ is the solution of the original problem (3.1), (3.2), then this function is a solution of the Cauchy problem (5.1), (5.2) with

$$\mu = 0 \quad \text{and} \quad z = x^0(a)\,. \tag{5.28}$$

As $x^0(\cdot)$ satisfies the integral boundary restrictions (3.2), by virtue of equality (5.22) of Theorem 3 the equality (5.26) holds.

Moreover,

$$\mu := \eta - z - \int_a^b f\left(s, x_\infty\,(s,z,\eta)\right) ds = 0\,, \tag{5.29}$$

where the pair of vectors (z,η) is defined by (5.27). From (5.29) we have that the first equation in the determining system (5.23) is satisfied, if (z,η) is given by (5.27). Using (3.2), we obtain that the second equation in the determining system (5.23) also holds.

Thus in (5.27) we have specified the values of (z,η) that satisfy the system of the determining equations (5.23), which proves the theorem. □

Similarly to [7], the solvability of the determining system (5.23) can be established by studying some its approximate versions

$$\Delta_m(z,\eta) := \eta - z - \int_a^b f\left(s, x_m\,(s,z,\eta)\right) ds = 0\,,$$

$$\Lambda_m(z,\eta) := \int_a^b \left[g\left(s, x_m\,(s,z,\eta)\right) + h\Big(s, f\left(s, x_m\,(s,z,\eta)\right)\Big)\right] ds - d = 0\,, \tag{5.30}$$

that can be constructed explicitly.

6. Approximation of a solution

Theorem 3 can be complemented by the following natural observation. Let $(\widehat{z},\widehat{\eta}) \in D_a \times D_b$ be a root of the approximate determining system (5.30) for a certain m. Then the function

$$\widetilde{x}(t) := x_m(t,\widehat{z},\widehat{\eta}), \qquad t \in [a,b]\,, \tag{6.1}$$

defined according to (4.1) can be regarded as the mth approximation to a solution of the integral boundary value problem (3.1), (3.2). This is justified by the next estimate following directly from inequality (4.8) of Theorem 1

$$\begin{aligned}&|x_\infty(t,\widehat{z},\widehat{\eta}) - x_m(t,\widehat{z},\widehat{\eta})|\\ &\leqslant \frac{10}{9}\alpha_1(t,a,b)Q^m\,(1_n - Q)^{-1}\,\delta_{[a,b],D}(f), \qquad t \in [a,b]\,,\ \ m \geq 0,\end{aligned} \tag{6.2}$$

where Q and $\delta_{[a,b],D}(f)$ are given in (3.4) and (2.7) respectively.

It is worth to emphasize the role of unknown parameters whose values appearing in (6.1) are determined from the approximate determining systems (5.30): $\widehat{z}$ is an approximation of the initial value at the point $t = a$ of the solution of integral boundary value problem (3.1), (3.2) and $\widehat{\eta}$ is that of its value at $t = b$.

The solvability analysis based on properties of the equations (5.30) can be carried out by analogy to [6] on the base of topological degree methods [2], but it is not treated here.

7. Example

Let us apply the numerical-analytic approach described above to the system of differential equations

$$\begin{cases} x_1'(t) = x_2^2(t) - \frac{t}{5}x_1(t) + \frac{t^3}{100} - \frac{t^2}{25} = f_1(t,x_1,x_2), \\ x_2'(t) = \frac{t^2}{10}x_2(t) + \frac{t}{8}x_1(t) - \frac{21}{800}t^3 + \frac{1}{16}t + \frac{1}{5} = f_2(t,x_1,x_2), \quad t \in \left[0,\frac{1}{2}\right], \end{cases} \tag{7.1}$$

considered for $t \in \left[0,\frac{1}{2}\right]$ with the integral boundary conditions

$$\begin{cases} \int_0^{\frac{1}{2}} \left[s x_1(s) x_2(s) + f_1^2(s,x_1(s),x_2(s))\right] ds = d_1, \\ \int_0^{\frac{1}{2}} \left[s^2 x_2^2(s) + f_2^2(s,x_1(s),x_2(s))\right] ds = d_2, \end{cases} \tag{7.2}$$

where

$$d = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} -59/16000 \\ 81/4000 \end{bmatrix}.$$

It is easy to check that

$$x_1^*(t) = \frac{t^2}{20} - \frac{1}{2}, \quad x_2^*(t) = \frac{t}{5} \tag{7.3}$$

is a continuously differentiable solution of the problem (7.1), (7.2).

Following to (3.6), (3.7), introduce the parameters

$$\begin{aligned} z &:= x(0) = \operatorname{col}\big(x_1(0), x_2(0)\big) = \operatorname{col}(z_1, z_2), \\ \eta &:= x\left(\frac{1}{2}\right) = \operatorname{col}\left(x_1\left(\frac{1}{2}\right), x_2\left(\frac{1}{2}\right)\right) = \operatorname{col}(\eta_1, \eta_2). \end{aligned} \tag{7.4}$$

Let us choose the sets D_a and D_b, where one looks for the values $x(a)$ and $x(b)$, as follows:

$$D_a = D_b = \big\{(x_1, x_2) : -10.3 \le x_1 \le 0.6, -0.01 \le x_2 \le 0.2\big\}. \tag{7.5}$$

In this case, a convex linear combination $D_{a,b}$ of the form (2.3) of vectors $z \in D_a$ and $\eta \in D_b$ will be

$$D_{a,b} = D_a = D_b. \tag{7.6}$$

In the inequality (3.3) of Theorem 1 let us choose

$$\rho := \operatorname{col}(0.2;\ 0.2). \tag{7.7}$$

Consequently ρ-neighhourhood D of the set $D_{a,b}$ is given as follows

$$D = \big\{(x_1, x_2) : -10.5 \le x_1 \le 0.8, -0.21 \le x_2 \le 0.4\big\}$$

Direct computations show that the Lipschitz condition (2.6) for the right hand side in (7.1) in the domain D holds with matrix

$$K = \begin{bmatrix} 1/10 & 9/10 \\ 1/16 & 1/40 \end{bmatrix}$$

and

$$Q = \frac{3}{20}\begin{bmatrix} 1/10 & 9/10 \\ 1/16 & 1/40 \end{bmatrix}, \quad r(Q) = 0.045 < 1, \tag{7.8}$$

$$\delta_{[a,b],D}(f) := \frac{1}{2}\left[\max_{(t,x)\in\left[0,\frac{1}{2}\right]\times D} f(t,x) - \min_{(t,x)\in\left[0,\frac{1}{2}\right]\times D} f(t,x)\right] = \begin{bmatrix} 0.645 \\ 0.36075 \end{bmatrix}, \tag{7.9}$$

$$\rho = \begin{bmatrix} 0.2 \\ 0.2 \end{bmatrix} \ge \frac{b-a}{2}\delta_{[a,b],D}(f) = \begin{bmatrix} 0.16125 \\ 0.0901875 \end{bmatrix}. \tag{7.10}$$

So, we check that all conditions of Theorem 1 are fulfilled, and the sequence of functions (4.1) for this example is convergent.

Using (4.1) and applying Maple 14 at the first iteration ($m = 1$) for the first and the second component gives the following results:

$$\begin{aligned} x_{11}(t,z,\eta) := {}& z_1 + 1/400t^4 + 1/3\big((-2z_2+2\eta_2)^2 + 2/5z_1 - 2/5\eta_1 - 1/25\big)t^3 \\ & + 1/2(2z_2(-2z_2+2\eta 2) - 1/5z_1)t^2 + z_2^2 t \\ & - 2t\Big(-29/19200 + 1/24(-2z_2+2\eta_2)^2 - 1/120z_1 \\ & - 1/60\eta_1 + 1/4z_2(-2z_2+2\eta_2) + 1/2z_2^2\Big) + 2t(\eta_1 - z_1), \end{aligned} \tag{7.11}$$

$$\begin{aligned} x_{12}(t,z,\eta) := {}& z_2 + 1/5t + 1/4(-1/5z_2 + 1/5\eta_2 - 21/800)t^4 \\ & + 1/3(-1/4z_1 + 1/4\eta_1 + 1/10z_2)t^3 + 1/2(1/8z_1 + 1/16)t^2 \\ & - 2t\Big(5499/51200 + (1/960)z_2 + (1/320)\eta_2 \\ & + (1/192)z_1 + (1/96)\eta_1\Big) + 2t(\eta_2 - z_2). \end{aligned}$$

The numerical computations show that the components of the solution of the approximate determining system

$$\begin{aligned} \Delta_m(z,\eta) &:= [\eta - z] - \int_a^b f\big(s, x_m(s,z,\eta)\big)\,ds = 0, \\ \Lambda_m(z,\eta) &:= \int_a^b \Big[g\big(s, x_m(s,z,\eta)\big) + h\Big(s, f\big(s, x_m(s,z,\eta)\big)\Big)\Big]ds - d = 0, \end{aligned} \tag{7.12}$$

of form (5.30) for $m = 1$ are

$$\begin{aligned} z_1 &:= z_{11} = -0.499994975, & z_2 &:= z_{12} = -0.6001633747 \cdot 10^{-5}, \\ \eta_1 &:= \eta_{11} = -0.4874955843, & \eta_2 &:= \eta_{12} = 0.09999409626. \end{aligned} \tag{7.13}$$

By putting (7.13) into (7.11), we obtain the first and second components of the first approximation to the solution of the given integral BVP (7.1), (7.2):

$$\begin{aligned} x_{11}(t) = {}& -0.4999949750 + (1/400)t^4 - 0.00166655933t^3 + 0.04999829717t^2 \\ & + 0.00010377264t, \\ x_{12}(t) = {}& -1.633747 \cdot 10^{-6} + 0.1999349966t - 0.001562495105t^4 \\ & + 0.001041415846t^3 + 3.140600000 \cdot 10^{-7}t^2. \end{aligned} \tag{7.14}$$

The errors of the first approximation are

$$\max_{t \in [0, \frac{1}{2}]} |x_1^*(t) - x_{11}(t)| \le 2.03 \cdot 10^{-6},$$

$$\max_{t \in [0, \frac{1}{2}]} |x_2^*(t) - x_{12}(t)| = 1.3 \cdot 10^{-5}.$$

Similarly, for the second approximation ($m = 2$) we get the following solutions of the approximate determining system (7.1)

$$\begin{aligned} z_1 &:= z_{21} = -0.5000000116, \quad z_2 := z_{22} = 3.028388758 \cdot 10^{-8}, \\ \eta_1 &:= \eta_{21} = -0.4875000081, \quad \eta_2 := \eta_{22} = 0.1000000296 \end{aligned} \tag{7.15}$$

and the second approximation to the solution

$$\begin{aligned} x_{21}(t) = {} & -0.5000000116 + 2.712673729 \cdot 10^{-7} t^9 - 4.069015552 \cdot 10^{-7} t^8 \\ & + 1.550106267 \cdot 10^{-7} t^7 - 0.0001874660935 t^6 + 0.00014997298 t^5 \\ & - 4.165 \cdot 10^{-10} t^4 - 0.1562396477 \cdot 10^{-4} t^3 + 0.0500000072 t^2 \\ & + 3.9414 \cdot 10^{-7} t, \end{aligned} \tag{7.16}$$

$$\begin{aligned} x_{22}(t) = {} & 3.028388758 \cdot 10^{-8} + 0.1999998998 t - 0.00002232142906 t^7 \\ & + 0.00006944446598 t^6 - 0.00004166669840 t^5 \\ & - 0.000001627411976 t^4 + 0.000004341407150 t^3 \\ & - 7.25 \cdot 10^{-10} t^2. \end{aligned}$$

For the second approximation ($m = 2$) the errors are

$$\max_{t \in [0, \frac{1}{2}]} |x_1^*(t) - x_{21}(t)| \le 10 \cdot 10^{-8},$$
$$\max_{t \in [0, \frac{1}{2}]} |x_2^*(t) - x_{22}(t)| \le 2.5 \cdot 10^{-8}.$$

For the third approximation ($m = 3$) the errors are

$$\max_{t \in [0, \frac{1}{2}]} |x_1^*(t) - x_{31}(t)| \le 2.3 \cdot 10^{-9},$$
$$\max_{t \in [0, \frac{1}{2}]} |x_2^*(t) - x_{32}(t)| \le 3 \cdot 10^{-10}.$$

The graphs of the first approximation and the exact solution of the original boundary-value problem are shown on Figure 1.

According to Theorems 3 and 4 the number of solutions of the algebraic determining system (5.23) coincides with the number of solutions of the given integral BVP.

Computations show that the approximate determining system of algebraic equations (7.12) side by side with the solution (7.13) for $m = 1$ has another solution

$$\begin{aligned} z_1 &:= z_{11} = -10.21326364, \quad z_2 := z_{12} = 0.1516110403, \\ \eta_1 &:= \eta_{11} = -9.951162260, \quad \eta_2 := \eta_{12} = 0.1020979352. \end{aligned} \tag{7.17}$$

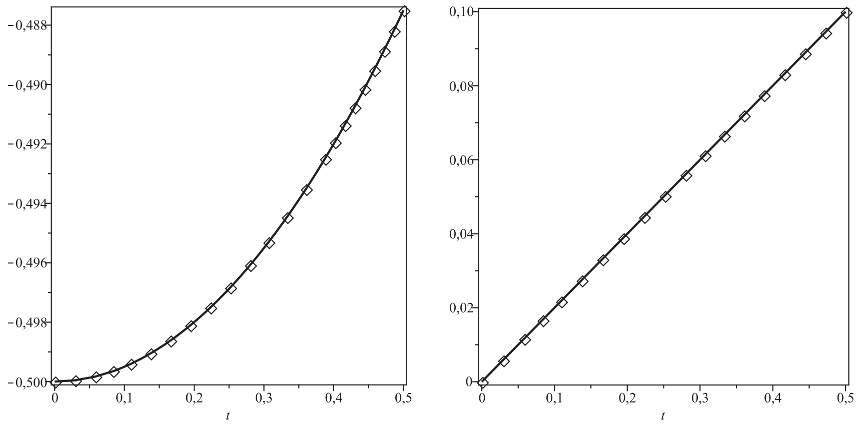

FIGURE 1. The components of the exact solution (solid line) and its first approximation (drawn with dots).

By substituting (7.17) into the first approximation (7.11) we obtain the following first approximation to the second solution of the given integral BVP (7.1), (7.2)

$$\begin{aligned} x_{11}(t) = &- 10.21326364 + 1/400t^4 - 0.04501145400t^3 \\ &+ 1.006312897t^2 + 0.0319866749t, \\ x_{12}(t) = &\, 0.1516110403 + 0.1989191771t - 0.009038155255t^4 \\ &+ 0.02689548301t^3 - 0.6070789775t^2. \end{aligned} \tag{7.18}$$

By analogy we obtain the second and the third approximations ($m = 2, m = 3$) to the second solution:

$$\begin{aligned} &z_1 := z_{21} = -10.20404878, \quad z_2 := z_{22} = 0.1514587161, \\ &\eta_1 := \eta_{21} = -9.942186105, \quad \eta_2 := \eta_{22} = 0.1020892875, \end{aligned} \tag{7.19}$$

$$\begin{aligned} x_{21}(t) = &- 10.20404878 + 0.000009062049441t^9 - 0.00006066670965t^8 \\ &+ 0.001668092557t^7 - 0.006114508143t^6 + 0.07696007972t^5 \\ &- 0.1060615932t^4 - 0.06351237337t^3 + 1.050533638t^2 \\ &+ 0.0229497715t, \\ x_{22}(t) = &\, 0.1514587161 + 0.2000037976t - 0.0001290138776t^7 \\ &+ 0.0004999252228t^6 - 0.01325502520t^5 + 0.02983091078t^4 \\ &+ 0.006379352260t^3 - 0.6065030490t^2. \end{aligned} \tag{7.20}$$

$$\begin{aligned} &z_1 := z_{31} = -10.20427860, \quad z_2 := z_{32} = 0.1514591964, \\ &\eta_1 := \eta_{31} = -9.942410605, \quad \eta_2 := \eta_{32} = 0.1020862671, \end{aligned} \tag{7.21}$$

$$
\begin{aligned}
x_{31}(t) = & -10.20427860 + 1.109681721 \cdot 10^{-9} t^{15} - 9.214218520 \cdot 10^{-9} t^{14} \\
& + 2.823261003 \cdot 10^{-7} t^{13} - 1.745922368 \cdot 10^{-6} t^{12} + 1.837022747 \cdot 10^{-5} t^{11} \\
& - 6.158409930 \cdot 10^{-5} t^{10} - 0.00003009699552 t^{9} + 0.002230452505 t^{8} \\
& - 0.008098511634 t^{7} + 0.003565338010 t^{6} + 0.07843090050 t^{5} \\
& - 0.1101976568 t^{4} - 0.06277119690 t^{3} + 1.050720292 t^{2} + 0.0229393376 t,
\end{aligned}
$$

$$
\begin{aligned}
x_{32}(t) = & \, 0.1514591964 + 0.2000001111 t + 1.029818264 \cdot 10^{-7} t^{11} \\
& - 2.048525315 \cdot 10^{-6} t^{10} + 0.00002872374699 t^{9} - 0.0002612343205 t^{8} \\
& + 0.001800515656 t^{7} - 0.002103346123 t^{6} - 0.01371819803 t^{5} \\
& + 0.03126749550 t^{4} + 0.006004863667 t^{3} - 0.6065174125 t^{2}. \qquad (7.22)
\end{aligned}
$$

If we substitute the third approximate solution (7.22) into the given system of equations (7.1), we obtain the following error:

$$
\begin{aligned}
& \max_{t \in \left[0, \frac{1}{2}\right]} \left| x'_{31}(t) - x^2_{32}(t) + \tfrac{t}{5} x_{31}(t) - \tfrac{t^3}{100} + \tfrac{t^2}{25} \right| = 5.7 \cdot 10^{-7}, \\
& \max_{t \in \left[0, \frac{1}{2}\right]} \left| x'_{32}(t) - \tfrac{t^2}{10} x_{32}(t) - \tfrac{t}{8} x_{31}(t) + \tfrac{21}{800} t^3 - \tfrac{1}{16} t - \tfrac{1}{5} \right| = 1.4 \cdot 10^{-7}.
\end{aligned} \qquad (7.23)
$$

The graphs of the first and the third approximations to the second solution of the given BVP are shown on Figure 2.

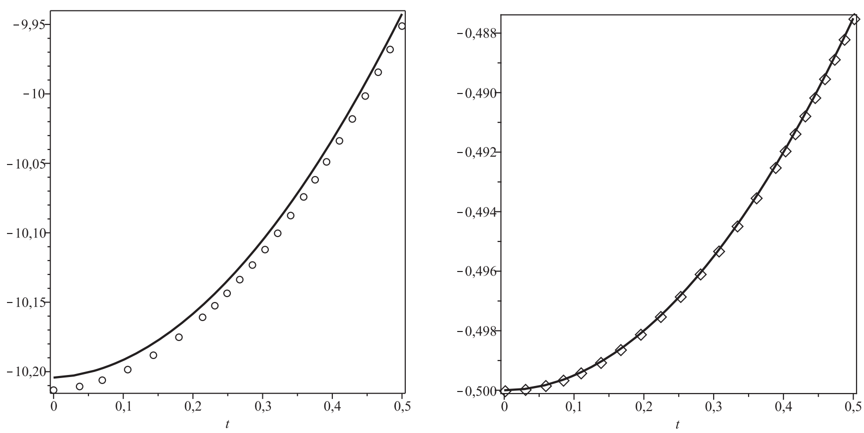

FIGURE 2. The components of the first (∘) and the third (solid line) approximations to the second solution.

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