DOI 10.1007/s10958-022-05810-2
Journal of Mathematical Sciences, *Vol. 262, No. 2, April, 2022*

Continued fraction representation of the generating function of Bernoulli polynomials

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(Presented by V.P. Motornyi)

Abstract. Continued fraction and quasi-reciprocal continued fraction expansions of the generating function of Bernoulli numbers have been obtained. The convergence and uniform convergence of continued fraction expansions have been proved. Representations of the generating function of Bernoulli polynomials in the form of the product of three continued fractions, as well as the product of three quasi-reciprocal continued fractions, have been found.

Keywords. Continued fraction, Bernoulli numbers, Bernoulli polynomials, generating function, continued fraction expansion of function, continued fraction representation of function.

1. Introduction

The function of a complex variable in the vicinity of a certain point can be expanded in a power series [1], approximated by Padé approximants [2, 3], and represented by a continued fraction [4].

There are several ways to expand functions in a continued fraction. Historically, the first of them is related to finding the solution of the Riccati differential equation in the form of an infinite continued fraction [5]. The function expansions are obtained from the representation of the ratio between hypergeometric functions via a continued fraction. If the expansion of a function in a formal power series in the vicinity of a certain point is known, then the determination of the coefficient corresponding to the formal power series of a regular C-fraction implies the calculation of four Hankel determinants formed from the coefficients of the formal series [6].

An analogous of the Taylor formula in the theory of continued fractions is the Thiele formula [7, 8]. The coefficients of function expansion in the continued fraction are determined by calculating the reciprocal Thiele derivatives. If the general formula for the coefficients of function expansion in a continued fraction is found, the regions of convergence and uniform convergence of continued fractions, as well as a priori and a posteriori estimates are determined. The reciprocal derivatives of the 2nd type and the methods of function expansion in quasi-reciprocal continued fractions were considered in [9]. The expansions of the generating functions of the Catalan, Motzkin, Euler, and other numbers in Jacobi continued fractions (J-fractions) and Stieltjes continued fractions (S-fractions) were studied in work [10]. The work [11] was devoted to the expansion of the generating functions of the generalized Bernoulli, Cauchy, and Euler numbers in continued T-fractions.

In this paper, the expansions of the generating functions of the Bernoulli numbers and the Bernoulli polynomials in the Thiele continued fractions, regular C-fractions, and quasi-reciprocal continued fractions are proposed.

Translated from Ukrains'kiĭ Matematychnyĭ Visnyk, Vol. 19, No. 1, pp. 88–102, January–March, 2022.
Original article submitted May 11, 2020

2. Formulation of the problem. Introductory concepts

It is known [12] that $\mathfrak{b}(z) = z/(e^z - 1)$ is the generating function of Bernoulli numbers $B_n, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, i.e.,

$$\mathfrak{b}(z) = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!}, \quad z \in \mathbb{C}. \tag{2.1}$$

In turn, $\mathfrak{B}(z, x) = ze^{xz}/(e^z - 1)$ is the generating function of Bernoulli polynomials, i.e.,

$$\mathfrak{B}(z, x) = \sum_{n=0}^{\infty} B_n(x) \frac{z^n}{n!}, \quad x \in \mathbb{R}. \tag{2.2}$$

Power series (2.1) and (2.2) converge for all $|z| < 2\pi$. At the same time, the function $\mathfrak{b}$ is defined in the domain $\mathbf{G} = \mathbb{C}\backslash\{2\pi ki, k \in \mathbb{Z}\backslash\{0\}\}$, whereas the function $\mathfrak{B}$ in the domain $\mathfrak{G} = \mathbf{G} \times \mathbb{R}$. The aim of this work is to obtain the expansions of the functions $\mathfrak{b}$ and $\mathfrak{B}$ in continued fractions the converge to functions in the domains $\mathbf{G}$ and $\mathfrak{G}$, respectively.

Let $b_0, a_k \neq 0, b_k, k \in \mathbb{N}$, be numbers, functions, and so forth. The infinite continued fraction

$$D = b_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \ddots + \cfrac{a_n}{b_n + \ddots}}}$$

can be briefly written in the form

$$D = b_0 + \underset{k=1}{\overset{\infty}{\mathrm{K}}} \frac{a_k}{b_k} = b_0 + \frac{a_1}{b_1 +} \frac{a_2}{b_2 +} \cdots + \frac{a_k}{b_k +} \cdots = b_0 + \mathbf{K}(a_k/b_k). \tag{2.3}$$

Analogously, the n-th approximant, i.e., the n-th approximation $D_n, n \geq 1$, of the infinite continued fraction D is briefly written as follows:

$$D_n = b_0 + \underset{k=1}{\overset{n}{\mathrm{K}}} \frac{a_k}{b_k} = b_0 + \frac{a_1}{b_1 +} \cdots + \frac{a_n}{b_n} = b_0 + \mathbf{K}_{k=1}^{n}(a_k/b_k), \ D_0 = b_0.$$

Definition 2.1. *The continued fractions $b_0 + \mathbf{K}(a_k/b_k)$ and $d_0 + \mathbf{K}(c_k/d_k)$ are said to be equivalent if the sequences of their approximants coincide, i.e., $b_0 + \mathbf{K}_{k=1}^{n}(a_k/b_k) = d_0 + \mathbf{K}_{k=1}^{n}(c_k/d_k), n \geq 0$.*

Theorem 2.1. [6] *The continued fractions $b_0 + \mathbf{K}(a_k/b_k)$ and $d_0 + \mathbf{K}(c_k/d_k)$ are equivalent if and only if there is a sequence of such numbers $\{r_k : r_0 = 1, r_k \neq 0, k \in \mathbb{N}\}$ for which the following relations hold: $d_0 = b_0, c_k = r_{k-1} r_k a_k, \ d_k = r_k b_k, k \in \mathbb{N}$.*

3. Thiele continued fraction. Regular C-fraction

Let the function f be analytic in the domain $\mathbf{K} \subset \mathbb{C}$. As ${}^{(k)}f(z_*)$, denote the reciprocal Thiele derivative of the k-th order of the function f at the point $z_* \in \mathbf{K}$. The reciprocal Thiele derivatives of the function f are determined using the recurrent formulas [7]

$$\begin{aligned} &{}^{(k)}f(z_*) = k \cdot {}^{(1)}({}^{(k-1)}f(z_*)) +{}^{(k-2)} f(z_*), \\ &{}^{(1)}f(z_*) = 1/f'(z_*), \quad {}^{(0)}f(z_*) = f(z_*), \quad k \in \mathbb{N}_2 = \mathbb{N}\backslash\{1\}. \end{aligned}$$

Theorem 3.1. [13] *If the function f is analytic in the domain $\mathbf{K} \subset \mathbb{C}$, then the reciprocal Thiele derivatives of the function f at the point $z_* \in \mathbf{K}$ are determined as follows:*

$$ {}^{(2k)}f(z_*) = \frac{H_{k+1}^{(0)}(z_*)}{H_k^{(2)}(z_*)}, \quad {}^{(2k-1)}f(z_*) = \frac{H_{k-1}^{(3)}(z_*)}{H_k^{(1)}(z_*)}, \qquad k \in \mathbb{N}, $$

where the Hankel determinants $H_k^{(m)}(z_*), k \geq 1,$ equal

$$ H_0^{(m)}(z_*) = 1, \quad H_k^{(m)}(z_*) = \begin{vmatrix} c_m & c_{m+1} & \cdots & c_{m+k-1} \\ c_{m+1} & c_{m+2} & \cdots & c_{m+k} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m+k-1} & c_{m+k} & \cdots & c_{m+2k-2} \end{vmatrix} \neq 0, $$

$c_m = f^{(m)}(z_*)/m!$ *if* $m \geq 0$, *and* $c_m = 0$ *if* $m < 0$.

It is known [7] that if $C = \text{const}$, then

$$ {}^{(2n)}(Cf(z)) = C \cdot {}^{(2n)}f(z), \quad {}^{(2n+1)}(Cf(z)) = \tfrac{1}{C} \cdot {}^{(2n+1)}f(z), \quad n \in \mathbb{N}_0. \tag{3.1} $$

Theorem 3.2. [14] *Let the function f have reciprocal Thiele derivatives up to the n-th order inclusive and $C = const$. Then, for $k = \overline{0, [n/2]}$,*

$$ {}^{(2k)}f(Cz) = {}^{(2k)}f(v)\big|_{v=Cz}, \quad {}^{(2k-1)}f(Cz) = \tfrac{1}{C} \cdot {}^{(2k-1)}f(v)\big|_{v=Cz}. \tag{3.2} $$

If there exist reciprocal Thiele derivatives ${}^{(n)}f(z)$, $n \geq 1$, in the vicinity of the point $z_* \in \mathbf{K}$, then the function $f(z)$ can be expanded in this vicinity in the Thiele continued fraction (TCF) of the form

$$ f(z) = b_0(z_*) + \mathop{\mathrm{K}}_{k=1}^{\infty} \frac{z - z_*}{b_k(z_*)}. \tag{3.3} $$

The coefficients of TCF are determined via the reciprocal Thiele derivatives of the function f using the recurrent formula ([7]),

$$ b_0 = f(z_*), \quad b_1 = {}^{(1)}f(z_*), \quad b_k = {}^{(k)}f(z_*) - {}^{(k-2)}f(z_*), \quad k \in \mathbb{N}_2. \tag{3.4} $$

The TCF (3.3) can be written in the form of an equivalent continued fraction with the partial denominators equal to unity,

$$ f(x) = a_0(z_*) + \mathop{\mathrm{K}}_{k=1}^{\infty} \frac{a_k(z_*)(z - z_*)}{1}. \tag{3.5} $$

The coefficients of continued fraction (3.5) are determined via the reciprocal Thiele derivatives at the point z_* as follows:

$$ \begin{aligned} a_0(z_*) &= f(z_*),\, a_1(z_*) = \frac{1}{{}^{(1)}f(z_*)}, \\ a_k(z_*) &= \frac{1}{k \cdot (k-1) \cdot {}^{(1)}({}^{(k-1)}f(z_*)) \cdot {}^{(1)}({}^{(k)}f(z_*))}, \quad k \in \mathbb{N}_2. \end{aligned} \tag{3.6} $$

It was proved (see [15]) that the continued fraction (3.5) coincides with the regular C-fraction (RCF) corresponding to the formal power series. Whence it follows that the TCF also corresponds to the formal power series.

Theorem 3.3. [6,9] *Let the function f have an expansion in RCF (3.5) with $\lim_{n\to\infty} a_n = 0$ and $a_n \neq 0$ in the vicinity of the point $z_* \in \mathbf{K}$, then (A) the RCF coneverges to function f; (B) the RCF converges uniformly on an arbitrary compact set $K \subset \mathbf{K}$ that does not contain the poles of the function f; and (C) the function f is holomorphic at the point z_*.*

4. Expansion of the generating function of Bernoulli numbers in a continued fraction

An expansion of the function $\mathfrak{b}$ in a continued T-fraction of the form

$$\mathfrak{b}(z) = 1 - \frac{z}{2+z-}\,\frac{2z}{3+z-}\,\cdots-\frac{nz}{n+1+z-}\,\cdots$$

was obtained in paper [11]. We rewrite the function $\mathfrak{b}$ in the form

$$\mathfrak{b}(z) = z/h(z), \quad \text{where } h(z) = e^z - 1. \tag{4.1}$$

Let us expand the auxiliary function h in TCF (3.3) in the vicinity of the point $z_* \in \mathbf{G}$. It is easy to see that the reciprocal Thiele derivatives of the function h are

$$^{(4k)}h(z) = e^z - 1, \quad ^{(4k+1)}h(z) = (2k+1)e^{-z}, \quad ^{(4k+2)}h(z) = -e^z - 1,$$
$$^{(4k+3)}h(z) = -2(k+1)e^{-z}, \quad k \in \mathbb{N}_0.$$

The expansion coefficients of the function h in TCF in the vicinity of the point z_* acquire the values

$$b_0(z_*) = e^{z_*} - 1,\ b_{2k-1}(z_*) = (-1)^{k+1}(2k-1)e^{-z_*},$$
$$b_{2k}(z_*) = (-1)^k 2e^{z_*},\ k \in \mathbb{N}.$$

Substituting them into (3.3), we obtain

$$h(z) = e^{z_*} - 1 + \frac{z-z_*}{e^{-z_*}+}\,\frac{z-z_*}{-2e^{z_*}+}\,\frac{z-z_*}{-3e^{-z_*}+}\,\frac{z-z_*}{2e^{z_*}+}\,\frac{z-z_*}{5e^{-z_*}+}+$$
$$+\frac{z-z_*}{-2e^{z_*}+}\cdots+\frac{z-z_*}{(-1)^{n-1}(2n-1)e^{-z_*}+}\,\frac{z-z_*}{(-1)^n 2e^{z_*}+}\cdots. \tag{4.2}$$

By performing in the continued fraction (4.2) equivalent transformations, when $r_0 = 1$, $r_{2k-1} = e^{z_*}$, and $r_{2k} = e^{-z_*}$, we get the function expansion

$$h(z) = e^{z_*} - 1 + \frac{e^{z_*}(z-z_*)}{1}+\frac{z-z_*}{-2}+\frac{z-z_*}{-3}+\frac{z-z_*}{2}+\frac{z-z_*}{5}+$$
$$+\frac{z-z_*}{-2}+\frac{z-z_*}{-7}+\cdots+\frac{z-z_*}{(-1)^{k-1}(2k-1)+}\,\frac{z-z_*}{(-1)^k 2+}\cdots. \tag{4.3}$$

Substitute (4.3) into (4.1). Then the expansion of the generating function $\mathfrak{b}$ in a continued fraction in the vicinity of the point z_* looks like

$$\mathfrak{b}(z) = \frac{z}{e^{z_*}-1+}\,\frac{e^{z_*}(z-z_*)}{1}+\frac{z-z_*}{-2}+\frac{z-z_*}{-3}+\frac{z-z_*}{2}+\frac{z-z_*}{5}+$$
$$+\frac{z-z_*}{-2}+\frac{z-z_*}{-7}+\cdots+\frac{z-z_*}{(-1)^{k-1}(2k-1)+}\,\frac{z-z_*}{(-1)^k 2+}\cdots. \tag{4.4}$$

According to (3.6), the coefficients of the equivalent RCF equal

$$a_0(z_*) = e^{z_*} - 1, \quad a_1(z_*) = e^{z_*}, \quad a_{2k}(z_*) = \frac{-1}{2(2k-1)},$$

$$a_{2k+1}(z_*) = \frac{1}{2(2k+1)}, \quad k \in \mathbb{N}.$$

The expansion of the function h in the RCF in the vicinity of the point z_* is

$$h(z) = e^{z_*} - 1 + \frac{e^{z_*}(z-z_*)}{1} \,\underset{+}{}\, \frac{-\frac{1}{2}(z-z_*)}{1} \,\underset{+}{}\, \frac{\frac{1}{6}(z-z_*)}{1} \,\underset{+}{}\, \frac{-\frac{1}{6}(z-z_*)}{1} \,\underset{+}{}$$

$$\underset{+}{}\, \frac{\frac{1}{10}(z-z_*)}{1} \,\underset{+}{}\, \cdots \underset{+}{}\, \frac{\frac{-1}{2(2k-1)}(z-z_*)}{1} \,\underset{+}{}\, \frac{\frac{1}{2(2k+1)}(z-z_*)}{1} \,\underset{+}{}\, \cdots.$$

So we obtain another expansion of the generating function $\mathfrak{b}$ in a continued fraction,

$$\mathfrak{b}(z) = \frac{z}{e^{z_*} - 1 +} \, \frac{e^{z_*}(z-z_*)}{1} \,\underset{+}{}\, \frac{-\frac{1}{2}(z-z_*)}{1} \,\underset{+}{}\, \frac{\frac{1}{6}(z-z_*)}{1} \,\underset{+}{}$$

$$\underset{+}{}\, \frac{-\frac{1}{6}(z-z_*)}{1} \,\underset{+}{}\, \cdots \underset{+}{}\, \frac{\frac{-1}{2(2k-1)}(z-z_*)}{1} \,\underset{+}{}\, \frac{\frac{1}{2(2k+1)}(z-z_*)}{1} \,\underset{+}{}\, \cdots. \tag{4.5}$$

Since $a_k \neq 0, k \in \mathbb{N}_0$, and $\lim\limits_{k\to\infty} a_k = 0$, then, by Theorem 3.3, the continued fraction (4.5) and the equivalent continued fraction (4.4) converge to the generating function $\mathfrak{b}$, and the continued fractions converge uniformly on an arbitrary compact set $K \subset \mathbf{G}$.

The value of e^{z_*} can be found with a required accuracy, e.g., from the expansion of the Lagrange function e^z in a regular C-fraction [5].

5. Quasi-reciprocal continued fractions

A function f analytical in the domain $\mathbf{K}$ can be expanded in the vicinity of the point $z_* \in \mathbf{K}$ in a continued fraction of the form

$$f = \left(d_0(z_*) + \mathop{\mathrm{K}}_{k=1}^{\infty} \frac{z - z_*}{d_k(z_*)} \right)^{-1}, \tag{5.1}$$

which is called the quasi-reciprocal continued fraction of the Thiele type (TQCF). From (5.1), it follows that the TQCF is a continued fraction of form (2.3) where $b_0 = 0, a_1 = 1, b_k = d_{k-1}(z_*)$, $a_{k+1} = z - z_*, k \in \mathbb{N}$.

Denote as ${}^{\{k\}}f(z_*)$ the value of the reciprocal derivative of the 2nd type of the k-th order of the function f at the point $z_* \in \mathbf{K}$ [9].

Theorem 5.1. [9] *If the function f is analytic in $\mathbf{K} \subset \mathbb{C}$ and the Hankel determinants $H_{k+1}^{(1)}(z_*)$, $H_k^{(2)}(z_*)$, $H_{k+2}^{(-1)}(z_*)$, $H_{k+1}^{(0)}(z_*)$, $k = \overline{0,n}$, differ from zero at the point $z_* \in \mathbf{K}$, then the function f has finite reciprocal derivatives of the 2-nd type to the $2n$-th order inclusive at the point z_*, which are determined either as the ratio between the Hankel determinants*

$${}^{\{2k+1\}}f(z_*) = \frac{H_{k+2}^{(-1)}(z_*)}{H_{k+1}^{(1)}(z_*)}, \ k = \overline{0, n-1}, \quad {}^{\{2k\}}f(z_*) = \frac{H_k^{(2)}(z_*)}{H_{k+1}^{(0)}(z_*)}, \quad k = \overline{1, n},$$

or using the recurrent formulas

$$\begin{aligned} {}^{\{k\}}f(z_*) &= \frac{k}{\left({}^{\{k-1\}}f(z_*)\right)'} + {}^{\{k-2\}}f(z_*), \quad k=\overline{2,2n}, \\ {}^{\{0\}}f(z_*) &= \frac{1}{f(z_*)}, \quad {}^{\{1\}}f(z_*) = \frac{-f^2(z_*)}{f'(z_*)}. \end{aligned} \tag{5.2}$$

Theorem 5.2. [9] *If the function f has reciprocal derivatives of the 2-nd type up to the n-th order inclusive and $C = const$, then, for $m = \overline{0, [\frac{n}{2}]}$,*

$$ {}^{\{2m\}}(Cf(z)) = \tfrac{1}{C}\cdot {}^{\{2m\}}f(z), \quad {}^{\{2m-1\}}(Cf(z)) = C\cdot {}^{\{2m-1\}}f(z)\,. \tag{5.3}$$

Theorem 5.3. *Let the function $w = f(u)$ have a reciprocal derivative of the 2-nd type at the point $u_0 \in \mathbf{K}$, and the function $u = g(z)$ a derivative at the point $z_0 \in \mathbb{C}$. Then the composite function $w = F(z) = f(g(z))$ will have a reciprocal derivative of the 2-nd type at the point z_0, which is determined by the formula ${}^{\{1\}}F(z_0) = {}^{\{1\}}f(g(z_0))/g'(z_0)$.*

Theorem 5.4. *If the function f has a reciprocal derivatives of the 2-nd type up to the n-th order inclusive and $C = const$, then for $k = \overline{0, [n/2]}$,*

$$ {}^{\{2k\}}f(Cz) = {}^{\{2k\}}f(v)\big|_{v=Cz}, \quad {}^{\{2k-1\}}f(Cz) = \tfrac{1}{C}\cdot {}^{\{2k-1\}}f(v)\big|_{v=Cz}. \tag{5.4}$$

Proof. We shall prove the theorem by induction. According to Theorem 5.3, we have

$$ {}^{\{1\}}f(Cz) = \tfrac{1}{C}{}^{\{1\}}f(v)\big|_{v=Cz}.$$

Then

$$ {}^{\{2\}}f(Cz) = \frac{2}{({}^{\{1\}}f(Cz))'} + f(Cz) = \left[\frac{2}{({}^{\{1\}}f(v))'} + f(v)\right]_{v=Cz} = {}^{\{2\}}f(v)\big|_{v=Cz}.$$

Let (5.4) be obeyed for $k = m-1$. If $k = m$, from (5.2), we get

$$\begin{aligned} {}^{\{2m\}}f(Cz) &= \frac{2m}{({}^{\{2m-1\}}f(Cz))'} + {}^{\{2m-2\}}f(Cz) = \\ &= \left[\frac{2m}{({}^{\{2m-1\}}f(v))'} + {}^{\{2m-2\}}f(v)\right]_{v=Cz} = {}^{\{2m\}}f(Cz)\big|_{v=Cz}, \end{aligned}$$

$$\begin{aligned} {}^{\{2m+1\}}f(Cz) &= \frac{2m+1}{({}^{\{2m\}}f(Cz))'} + {}^{\{2m-1\}}f(Cz) = \\ &= \left[\frac{2m+1}{C({}^{\{2m-1\}}f(v))'} + \frac{1}{C}\cdot {}^{\{2m-1\}}f(v)\right]_{v=Cz} = \tfrac{1}{C}{}^{\{2m+1\}}f(v)\big|_{v=Cz}. \end{aligned}$$

Hence, (5.4) are obeyed for any k-value. □

Suppose that there are reciprocal derivatives of the 2-nd type ${}^{\{n\}}f(z)$ in a certain vicinity of the point $z_* \in \mathbf{K}$. Then the function $f(z)$ can be expanded in TQCF (5.1) with the coefficients

$$\begin{aligned} d_0(z_*) &= \frac{1}{f(z_*)}, \quad d_1(z_*) = {}^{\{1\}}f(z_*), \\ d_k(z_*) &= \frac{k}{\left({}^{\{k-1\}}f(z_*)\right)'} = {}^{\{k\}}f(z_*) - {}^{\{k-2\}}f(z_*), \quad k \in \mathbb{N}_2. \end{aligned} \tag{5.5}$$

The function f can be expanded in the vicinity of the point $z_* \in \mathbf{K}$ in a quasi-reciprocal continued fraction of the C-fraction type (CQCF), which is equivalent to TQCF (5.1) and looks like

$$f = \left(e_0(z_*) + \mathop{\mathrm{K}}_{k=1}^{\infty} \frac{e_k(z_*)(z-z_*)}{1} \right)^{-1}. \tag{5.6}$$

The CQCF coefficients are determined via the reciprocal derivatives of the 2-nd type of the function f at the point $z_* \in \mathbf{K}$ according to the formulas

$$\begin{aligned} e_0(z_*) &= \frac{1}{f(z_*)}, \ e_1(z_*) = \frac{1}{{}^{\{1\}}f(z_*)}, \\ e_k(z_*) &= \frac{\left({}^{\{k-2\}}f(z_*)\right)' \left({}^{\{k-1\}}f(z_*)\right)'}{(n-1)n}, \qquad k \in \mathbb{N}_2. \end{aligned} \tag{5.7}$$

Theorem 5.5. [9] *Let the elements of CQCF* (5.6) *be such that* $e_n \neq 0$, $\lim\limits_{n\to\infty} e_n = 0$, $n \in \mathbb{N}_0$. *Then (A) CQCF* (5.6) *and equivalent to it TQCF* (5.1) *converge to a meromorphic function* f*; (B) the convergence of the continued fractions* (5.1) *and* (5.6) *is uniform on each compact set* $K \subset \mathbb{C}$ *that does not contain poles of* f*; and (C) the function* f *is holomorphic at the point* $z = z_*$*, and* $f(z_*) = 1/e_0(z_*)$.

6. Expansion of the generating function of Bernoulli numbers in quasi-reciprocal continued fractions

Let us expand the auxiliary function h defined in (4.1) in the TQCF and CQCF. It is easy to verify that the reciprocal derivatives of the 2-nd type of the function h are determined by the formulas

$${}^{\{4k\}}h(z) = \frac{1}{e^z-1}, \quad {}^{\{4k+1\}}h(z) = -(2k+1)e^{-z}(e^{2z} - 2(2k+1)e^z + 1),$$

$${}^{\{4k+2\}}h(z) = \frac{-1}{e^z+1}, \quad {}^{\{4k+3\}}h(z) = 2(k+1)e^{-z}(e^{2z} + 4(k+1)e^z + 1), \ k \in \mathbb{N}_0.$$

Then, according to (5.5), the coefficients of the function h expansion in a TQCF in the vicinity of the point z_* acquire the values

$$d_0(z_*) = \frac{1}{e^{z_*}-1}, \quad d_{4k-3}(z_*) = \frac{-(4k-3)(e^{z_*}-1)^2}{e^{z_*}}, \quad d_{4k-2}(z_*) = \frac{-2e^{z_*}}{e^{2z_*}-1},$$

$$d_{4k-1}(z_*) = (4k-1)e^{-z_*}(e^{z_*}+1)^2, \quad d_{4k}(z_*) = \frac{2e^{z_*}}{e^{2z_*}-1}, \quad k \in \mathbb{N}.$$

Substituting them into TQCF (5.1), after equivalent transformations, we obtain the expansion of the generating function $\mathfrak{b}$,

$$\mathfrak{b}(z) = \frac{((e^{2z_*}-1)/e^{z_*})z}{(e^{z_*}+1)/e^{z^*}} + \frac{z-z_*}{-\frac{e^{z_*}-1}{e^{z_*}+1}} + \frac{z-z_*}{-2} + \frac{z-z_*}{3\frac{e^{z_*}+1}{e^{z_*}-1}} + \frac{z-z_*}{2} +$$

$$+ \ \cdots + \frac{z-z_*}{(-1)^k(2k-1)\left(\frac{e^{z_*}-1}{e^{z_*}+1}\right)^{(-1)^{k+1}}} + \frac{z-z_*}{(-1)^k 2} + \cdots. \tag{6.1}$$

Using formulas (5.7), we find the coefficients of the function h expansion in CQCF (5.6) in the vicinity of the point $z_* \in \mathbf{K}$,

$$e_0(z_*) = \frac{1}{e^{z_*}-1}, \quad e_1(z_*) = \frac{-e^{z_*}}{(e^{z_*}-1)^2}, \quad e_{2k-1}(z_*) = \frac{\left(\frac{e^{z_*}-1}{e^{z_*}+1}\right)^{(-1)^{k+1}}}{2(2k-1)},$$

$$e_{2k}(z_*) = \frac{-\left(\frac{e^{z_*}-1}{e^{z_*}+1}\right)^{(-1)^{k}}}{2(2k-1)}, \qquad k \in \mathbb{N}.$$

As a result, we obtain

$$\mathfrak{b}(z) = \frac{(e^{z_*}-1)z}{1} + \frac{-\frac{e^{z_*}(z-z_*)}{e^{z_*}-1}}{1} + \frac{\frac{(e^{z_*}+1)(z-z_*)}{2(e^{z_*}-1)}}{1} + \frac{-\frac{(e^{z_*}-1)(z-z_*)}{6(e^{z_*}+1)}}{1} + \frac{\frac{(e^{z_*}-1)(z-z_*)}{6(e^{z_*}+1)}}{1} +$$
$$+ \cdots + \frac{-\frac{z-z_*}{2(2k-1)}\left(\frac{e^{z_*}-1}{e^{z_*}+1}\right)^{(-1)^{k+1}}}{1} + \frac{\frac{z-z_*}{2(2k-1)}\left(\frac{e^{z_*}-1}{e^{z_*}+1}\right)^{(-1)^{k}}}{1} + \cdots. \tag{6.2}$$

Since $\lim\limits_{k\to\infty} e_{2k-1}(z_*) = \lim\limits_{k\to\infty} e_{2k}(z_*) = 0$, then, by Theorem 5.5, the continued fractions (6.1) and (6.2) converge to the function $\mathfrak{b}$, with the convergence being uniform on an arbitrary compact set $K \subset \mathbf{G}$.

7. Representation of the generating function of Bernoulli polynomials by the Thiele continued fraction and the regular C-fraction

Let us fix some point z_0 in the set $\mathbf{G}$ and consider the auxiliary function

$$\mathbf{B}(x) = \mathfrak{B}(z_0, x) = \mathfrak{b}(z_0)e^{z_0 x}. \tag{7.1}$$

Let us find the reciprocal Thiele derivatives of the function $\mathbf{B}$. It is known [13] that the reciprocal Thiele derivatives of the function $e^t, t \in \mathbb{R}$, are determined by the formulas

$${}^{(2k)}e^t = (-1)^k e^t, \quad {}^{(2k+1)}e^t = \frac{(-1)^k(k+1)}{e^t}, \qquad k \in \mathbb{N}_0.$$

According to (3.2), we have

$${}^{(2k)}\big(e^{z_0 x}\big) = (-1)^k e^{z_0 x}, \quad {}^{(2k+1)}\big(e^{z_0 x}\big) = \frac{(-1)^k(k+1)}{z_0\, e^{z_0 x}}, \qquad k \in \mathbb{N}_0.$$

Taking the last equality in (3.1) into account, we obtain that the reciprocal Thiele derivatives of the function $\mathbf{B}$ are

$${}^{(2k)}\big(\mathbf{B}(x)\big) = (-1)^k \cdot \mathfrak{b}(z_0)e^{z_0 x} = (-1)^k \cdot \mathbf{B}(x),$$
$${}^{(2k+1)}\big(\mathbf{B}(x)\big) = \frac{1}{\mathfrak{b}(z_0)} \cdot \frac{(-1)^k \cdot (k+1)}{z_0 \cdot e^{z_0 x}} = \frac{(-1)^k \cdot (k+1)}{z_0 \cdot \mathbf{B}(x)}, \qquad k \in \mathbb{N}_0.$$

According to (3.4), the expansion coefficients of the function $\mathbf{B}$ in the TCF in the vicinity of the point x_* equal

$$b_0(x_*) = \mathbf{b}_*, \quad b_{2k-1}(x_*) = \frac{(-1)^k \cdot (2k-1)}{z_0 \cdot \mathbf{b}_*},$$
$$b_{2k}(x_*) = (-1)^k \cdot 2 \cdot \mathbf{b}_*, \quad \mathbf{b}_* = \mathbf{B}(x_*), \qquad k \in \mathbb{N}. \tag{7.2}$$

Substituting coefficients (7.2) into TCF (3.3), after equivalent transformations, we obtain the following expansion of the function $\mathbf{B}$:

$$\mathbf{B}(x) = \mathbf{b}_*\Big(1 + \frac{x - x_*}{1/z_0 +} \frac{x - x_*}{-2 +} \frac{x - x_*}{-3/z_0 + +} \frac{x - x_*}{2 +} \frac{x - x_*}{5/z_0 +}$$

$$+ \cdots + \frac{x - x_*}{(-1)^{k-1}(2k-1)/z_0 +} \frac{x - x_*}{(-1)^k 2 + \ldots}\Big).$$

Since z_0 is an arbitrary point of the set $\mathbf{G}$, we obtain from (7.1) that

$$\mathfrak{B}(z, x) = \mathfrak{b}(z)e^{x_* z}\Big(1 + \frac{x - x_*}{1/z +} \frac{x - x_*}{-2 +} \frac{x - x_*}{-3/z +} \frac{x - x_*}{2 +} \frac{x - x_*}{5/z +}$$

$$+ \cdots + \frac{x - x_*}{(-1)^{k-1}(2k-1)/z +} \frac{x - x_*}{(-1)^k 2 + \ldots}\Big). \tag{7.3}$$

Repeating the arguments similar to the above, it is easy to demonstrate that the function $e^{x_* z}$, where x_* is fixed, has the following expansion in the TCF in the vicinity of the point z_*:

$$e^{x_* z} = e^{x_* z_*}\Big(1 + \frac{z - z_*}{1/x_* +} \frac{z - z_*}{-2 +} \frac{z - z_*}{-3/x_* +} \frac{z - z_*}{2 +} \frac{z - z_*}{5/x_* +}$$

$$+ \cdots + \frac{z - z_*}{(-1)^k 2 +} \frac{z - z_*}{(-1)^k(2k+1)/x_* + \ldots}\Big). \tag{7.4}$$

Substituting expansions (4.4) and (7.4) into (7.3), we obtain the representation of the generating function of the Bernoulli polynomials $\mathfrak{B}$ in the form of the product of three continued fractions, namely,

$$\mathfrak{B}(z, x) = e^{x_* z_*}\Big(\frac{z}{e^{z_*} - 1 +} \frac{e^{z_*}(z - z_*)}{1 +} \frac{z - z_*}{-2 +} \frac{z - z_*}{-3 +} \cdots +$$

$$+ \frac{z - z_*}{(-1)^{k-1}(2k-1) +} \frac{z - z_*}{(-1)^k 2 + \ldots}\Big) \cdot \Big(1 + \frac{x - x_*}{1/z +} \frac{x - x_*}{-2 +} \frac{x - x_*}{-3/z +}$$

$$+ \cdots + \frac{x - x_*}{(-1)^{k-1}(2k-1)/z +} \frac{x - x_*}{(-1)^k 2 + \ldots}\Big) \cdot \Big(1 + \frac{z - z_*}{1/x_* +} \frac{z - z_*}{-2 +}$$

$$+ \frac{z - z_*}{-3/x_* +} \cdots + \frac{z - z_*}{(-1)^{k-1}(2k-1)/x_* +} \frac{z - z_*}{(-1)^k 2 + \ldots}\Big). \tag{7.5}$$

Now changing each continued fraction in product (7.5) into an equivalent regular C-fraction, we obtain

$$\mathfrak{B}(z, x) = e^{x_* z_*}\Big(\frac{z}{e^{z_*} - 1 +} \frac{e^{z_*}(z - z_*)}{1 +} \frac{\frac{-1}{2}(z - z_*)}{1 +} \frac{\frac{1}{6}(z - z_*)}{1 +}$$

$$+ \cdots + \frac{\frac{-1}{2(2k-1)}(z - z_*)}{1 +} \frac{\frac{1}{2(2k+1)}(z - z_*)}{1 + \ldots}\Big) \cdot \Big(1 + \frac{z(x - x_*)}{1 +}$$

$$+ \frac{\frac{-1}{2}z(x - x_*)}{1 +} \frac{\frac{1}{6}z(x - x_*)}{1 +} \cdots + \frac{\frac{-1}{2(2k-1)}z(x - x_*)}{1 +}$$

$$+ \frac{\frac{1}{2(2k+1)}z(x - x_*)}{1 + \ldots}\Big) \cdot \Big(1 + \frac{x_*(z - z_*)}{1 +} \frac{\frac{-x_*}{2}(z - z_*)}{1 +}$$

$$+\frac{\frac{x_*}{6}(z-z_*)}{1}+\cdots+\frac{\frac{-x_*}{2(2k-1)}(z-z_*)}{1}+\frac{\frac{x_*}{2(2k+1)}(z-z_*)}{1}+\cdots\Big). \tag{7.6}$$

Since, for fixed x_* and z,

$$\lim_{k\to\infty}\frac{-1}{2(2k-1)}=\lim_{k\to\infty}\frac{1}{2(2k+1)}=\lim_{k\to\infty}\frac{-x_*}{2(2k-1)}=0,$$

$$\lim_{k\to\infty}\frac{x_*}{2(2k+1)}=\lim_{k\to\infty}\frac{-z}{2(2k-1)}=\lim_{k\to\infty}\frac{z}{2(2k+1)}=0,$$

then, according to Theorem 3.3, the products of continued fractions (7.5) and (7.6) converge to the generating function of Bernoulli polynomials $\mathfrak{B}(z,x)$ on the set $\mathfrak{G}$ free of the poles of the examined function. Products (7.5) and (7.6) converge uniformly on an arbitrary compact set $K\subset\mathfrak{G}$.

8. Representations of the generating function of Bernoulli polynomials via quasi-reciprocal continued fractions

Let $z_0\in\mathbf{G}$ be a certain fixed point. Let us determine the reciprocal derivatives of the 2-nd type of the auxiliary function $\mathbf{B}$ defined in (7.1). In [9] , it was proved that the reciprocal derivatives of the 2-nd type of the function $e^t, t\in\mathbb{R}$, are defined as follows:

$${}^{\{2k-1\}}e^t=(-1)^k\cdot k\cdot e^t,\quad {}^{\{2k\}}e^t=\frac{(-1)^k}{e^t},\qquad k\in\mathbb{N}.$$

Whence and from Theorem 5.4, we have

$${}^{\{2k-1\}}e^{z_0x}=\frac{(-1)^k}{z_0}\cdot k\cdot e^{z_0x},\quad {}^{\{2k\}}e^{z_0x}=\frac{(-1)^k}{e^{z_0x}},\qquad k\in\mathbb{N}.$$

Taking property (5.3) of the reciprocal derivatives of the 2nd type into account, we obtain

$${}^{\{2k-1\}}\mathbf{B}(x)=\frac{(-1)^kk}{z_0}\,\mathfrak{b}(z_0)e^{z_0x}=\frac{(-1)^kk}{z_0}\mathbf{B}(x),$$

$${}^{\{2k\}}\mathbf{B}(x)=\frac{(-1)^k}{\mathfrak{b}(z_0)\cdot e^{z_0x}}=\frac{(-1)^k}{\mathbf{B}(x)},\quad k\in\mathbb{N}.$$

According to (5.5), the expansion coefficients of the function $\mathbf{B}$ in TQCF (5.1) in the vicinity of the point $x=x_*$ are

$$d_0(x_*)=\frac{1}{\mathbf{b}_*},\quad d_{2k-1}(x_*)=\frac{(-1)^k(2k-1)}{z_*}\mathbf{b}_*,$$

$$d_{2k}(x_*)=\frac{(-1)^k\cdot 2}{\mathbf{b}_*},\quad \mathbf{b}_*=\mathbf{B}(x_*),\quad k\in\mathbb{N}.$$

Substituting them into (5.1), after equivalent transformations, we obtain

$$\mathbf{B}(x)=\mathbf{b}_*\Big(1+\frac{x-x_*}{-1/z_0}+\frac{x-x_*}{-2}+\frac{x-x_*}{3/z_0}+\frac{x-x_*}{2}+\frac{x-x_*}{-5/z_0}+$$

$$+\frac{x-x_*}{-2}+\cdots+\frac{x-x_*}{(-1)^k(2k-1)/z_0}+\frac{x-x_*}{(-1)^k2}+\cdots\Big)^{-1}.$$

The point z_0 is an arbitrary point of the set $\mathbf{G}$. Then

$$\mathfrak{B}(z,x)=\mathfrak{b}(z)\cdot e^{x_*z}\Big(1+\frac{x-x_*}{-1/z\ +}\ \frac{x-x_*}{-2\ +}\ \frac{x-x_*}{3/z\ +}\ \frac{x-x_*}{2\ +}\ \frac{x-x_*}{-5/z\ +}$$

$$+\ \frac{x-x_*}{-2\ +}\ \cdots+\ \frac{x-x_*}{(-1)^k(2k-1)/z+}\ \frac{x-x_*}{(-1)^k2+\ \cdots}\Big)^{-1}. \tag{8.1}$$

Repeating the arguments similar to the above, we obtain that the expansion of the function e^{x_*z} with a fixed x_* value in the TQCF the vicinity of the point $z=z_*$, after equivalent transformations, has the form

$$e^{x_*z}=e^{x_*z_*}\Big(1+\frac{z-z_*}{-1/x_*+}\ \frac{z-z_*}{-2\ +}\ \frac{x-x_*}{3/x_*\ +}\ \frac{x-x_*}{2\ +}$$

$$+\ \cdots+\ \frac{z-z_*}{(-1)^k(2k-1)/x_*+}\ \frac{z-z_*}{(-1)^k2+\ \cdots}\Big)^{-1}. \tag{8.2}$$

Substituting expansion (8.2) and representation (6.1) of the generating function of Bernoulli numbers into (8.1), we obtain the representation of the generating function of Bernoulli polynomials $\mathfrak{B}$ in the form of the product of three quasi-reciprocal Thiele continued fractions,

$$\mathfrak{B}(z,x)=e^{x_*z_*}\Big[\frac{((e^{2z_*}-1)/e^{z_*})z}{(e^{z_*}+1)/e^{z^*}\ +}\ \frac{z-z_*}{-\frac{e^{z_*}-1}{e^{z_*}+1}+}\ \frac{z-z_*}{-2\ +}\ \frac{z-z_*}{3\frac{e^{z_*}+1}{e^{z_*}-1}+}$$

$$+\ \frac{z-z_*}{2\ +}\ \cdots+\ \frac{z-z_*}{(-1)^k(2k-1)\left(\frac{e^{z_*}-1}{e^{z_*}+1}\right)^{(-1)^{k+1}}+}\ \frac{z-z_*}{(-1)^k2+\ \cdots}\Big]\times$$

$$\times\Big[1+\frac{z-z_*}{-1/x_*+}\ \frac{z-z_*}{-2\ +}\ \frac{z-z_*}{-3/x_*+}\ \frac{z-z_*}{2\ +}\ \frac{z-z_*}{5/x_*\ +}\ \cdots+$$

$$+\ \frac{z-z_*}{(-1)^k(2k-1)/x_*+}\ \frac{z-z_*}{(-1)^k2+\ \cdots}\Big]^{-1}\times\Big[1+\frac{x-x_*}{-1/z\ +}\ \frac{x-x_*}{-2\ +}$$

$$+\ \cdots+\ \frac{x-x_*}{(-1)^k(2k-1)/z+}\ \frac{x-x_*}{(-1)^k2+\ \cdots}\Big]^{-1}. \tag{8.3}$$

Each of the quasi-reciprocal continued fractions can be written via equivalent continued fractions with partial denominators equal to unity, i.e.,

$$\mathfrak{B}(z,x)=e^{x_*z_*}\Big(\frac{(e^{z_*}-1)z}{1\ \ +}\ \frac{-\frac{e^{z_*}}{e^{z_*}-1}(z-z_*)}{1\ \ +}\ \frac{\frac{e^{z_*}+1}{2(e^{z_*}-1)}(z-z_*)}{1\ \ +}$$

$$+\ \frac{-\frac{e^{z_*}-1}{6(e^{z_*}+1)}(z-z_*)}{1\ \ +}\ \frac{\frac{e^{z_*}-1}{6(e^{z_*}+1)}(z-z_*)}{1\ \ +\ \cdots+}$$

$$+\ \frac{\frac{-1}{2(2k-1)}\left(\frac{e^{z_*}-1}{e^{z_*}+1}\right)^{(-1)^{k+1}}(z-z_*)}{1\ \ +}\ \frac{\frac{1}{2(2k-1)}\left(\frac{e^{z_*}-1}{e^{z_*}+1}\right)^{(-1)^{k}}(z-z_*)}{1\ \ +\ \cdots}\Big)\times$$

$$\times\Big(1+\frac{-x_*(z-z_*)}{1\ \ +}\ \frac{\frac{x_*}{2}(z-z_*)}{1\ \ +}\ \frac{\frac{-x_*}{6}(z-z_*)}{1\ \ +}\ \frac{\frac{x_*}{6}(z-z_*)}{1\ \ +\ \cdots+}$$

$$+\ \frac{\frac{-x_*}{2(2k-1)}(z-z_*)}{1\ \ +}\ \frac{\frac{x_*}{2(2k-1)}(z-z_*)}{1\ \ +\ \cdots}\Big)\cdot\Big(1+\frac{-z(x-x_*)}{1\ \ +}$$

$$+\frac{\frac{z}{2}(x-x_*)}{1}+\frac{\frac{-z}{6}(x-x_*)}{1}+\frac{\frac{z}{6}(x-x_*)}{1}+$$

$$+\cdots+\frac{\frac{-z}{2(2k-1)}(x-x_*)}{1}+\frac{\frac{z}{2(2k-1)}(x-x_*)}{1}+\cdots\Big). \tag{8.4}$$

Since $\lim_{k\to\infty}\frac{1}{2(2k-1)}=0$, then, by Theorem 5.5, the continued fractions from product (8.4) and their equivalent continued fractions from product (8.3) converge, and this convergence is uniform on any compact set $\mathfrak{K}\subset\mathfrak{G}$.

9. Final remarks

In this paper, expansions of the generating function of Bernoulli numbers in the vicinity of a certain point $z_*\in\mathbf{G}$ in the Thiele continued fraction and the regular C-fraction were obtained. The areas of convergence and uniform convergence of expansions were determined.

An expansion of the generating function of Bernoulli numbers in quasi-reciprocal continued fractions was also proposed. The convergence and uniform convergence of such expansions were proved.

A representation of the generating function of Bernoulli polynomials was obtained in the form of the product of three continued fractions and the product of three quasi-reciprocal continued fractions.

The author is grateful to the Reviewer for valuable comments and advice.

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