УДК 519.2

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Генератриси розподілу екстремумів та їх доповнень для напівнеперервних зверху гратчастих пуассонівських процесів на ланцюгу Маркова

На скінченному регулярному ланцюгу Маркова (ЛМ) розглядається гратчастий пуассонівський процес; стрибки якого приймають довільні цілі від'ємні значення, а додатні стрибки тільки одиничні. Такі процеси називаються напівнеперервними зверху. Для цих процесів встановлюються співвідношення для генератрис мінімуму та доповнення до максимуму процесу без застосування операції проектування. Одержані співвідношення для досліджуваних генератирис визначалися в [6] у термінах проекцій відповідної компоненти факторизації. Нові співвідношення для цих генератрис встановлюються оберненням кумулянти, яка виражається через твірні перетворення функції розподілу від'ємних стрибків.

Ключові слова: напівнеперервні зверху процеси, генератриси мінімуму та доповнення до максимуму, кумулянта.

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Moment generating funtions of extremums and their complemets for upper semi-continuous lattice Poison process on Markov chain

On the finite regular Markov chain it is considered latticed Poisson process, jumps of which take arbitrary integer negative values, and positive jumps are equal 1. Such processes are called upper semi-continuous. For these processes the relations for moment generating funtions (m. g. f.) of minimum and complements to maximum of the process are established without proective operation. Obtained relations for considered m. g. f. were defined in [6] in terms of projection of coresponding component of factorization. New relations for these m. g. f. are established by the invertions of the cumulant function, which is represented in terms of generating transformation for distribution function of negative jumps.

Key Words: upper semi-continuous processes, moment generating funtions of minimum and complements to maximum, cumulant function.

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Communicated by Prof. Kozachenko Yu.V.

1 Introduction

The description of the processes defined on Markov chain (MCh) was given in papers of I.I. Yezhov, A.B. Skorokhod [1]. Boundary problems for such processes with continuously distributed jumps on MCh were investigated in [2, 3], where matrix analogues of basic factorization identity (b. f. i.) and 2-nd factorization identity for almost semi-continuous processes were obtained. For semi-continuous processes refinements of some results in lattice case were obtained in papers D.V. Husak, A. I. Tureniyazova [4, 5] and M.S. Gerich [7, 6].

In this paper, our task is to find:

- relations for moment generating functions (m. g. f.) of the minimum distribution (including absolute);
- relations for m. g. f. of complements to maximum and its limit relations when $s \to 0$;
- and express these m. g. f. directly through the m. g. f. of distribution‘s tail for negative jumps of processes.

2 Upper semi continuous lattice Poison process on Markov chain

To do this, first consider two-dimensional Markov integral-valued process $\mathbf{Z}(t) =$

$\{\xi(t), x(t)\}(t \geqslant 0, \xi(0) = 0)$, where $x(t)$ is finite ergodic MCh with values in $\mathbb{E} = \{1, ..., m\}$ and the generating matrix $\mathbf{Q} = \mathbf{N}(\mathbf{P} - \mathbf{I})$; $\xi(t)$ of lattice Poison process given on MCh with values in $\mathbb{Z}$. Its m. g. f. has an exponential form

$$\mathbf{g}_t(z) = \|E[z^{\xi(t)}, x(t) = r|\, x(0) = k]\|$$
$$= \mathbf{E}z^{\xi(t)} = e^{t\mathbf{K}(z)},$$

where matrix cumulant function $\mathbf{K}(z)$ is determined by general matrix relation

$$\mathbf{K}(z) = \sum_{x \neq 0} (z^x - 1)\mathbf{K}_0(x) + \mathbf{Q},$$
$$\mathbf{K}_0(x) = \mathbf{\Lambda p}(x) + \mathbf{Nf}(x). \tag{1}$$

$\mathbf{\Lambda} = \|\delta_{kr}\lambda_k\|$ $(k = 1, ..., m)$, λ_k are jumps intensitis of Poisson processes $\{\xi_k(t)\}_{k=1}^m$ with jumps distribution $\mathbf{p}(x) = \|\delta_{kr} P\{\xi_1^{(k)} = x\}\|$, $\mathbf{N} = \|\delta_{kr} n_k\|_{k,r \in \mathbb{E}}$, where $\{n_k > 0, k \in \mathbb{E}\}$ – parameters of the exponential distributed random variables ζ_k are sojour time $x(t)$ in the state k. Matrix of transition probabilities of embedded MCh $y_n = x(\sigma_n + 0)$, where σ_n are moments of n-th change of states for $x(t)$: $\mathbf{P} = \|p_{kr}\|$, $(k, r = \overline{1, m})$, $\mathbf{f}(x) = \|p_{kr} P\{\chi_{kr} = x\}\|$ is a distribution of jumps on transitions of $x(t)$.

To reduce the notation of integral transformations on t it is necessary to introduce exponential distributive random variable θ_s $(P\{\theta_s > t\} = e^{-st}$, $s > 0)$.

For extremums of process and their complements, as well as for intersection functionals of positive (negative) level introduce the following notations:

$$\xi^{\pm}(t) = \sup_{0 \leqslant t' \leqslant t} (\inf) \xi(t'),\ \xi^{\pm} = \sup_{0 \leqslant t \leqslant \infty} (\inf) \xi(t),$$

$$\bar{\xi}(t) = \xi(t) - \xi^{+}(t),\ \check{\xi}(t) = \xi(t) - \xi^{-}(t);$$

$$\tau^{+}(x) = \inf\{t > 0, \xi(t) > x\}, \gamma^{+}(x) = \xi(\tau^{+}(x)) - x,$$
$$x \geqslant 0;$$

$$\tau^{-}(x) = \inf\{t > 0, \xi(t) < x\}, \gamma^{-}(x) = x - \xi(\tau^{-}(x)),$$
$$x \leqslant 0.$$

Denote (in pursuance of averaging over distribution θ_s)

$$\mathbf{g}(s, z) = \mathbf{E}z^{\xi(\theta_s)} = s \int_0^{+\infty} e^{-st} \mathbf{g}_t(z) dt =$$

$$= s(s\mathbf{I} - \mathbf{K}(z))^{-1}. \tag{2}$$

$$\mathbf{g}_{\pm}(s, z) = \mathbf{E}z^{\xi^{\pm}(\theta_s)} =$$
$$= \|E[z^{\xi^{\pm}(\theta_s)},\, x(\theta_s) = r|x(0) = k]\|,\ k, r = \overline{1, m},$$
$$\mathbf{g}^{-}(s, z) = \mathbf{E}z^{\bar{\xi}(\theta_s)},\ \mathbf{g}^{+}(s, z) = \mathbf{E}z^{\check{\xi}(\theta_s)},$$
$$\mathbf{P}_s = s(s\mathbf{I} - \mathbf{Q})^{-1}. \tag{3}$$

In [3], [4, 6] b. f. i. matrix analogue that establishes a relationship between $\mathbf{g}(s, z)$ and $\mathbf{g}_{+}(s, z)$, $\mathbf{g}^{-}(s, z)$, $(\mathbf{g}_{-}(s, z)$, $\mathbf{g}^{+}(s, z))$ was obtained .

$$\mathbf{g}(s, z) = \mathbf{E}\, z^{\xi(\theta_s)} =$$
$$= \begin{cases} \mathbf{g}_{+}(s, z)\mathbf{P}_s^{-1}\mathbf{g}^{-}(s, z), \\ \mathbf{g}_{-}(s, z)\mathbf{P}_s^{-1}\mathbf{g}^{+}(s, z). \end{cases} \tag{4}$$

All next probabilities are strictly positive:

$$\mathbf{p}_{\pm}(s) = \|P\{\xi^{\pm}(\theta_s) = 0,\, x(\theta_s) = r|x(0) = k\}\|,$$

$$\mathbf{q}_{\pm}(s) = \mathbf{P}_s - \mathbf{p}_{\pm}(s),$$

$$\mathbf{p}^{+}(s) = \|P\{\check{\xi}(\theta_s) = 0,\, x(\theta_s) = r|x(0) = k\}\|,$$

$$\mathbf{p}^{-}(s) = \|P\{\bar{\xi}(\theta_s) = 0,\, x(\theta_s) = r|x(0) = k\}\|,$$

$$\mathbf{q}^{\pm}(s) = \mathbf{P}_s - \mathbf{p}^{\pm}(s).$$

Next, consider the lattice Poisson processes on MCh with cumulant function:

$$\mathbf{K}(z) = \mathbf{\Lambda}_1(z - 1) +$$
$$\sum_{x<0} (z^x - 1)(\mathbf{\Lambda}_2 \mathbf{p}_2(x) + \mathbf{Nf}(x)) + \mathbf{Q}. \tag{5}$$

In [5, 7] it is shown that one among the b. f. i. component pair in (4) is matrix fractionall linear function relatively to z, and the other component of these pairs is determined by application of some projective procedures to m.g.f. of process itself. Our task is to express "notsimple"m. g. f. from paires $\{\mathbf{g}_{+}(s, z), \mathbf{g}^{-}(s, z)\}$ and $\{\mathbf{g}_{-}(s, z), \mathbf{g}^{+}(s, z)\}$ without projective operation. To do this, introduce the inverse MCh concept (see [9]) and auxiliary assertions on singularly perturbed matrices inversion (see [10]).

If $x(t)$ is homogeneous regular MCh with generating (degenerate) matrix $\mathbf{Q}$ and appropriate matrix of transition probabilities

$$\mathbf{P}(t) = \|P\{x(t) = r|x(0) = k\| = e^{t\mathbf{Q}} \tag{6}$$

and its Laplace - Carson transform

$$\widetilde{\mathbf{P}}(s) = s\int\limits_0^{+\infty} e^{-st}\mathbf{P}(t)dt = s(s\mathbf{I}-\mathbf{Q})^{-1},\ s>0 \quad (7)$$

is expressed through the inversion of singular perturbed matrix $(s\mathbf{I}-\mathbf{Q})^{-1}$ (at $s>0$ i $|s\mathbf{I}-\mathbf{Q}|\neq 0$).

For $x(t)$ the probabilities of stationary distribution exists

$$\lim_{t\to+\infty}\mathbf{P}(t) = \mathbf{P}_0 = \|p^0_{kr}\|,\ p^0_{kr} = \pi_r, \forall\, k,$$

that defines as the boundary of inversion in (7) for $s\to 0$

$$\mathbf{P}_0 = \lim_{s\to 0} s(s\mathbf{I}-\mathbf{Q})^{-1}. \quad (8)$$

Thus the relations (9) take place

$$\mathbf{Q}\mathbf{P}_0 = \mathbf{0},\ \mathbf{P}_0\mathbf{Q} = \mathbf{0}. \quad (9)$$

The first one is obvious, and the second defines the unique solution of corresponding system of linear equations for the values of stationary probabilities $\{\pi_k\}$ (see [8]).

We need the generalization of relations (8)-(9), which are given in Lemmas 1-2.

Lemma 1. *[10] Let $\mathbf{Q}_0$ be degenerate matrix of m-order, $\nu\neq 0$. Then the perturbed matrix inversion $\mathbf{Q}_0+\nu\mathbf{I}$ has the splitting*

$$(\mathbf{Q}_0+\nu\mathbf{I})^{-1} = \nu^{-1}\mathbf{\Pi}_1 + \mathbf{T}_0(\mathbf{I}+\nu\mathbf{T}_0)^{-1}, \quad (10)$$

where $\mathbf{\Pi}_1$ is matrix eigen projector of $\mathbf{Q}_0$, $r = dimN(\mathbf{Q}_0) < m$, $\mathbf{\Pi}_1 = \sum\limits_{k=1}^{r} u^k\bigotimes\rho^k$; u^k, ρ^k are right and left eigenvectors of the operator $\mathbf{Q}_0$, that correspond to the zero eigenvalue: $(\rho^{(i)}, u^{(j)}) = \delta_{ij}$, $i, j = \overline{1,r}$. Besides $\mathbf{T}_0 = (\mathbf{I}-\mathbf{\Pi}_1)((\mathbf{\Pi}_1-\mathbf{Q}_0)^{-1}-\mathbf{\Pi}_1)(\mathbf{I}-\mathbf{\Pi}_1)$,

$$\mathbf{\Pi}_1\mathbf{Q}_0 = \mathbf{Q}_0\mathbf{\Pi}_1 = 0,\ \mathbf{\Pi}_1\mathbf{T}_0 = \mathbf{T}_0\mathbf{\Pi}_1 = 0, \quad (11)$$

and (11) is the analogue of relations (9). Denote for the process $\xi(t)$ with cumulant function (1)

$$\mathbf{M}^0_1 = \mathbf{E}\xi(1) = \sum_{x\leqslant 1,\, x\neq 0} x\mathbf{K}_0(x),$$

$$\mathbf{D}_0 = \mathbf{D}\xi(1) = \sum_{x\leqslant 1,\, x\neq 0} x^2\mathbf{K}_0(x),$$

which further will consider as finite. Denote $\widetilde{\mathbf{K}}_0(z) = \sum\limits_{x\leqslant 1,\, x\neq 0}(z^x-1)\mathbf{K}_0(x)$, at $z = 1+\varepsilon$, $\widetilde{\mathbf{K}}_0(1+\varepsilon) = \varepsilon\widetilde{\mathbf{K}}'_0(1) + \frac{\varepsilon^2}{2}\widetilde{\mathbf{K}}''_0(1)$, where $\widetilde{\mathbf{K}}'_0(1) = \mathbf{E}\xi(1) = \mathbf{M}^0_1$, $\widetilde{\mathbf{K}}''_0(1) = \mathbf{D}\xi(1) = \mathbf{D}_0$.

Then at $z = 1+\varepsilon$ the approximation

$$-\mathbf{K}(z) = (1-z)\mathbf{M}^0_1 - \frac{1}{2}(1-z)^2\mathbf{D}_0 - \mathbf{Q} + o(\varepsilon^2) \quad (12)$$

takes place and respectively averaged over the stationary distribution $\mathbf{P}_0$ moments denote as

$$m^0_1 = \sum_{k=1}^{m}\pi_k\sum_{r=1}^{m}[\delta_{kr}E\xi^{(k)}(1) + \sum_{r\neq k}^{m} n_k p_{kr}E\chi_{kr}], \quad (13)$$

$$\sigma^2_0 = \sum_{k=1}^{m}\pi_k\sum_{r=1}^{m}[\delta_{kr}D\xi^{(k)}(1) + \sum_{r\neq k}^{m} n_k p_{kr}D\chi_{kr}]. \quad (14)$$

Based on (12)-(13) and (14) and some results in [9] the following lemma, which generalizes the relation (8), is proving.

Lemma 2. *For the process $\mathbf{Z}(t)$ with cumulant function (5) from $|m^0_1| < \infty$ the following relations take place*

$$\lim_{z\to 1}(-(1-z)\mathbf{K}^{-1}(z)) = \lim_{z\to 1}(1-z)((1-z)\mathbf{M}^0_1-\mathbf{Q})^{-1} = \frac{1}{m^0_1}\mathbf{P}_0,\ m^0_1\neq 0, \quad (15)$$

$$\lim_{z\to 1}(1-z)^2\mathbf{K}^{-1}(z) = \lim_{z\to 1}(1-z)^2(\frac{1}{2}(1-z)^2\mathbf{D}_0-\mathbf{Q})^{-1} = \frac{2}{\sigma^2_0}\mathbf{P}_0,\ m^0_1 = 0,\ \sigma^2_0 < \infty. \quad (16)$$

For ergodic MCh $x(t)$ in [9] the inverse MCh $\widehat{x}(t)$ concept is introduced.

Definition 2.1. If $x(t)$ is ergodic MCh with generating matrix $\mathbf{Q} = \mathbf{N}(\mathbf{P}-\mathbf{I})$ with diagonally written distribution $\mathbf{R} = \|\delta_{kr}\pi_r\|$, then the inverse to MCh $x(t)$ is called MCh $\widehat{x}(t)$ which is defined by generating matrix $\widehat{\mathbf{Q}} = \mathbf{S}\mathbf{Q}^T\mathbf{S}^{-1} = \mathbf{N}(\widehat{\mathbf{P}}-\mathbf{I})$, where $\mathbf{S} = \mathbf{N}\mathbf{R}^{-1}$.

Accordingly reverse process $\widehat{\xi}(t)(t \geqslant 0)$ on MCh $\widehat{x}(t)$ is defined by cumulant function and moment generating function $\widehat{\xi}(\theta_s)$:

$$\widehat{\mathbf{K}}(z) = \mathbf{S}\mathbf{K}^T(z)\mathbf{S}^{-1},$$

$$\widehat{\mathbf{g}}(s,z) = \mathbf{E}z^{\widehat{\xi}(\theta_s)} = s(s\mathbf{I} - \widehat{\mathbf{K}}(z))^{-1}.$$

Theorem 1. *[4] Basic factorization identity (4) in terms of moment generating functions extremums for inverse process $\widehat{\mathbf{g}}_{\pm}(s,z) = \mathbf{E}z^{\widehat{\xi}^{\pm}(\theta_s)}$ takes the form*

$$\mathbf{g}(s,z) = \begin{cases} \mathbf{g}_+(s,z)\mathbf{P}_s^{-1}\mathbf{S}\,\widehat{\mathbf{g}}_-^T(s,z)\mathbf{S}^{-1}; \\ \mathbf{g}_-(s,z)\mathbf{P}_s^{-1}\mathbf{S}\,\widehat{\mathbf{g}}_+^T(s,z)\mathbf{S}^{-1}. \end{cases} \quad (17)$$

Note that from (4) and (17) the relations of connection between the extremums distributions of direct and inverse processes follow

$$\mathbf{g}^{\pm}(s,z) = \mathbf{S}\,\widehat{\mathbf{g}}_{\pm}^T(s,z)\mathbf{S}^{-1},\ \mathbf{P}_s = \mathbf{S}\,(\widehat{\mathbf{P}}_s)^T\mathbf{S}^{-1};$$

$$\mathbf{q}^+(s) = \mathbf{S}\,\widehat{\mathbf{q}}_+^T(s)\mathbf{S}^{-1},\ \mathbf{q}^-(s) = \mathbf{S}\,\widehat{\mathbf{q}}_-^T(s)\mathbf{S}^{-1}.$$

As well as in [3] in case of lattice Poison processes the similar result takes place.

Lemma 3. *For lattice Poison process defined on Markov chain, the following matrix representations take place for $\mathbf{p}_*^{\pm}(s)$ and $\mathbf{p}_{\pm}^*(s)$*

$$\mathbf{p}_{\pm}^*(s) = \mathbf{p}_{\pm}(s)\mathbf{P}_s^{-1} = \mathbf{I} - \mathbf{E}[e^{-s\tau^{\pm}(0)},\ \tau^{\pm}(0) < \infty] = \mathbf{I} - \mathbf{T}_*^{\pm}(s,0); \quad (18)$$

$$\mathbf{p}_*^{\pm}(s) = \mathbf{P}_s^{-1}\mathbf{p}^{\pm}(s) = \mathbf{I} - \mathbf{S}(\mathbf{E}[e^{-s\widehat{\tau}^{\pm}(0)},\ \widehat{\tau}^{\pm}(0) < \infty])^T\mathbf{S}^{-1} = \mathbf{I} - \mathbf{S}(\widehat{\mathbf{T}}_*^{\pm}(s,0))^T\mathbf{S}^{-1}; \quad (19)$$

$$\mathbf{T}_*^{\pm}(s,0) = \mathbf{q}_{\pm}(s)\mathbf{P}_s^{-1},\ \widehat{\mathbf{T}}_*^{\pm}(s,0) = \widehat{\mathbf{q}}_{\pm}(s)\widehat{\mathbf{P}}_s^{-1}.$$

For upper semi-continuous processes $\xi(t)$ according to [7] the "not complicated"components of b.f.t. in (4) are as follows

$$\mathbf{g}_+(s,z) = (\mathbf{I} - \mathbf{Z}_s^{-1}z)^{-1}\mathbf{p}_+(s),\quad \mathbf{p}_+(s) = (\mathbf{I} - \mathbf{Z}_s^{-1})\mathbf{P}_s,\quad \mathbf{Z}_s^{-1} = \mathbf{q}_+(s)\mathbf{P}_s^{-1}, \quad (20)$$

$$\mathbf{g}^+(s,z) = \mathbf{p}_+(s)(\mathbf{I} - \mathbf{Q}_s^{-1}z)^{-1},\quad \mathbf{p}^+(s) = \mathbf{P}_s(\mathbf{I} - \mathbf{Q}_s^{-1}),\quad \mathbf{Q}_s^{-1} = \mathbf{P}_s^{-1}\mathbf{q}^+(s). \quad (21)$$

Further suppose $m_1^0 > 0$. Consider the embedded MCh ($y_*^1 = x(\tau^+(0))$, $y_*^0 = x(0)$) with matrix of transition probabilities $\mathbf{P}_* = \|P\{y_*^1 = r|y_*^0 = k\}\|$ and generating matrix $\mathbf{Q}_* = \mathbf{P}_* - \mathbf{I}$, де $\mathbf{P}\{\tau^+(0) < \infty\} = \|P\{\tau^+(0) < \infty,\ y_*^1 = r|y_*^0 = k\}\| = \mathbf{P}_*$, $\mathbf{T}_*^+(s,0) = \mathbf{E}[e^{-s\tau^+(0)},\ \tau^+(0) < \infty] = \|E[e^{-s\tau^+(0)},\ y_*^1 = r|y_*^0 = k]\|$, $\mathbf{P}_* = \mathbf{T}_*^+(0,0)$. The moment generating function $\tau^+(0)$ at $s \to 0$ satisfies the approximation

$$\mathbf{I} - \mathbf{T}_*^+(s,0) = -\mathbf{Q}_* + s\mathbf{M}_* + o(s),\quad \mathbf{M}_* = \mathbf{E}\tau^+(0) > 0. \quad (22)$$

Denote the stationary distribution of the embedded MCh with generating matrix $\mathbf{Q}_*$ through $\mathbf{\Pi}_* = \lim_{s\to 0} s(s\mathbf{I} - \mathbf{Q}_*)^{-1} = \|\pi_{*kr}\|$, $\pi_{*kr} = \pi_{*r}$, $k,r = \overline{1,m}$, and appropriate averaging on it for $\mathbf{M}_*$

$$\mu_*^+ = \sum_{k=1}^m \pi_{*k} \sum_{r=1}^m E[\tau^+(0),\ y_*^1 = r|y_*^0 = k].$$

Similarly, introduce notion of inverse MCh for embedded with generating matrix $\widehat{\mathbf{Q}}_* = \mathbf{S}\mathbf{Q}_*^T\mathbf{S}^{-1}$, $\widehat{\mathbf{Q}}_* = \widehat{\mathbf{P}}_* - \mathbf{I}$, $\mathbf{P}\{\widehat{\tau}^+(0) < \infty\} = \|P\{\widehat{\tau}^+(0) < \infty,\ \widehat{y}_*^1 = r|\widehat{y}_*^0 = k\}\| = \widehat{\mathbf{P}}_*$.

$$\widehat{\mathbf{T}}_*^+(s,0) = \mathbf{E}[e^{-s\widehat{\tau}^+(0)},\ \widehat{\tau}^+(0) < \infty] = \|E[e^{-s\widehat{\tau}^+(0)},\ \widehat{y}_*^1 = r|\widehat{y}_*^0 = k]\|;$$

$$\widehat{\mathbf{P}}_* = \|P\{\widehat{y}_*^1 = r|\widehat{y}_*^0 = k\}\|,\ k,r = \overline{1,m},\quad \widehat{\mathbf{P}}_* = \widehat{\mathbf{T}}_*^+(0,0).$$

M. g. f. $\widehat{\tau}^+(0)$ for $s \to 0$ satisfies the approximation

$$\mathbf{I} - \widehat{\mathbf{T}}_*^+(s,0) = -\widehat{\mathbf{Q}}_* + s\widehat{\mathbf{M}}_* + o(s),\quad \widehat{\mathbf{M}}_* = \mathbf{E}\widehat{\tau}^+(0) > 0. \quad (23)$$

Introduce the following notations

$$\widehat{\mathbf{P}}_{*\mathbf{S}} = \mathbf{S}(\widehat{\mathbf{P}}_*)^T\mathbf{S}^{-1},\ \widehat{\mathbf{Q}}_{*\mathbf{S}} = \mathbf{S}(\widehat{\mathbf{Q}}_*)^T\mathbf{S}^{-1};$$

$$\widehat{\mathbf{M}}_{*\mathbf{S}} = \mathbf{S}(\mathbf{E}[\widehat{\tau}^+(0)])^T\mathbf{S}^{-1};$$

$$\widehat{\mathbf{T}}_{*\mathbf{S}}^+(s,0) = \mathbf{S}(\mathbf{E}[e^{-\widehat{\tau}^+(0)},\ \widehat{\tau}^+(0) < \infty])^T\mathbf{S}^{-1}.$$

After performing of transposition operation on (23), then multiplying on $\mathbf{S}$ the left and on $\mathbf{S}^{-1}$ on the right and taking into account the necessary notations we obtain (24)

$$\mathbf{I} - \widehat{\mathbf{T}}_{*\mathbf{S}}^+(s,0) = -\widehat{\mathbf{Q}}_{*\mathbf{S}} + s\widehat{\mathbf{M}}_{*\mathbf{S}} + o(s). \quad (24)$$

Denote the stationary distribution of inverse MCh with generating matrix $\widehat{\mathbf{Q}}_*$ through $\widehat{\mathbf{\Pi}}_* = \lim_{s\to 0} s(s\mathbf{I} - \widehat{\mathbf{Q}}_*)^{-1} = \|\widehat{\pi}_{*kr}\|$, $\widehat{\pi}_{*kr} = \widehat{\pi}_{*r}$, $k, r = \overline{1, m}$, and appropriate averaging for $\widehat{\mathbf{M}}_*$ by corresponding stationary distribution

$$\widehat{\mu}_*^+ = \sum_{k=1}^{m} \widehat{\pi}_{*k} \sum_{r=1}^{m} E[\widehat{\tau}^+(0),\, \widehat{y}_*^1 = r | \widehat{y}_*^0 = k].$$

For upper semi-continuous process $\xi(t)$ according with (18), (19), (20), (21) and previously introduced concepts and notations we found $\mathbf{Z}_s^{-1}$, $\mathbf{Q}_s^{-1}$:

$$\mathbf{Z}_s^{-1} = \mathbf{T}_*^+(s,0);\; \mathbf{Q}_s^{-1} = \widehat{\mathbf{T}}_{*\mathbf{S}}^+(s,0). \quad (25)$$

From Lemma 2 and the relations (22), (24) and (25) the next lemma follows

Lemma 4. *If $\xi(t)$ is upper semi-continuous process and $0 < m_1^0 < \infty$, then $0 < \mu_*^+ < \infty$, $0 < \widehat{\mu}_*^+ < \infty$, and the following limit relations take place*

$$\lim_{s\to 0} s(\mathbf{I} - \mathbf{Z}_s^{-1})^{-1} = \lim_{s\to 0} s(\mathbf{I} - \mathbf{T}_*^+(s,0))^{-1} = \frac{1}{\mu_*^+}\mathbf{\Pi}_*, \quad (26)$$

$$\lim_{s\to 0} s(\mathbf{I} - \mathbf{Q}_s^{-1})^{-1} = \lim_{s\to 0} s(\mathbf{I} - \widehat{\mathbf{T}}_{*\mathbf{S}}^+(s,0))^{-1} = \frac{1}{\widehat{\mu}_*^+}\widehat{\mathbf{\Pi}}_{*\mathbf{S}}. \quad (27)$$

Proof. Using relation (25), (22) and (15) we obtain (26)

$$\lim_{s\to 0} s(\mathbf{I} - \mathbf{Z}_s^{-1})^{-1} = \lim_{s\to 0} s(\mathbf{I} - \mathbf{T}_*^+(s,0))^{-1} = \lim_{s\to 0} s(s\mathbf{M}_* - \mathbf{Q}_*)^{-1} = \frac{1}{\mu_*^+}\mathbf{\Pi}_*.$$

Similarly (27) is proved after using (25), (24) and taking into account the result of Lemma 2.

From dual relations

$$\mathbf{Z}_s - \mathbf{I} = \begin{cases} \mathbf{Z}_s(\mathbf{I} - \mathbf{Z}_s^{-1}), \\ (\mathbf{I} - \mathbf{Z}_s^{-1})\mathbf{Z}_s. \end{cases}$$
$$\mathbf{Q}_s - \mathbf{I} = \begin{cases} \mathbf{Q}_s(\mathbf{I} - \mathbf{Q}_s^{-1}), \\ (\mathbf{I} - \mathbf{Q}_s^{-1})\mathbf{Q}_s. \end{cases} \quad (28)$$

at $s > 0$ follows

Lemma 5. *If $\xi(t)$ is upper semi-continuous process on Markov chain $x(t)$, then at $0 < m_1^0 < \infty$, according to (26) and (27) the following relations take place*

$$\lim_{s\to 0} s(\mathbf{Z}_s - \mathbf{I})^{-1} = \frac{1}{\mu_*^+}\mathbf{\Pi}_*^{'},$$
$$\mathbf{\Pi}_*^{'} = \mathbf{\Pi}_*\mathbf{P}_* = \mathbf{P}_*\mathbf{\Pi}_* = \mathbf{\Pi}_*. \quad (29)$$

$$\lim_{s\to 0} s(\mathbf{Q}_s - \mathbf{I})^{-1} = \frac{1}{\widehat{\mu}_*^+}\widehat{\mathbf{\Pi}}_{*\mathbf{S}}^{'},$$
$$\widehat{\mathbf{\Pi}}_{*\mathbf{S}} = \mathbf{\Pi}_*, \quad (30)$$
$$\mathbf{\Pi}_{*\mathbf{S}}^{'} = \widehat{\mathbf{\Pi}}_{*\mathbf{S}}\widehat{\mathbf{P}}_{*\mathbf{S}} = \widehat{\mathbf{P}}_{*\mathbf{S}}\widehat{\mathbf{\Pi}}_{*\mathbf{S}} = \widehat{\mathbf{\Pi}}_{*\mathbf{S}}.$$

Proof. The first relations in (29) and (30) follow from (28) taking to the account the conditions (25)-(27). Second relation in (30) follows from early mentiond inversity notations. Last relations are evident.

Consider cumulant function (5) and present it as

$$\mathbf{K}(z) = (z-1)[\mathbf{\Lambda}_1 - z^{-1}(\mathbf{\Lambda}_2\widetilde{\mathbf{F}}_2(z) + \mathbf{N}\widetilde{\mathbf{F}}(z))] + \mathbf{Q} \quad (31)$$

$$\widetilde{\mathbf{F}}_2(z) = \sum_{x\leqslant 0} z^x \mathbf{P}\{\xi_1^{(k)} < x\},\; |z| \geqslant 1.$$

$$\widetilde{\mathbf{F}}(z) = \sum_{x\leqslant 0} z^x \|p_{kr} P\{\chi_{kr} < x\}\|,\; |z| \geqslant 1.$$

Should be noted that according to (25) $\mathbf{K}(z)$ is expressed through generating transform of the negative jumps distribution functions and $\mathbf{K}(1) = \mathbf{Q}$.

Theorem 2. *If $\xi(t)$ is upper semi-continuous process on Markov chain $x(t)$ і $0 < m_1^0 < \infty$, and then at $|z| \geqslant 1$ moment generating function of $\xi^-(\theta_s)$ looks as*

$$\mathbf{g}_-(s,z) = s(s\mathbf{I} - \mathbf{K}(z))^{-1}(\mathbf{I} - \mathbf{Q}_s^{-1}z)(\mathbf{I} - \mathbf{Q}_s^{-1})^{-1} = s(s\mathbf{I} - \mathbf{K}(z))^{-1}[\mathbf{I} + (1-z)(\mathbf{Q}_s - \mathbf{I})^{-1}], \quad (32)$$

$$\mathbf{p}_-(s) = s\mathbf{\Lambda}_1^{-1}(\mathbf{Q}_s - \mathbf{I})^{-1}. \quad (33)$$

If $0 < m_1^0 < \infty$, and according to (30) moment generating function of absolute minimum ξ^- looks as

$$\mathbf{g}_-(z) = \lim_{s\to 0} \mathbf{g}_-(s,z) = m_1^0[\mathbf{\Lambda}_1 - \mathbf{Q}(1-z)^{-1} - (\mathbf{\Lambda}_2\widetilde{\mathbf{F}}_2(z) + \mathbf{N}\widetilde{\mathbf{F}}(z))z^{-1}]^{-1}\mathbf{P}_0, \quad (34)$$

$$\mathbf{g}_-(1) = \frac{1}{\widehat{\mu}_*^+ m_1^0}\mathbf{P}_0\widehat{\mathbf{\Pi}}_*\mathbf{S} = \frac{1}{\widehat{\mu}_*^+ m_1^0}\mathbf{P}_0\mathbf{\Pi}_* = = \frac{1}{\widehat{\mu}_*^+ m_1^0}\mathbf{\Pi}_* = \mathbf{P}_0, \quad (35)$$

$$\widehat{\mathbf{\Pi}}_*\mathbf{S} = \mathbf{\Pi}_* = \mathbf{P}_0, \widehat{\mu}_*^+ = \frac{1}{m_1^0}, \quad (36)$$

$$\mathbf{p}_- = \mathbf{P}\{\xi^- = 0\} = m_1^0\mathbf{\Lambda}_1^{-1}\mathbf{P}_0. \quad (37)$$

Proof. From the second equality in (4), taking into account (2), (3) and (21) we get the first relation in(32)
$\mathbf{g}_-(s,z) = s(s\mathbf{I}-\mathbf{K}(z))^{-1}(\mathbf{I}-\mathbf{Q}_s^{-1}z)(\mathbf{I}-\mathbf{Q}_s^{-1})^{-1}$.
In order to obtain the second relation in (32) the last two multipliers $(\mathbf{I}-\mathbf{Q}_s^{-1}z)(\mathbf{I}-\mathbf{Q}_s^{-1})^{-1}$ are reduced to

$$(\mathbf{I}-\mathbf{Q}_s^{-1}z)(\mathbf{I}-\mathbf{Q}_s^{-1})^{-1} = (\mathbf{Q}_s-\mathbf{I}z)(\mathbf{Q}_s-\mathbf{I})^{-1} = (\mathbf{Q}_s-\mathbf{I})(\mathbf{Q}_s-\mathbf{I})^{-1} + (1-z)(\mathbf{Q}_s-\mathbf{I})^{-1} = [\mathbf{I}+(1-z)(\mathbf{Q}_s-\mathbf{I})^{-1}].$$

To receive (33) proceed to limit in the first relation (32) for $z\to\infty$, pre-substituting $\mathbf{K}(z)$ from (31).

M. g. f. of absolute minimum in accordance with (27) is determined from (32) for $s\to 0$ and $0<m_1^0<\infty$ after accounting the second relation in (30) and (31)

$$\mathbf{g}_-(z) = \lim_{s\to 0}\mathbf{g}_-(s,z) = \lim_{s\to 0} s(s\mathbf{I}-\mathbf{K}(z))^{-1}(\mathbf{I}-\mathbf{Q}_s^{-1}z)(\mathbf{I}-\mathbf{Q}_s^{-1})^{-1} = (-\mathbf{K}(z))^{-1}(\mathbf{I}-\widehat{\mathbf{P}}_*\mathbf{S}z)\frac{1}{\widehat{\mu}_*^+}\widehat{\mathbf{\Pi}}_*\mathbf{S} = \frac{1}{\widehat{\mu}_*^+}[\mathbf{\Lambda}_1-\mathbf{Q}(1-z)^{-1}-(\mathbf{\Lambda}_2\widetilde{\mathbf{F}}_2(z)+\mathbf{N}\widetilde{\mathbf{F}}(z))z^{-1}]^{-1}\widehat{\mathbf{\Pi}}_*\mathbf{S}.$$

The first equality in (35) we receive from $\mathbf{g}_-(z)$ after limited transition at $z\to 1$ and taking into account conditions (5) and (15). Since $\widehat{\mathbf{\Pi}}_*\mathbf{S} = \mathbf{\Pi}_*$, then

$$\mathbf{g}_-(1) = \frac{1}{\widehat{\mu}_*^+ m_1^0}\mathbf{P}_0\mathbf{\Pi}_* = \frac{1}{\widehat{\mu}_*^+ m_1^0}\mathbf{\Pi}_*.$$

By carrying out the limiting transition in (32) for $z\to 1$ and $s\to 0$ in the general case, we get $\mathbf{g}_-(1) = \mathrm{P}_0$. Thus, $\mathbf{\Pi}_* = \mathbf{P}_0$, and $\widehat{\mu}_*^+ m_1^0 = 1$. So the relations (35) and (36) are proved.

Taking into account the conditions (36) in the last relation for $\mathbf{g}_-(z)$ we get (34).

The value $\mathbf{p}_-$ in (37) is determined from (34) for $z\to\infty$. So the theorem is proved.

Almost similarly the following theorem is established.

Theorem 3. *If $\xi(t)$ is upper semi-continuous process on Markov chain $x(t)$ and $0<m_1^0<\infty$, then for $|z|\geqslant 1$ moment generating function $\overline{\xi}(\theta_s)$ satisfies the following relation*

$$\mathbf{g}^-(s,z) = (\mathbf{I}-\mathbf{Z}_s^{-1})^{-1}(\mathbf{I}-\mathbf{Z}_s^{-1}z)s(s\mathbf{I}-\mathbf{K}(z))^{-1} = = [\mathbf{I}+(1-z)(\mathbf{Z}_s-\mathbf{I})^{-1}]s(s\mathbf{I}-\mathbf{K}(z))^{-1}, \quad (38)$$

$$\mathbf{p}^-(s) = (\mathbf{Z}_s-\mathbf{I})^{-1}s\mathbf{\Lambda}_1^{-1}. \quad (39)$$

If $0<m_1^0<\infty$, then according to (29) moment generating function $\overline{\xi} = \lim_{s\to 0}(\overline{\xi}(\theta_s) - \xi^+(\theta_s))$ is defined by limit relation for $s\to 0$

$$\mathbf{g}^-(z) = \lim_{s\to 0}\mathbf{g}^-(s,z) = m_1^0\mathbf{P}_0[\mathbf{\Lambda}_1-\mathbf{Q}(1-z)^{-1} - (\mathbf{\Lambda}_2\widetilde{\mathbf{F}}_2(z)+\mathbf{N}\widetilde{\mathbf{F}}(z))z^{-1}]^{-1}, \quad (40)$$

$$\mathbf{g}^-(1) = \frac{1}{\mu_*^+}\mathbf{\Pi}_*\frac{1}{m_1^0}\mathbf{P}_0 = \frac{1}{\mu_*^+ m_1^0}\mathbf{P}_0 = \mathbf{P}_0, \quad (41)$$

$$\mu_*^+ = \frac{1}{m_1^0}, \quad (42)$$

$$\mathbf{p}^- = \mathbf{P}\{\overline{\xi} = 0\} = m_1^0\mathbf{P}_0\mathbf{\Lambda}_1^{-1}. \quad (43)$$

Proof. From the first equality in (4) we receive $\mathbf{g}^-(s,z) = \mathbf{P}_s((\mathbf{g}_+(s,z))^{-1}\mathbf{g}(s,z)$. Considering (2), (3) and (20) we rewrite $\mathbf{g}^-(s,z)$ as
$\mathbf{g}^-(s,z) = (\mathbf{I}-\mathbf{Z}_s^{-1})^{-1}(\mathbf{I}-\mathbf{Z}_s^{-1}z)s(s\mathbf{I}-\mathbf{K}(z))^{-1}$.
So the first relation in (38)is proved. In order to get the second relation in (38) the first two multipliers $(\mathbf{I}-\mathbf{Z}_s^{-1})^{-1}(\mathbf{I}-\mathbf{Z}_s^{-1}z)$ will be transformed similarly as in the previous theorem. Thus the second relation in (38) is also proved.

To prove (39) proceed to limit in the first relation (38) for $z\to\infty$, and , pre-substituting $\mathbf{K}(z)$ from (31).

$$\mathbf{g}^-(s,z) = (\mathbf{I}-\mathbf{Z}_s^{-1})^{-1}(\mathbf{I}-\mathbf{Z}_s^{-1}z)s(s\mathbf{I}-\mathbf{K}(z))^{-1} = = (\mathbf{I}-\mathbf{Z}_s^{-1})^{-1}(\mathbf{I}-\mathbf{Z}_s^{-1}z)s(s\mathbf{I}+ [\mathbf{\Lambda}_1 - z^{-1}(\mathbf{\Lambda}_2\widetilde{\mathbf{F}}_2(z)-\mathbf{N}\widetilde{\mathbf{F}}(z))]-\mathbf{Q})^{-1} \longrightarrow_{z\to\infty} s(\mathbf{Z}_s-\mathbf{I})^{-1}\mathbf{\Lambda}_1^{-1} = \mathbf{p}^-(s)$$

M. g. f. $\mathbf{g}^-(z)$ according to (26) is defined from (38) for $s\to 0$ and $0<m_1^0<\infty$ after accounting of second relation in (29) and (31).

$$\mathbf{g}^-(z) = \lim_{s\to 0}\mathbf{g}^-(s,z) =$$

$$\lim_{s\to 0}(\mathbf{I}-\mathbf{Z}_s^{-1})^{-1}(\mathbf{I}-\mathbf{Z}_s^{-1}z)s(s\mathbf{I}-\mathbf{K}(z))^{-1} = \frac{1}{\mu_*^+}\mathbf{\Pi}_*(\mathbf{I}-\mathbf{P}_*z)(-\mathbf{K}(z))^{-1} =$$

$$\frac{1}{\mu_*^+}\mathbf{\Pi}_*[\mathbf{\Lambda}_1-\mathbf{Q}(1-z)^{-1}-(\mathbf{\Lambda}_2\widetilde{\mathbf{F}}_2(z)+\mathbf{N}\widetilde{\mathbf{F}}(z))z^{-1}]^{-1}.$$

First equality in (41) we obtain from the previous relation after the limit passage for $z \to 1$ and considering (5) and (15). From the obvious relation $\mathbf{\Pi}_*\mathbf{P}_0 = \mathbf{P}_0$ we receive the second expression in (41). To deduce the last expression in (41) we will accomplish the limit passage transition in (38) at $z \to 1$, taking into account the condition $\mathbf{K}(1) = \mathbf{Q}$, and then, proceeding to limit for $s \to 0$, we'll obtain the necessary. Equating these three expressions we receive (42).

Validity (40) follows from the previous relation for $\mathbf{g}^-(z)$ after consideration of the conditions (42) and (36).

For $z \to \infty$ from (40) receive (43), previously taking into account the conditions (36) and (42)

$$\mathbf{p}^- = \lim_{z\to\infty}\frac{1}{\mu_*^+}\mathbf{\Pi}_*[\mathbf{\Lambda}_1- \mathbf{Q}(1-z)^{-1}-(\mathbf{\Lambda}_2\widetilde{\mathbf{F}}_2(z)+\mathbf{N}\widetilde{\mathbf{F}}(z))z^{-1}]^{-1} = \frac{1}{\mu_*^+}\mathbf{\Pi}_*\mathbf{\Lambda}_1^{-1}.$$

The theorem is proved.

3 Conclusions

In paper the limiting relations (34) and (40). were obtained. They are appropriate matrix analogues of Polyaczek-Khinchin formula that in the scalar case ($P\{\xi^- = \overline{\xi}\} = 1$, $\mu_*^+ = \int\limits_0^{+\infty} P\{\tau^+(0) > t\}\,dt$), for upper semi-continuous lattice process looks

$$Ez^{\xi^-} = \frac{1}{\mu_*^+}(\lambda_1 - \lambda_2\widetilde{\widetilde{F}}_2(z))^{-1},$$

$$\widetilde{\widetilde{F}}_2(z) = z^{-1}\widetilde{F}_2(z),\ \widetilde{\widetilde{F}}_2(z) = \sum_{z\leqslant k} z^k p_k,\ (k \geqslant 0)$$

$$\frac{1}{\mu_*^+} = \lambda_1 - \lambda_2\widetilde{F}_2(1) = m_1.$$

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Received: 19.10.2012