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SEMICLASSICAL APPROXIMATION IN THE RELATIVISTIC POTENTIAL MODEL OF B AND D MESONS

V. Yu. Lazur,* A. K. Reity,* and V. V. Rubish*

We construct a relativistic potential quark model of D, D_s, B, and B_s mesons in which the light quark motion is described by the Dirac equation with a scalar–vector interaction and the heavy quark is considered a local source of the gluon field. The effective interquark interaction is described by a combination of the perturbative one-gluon exchange potential $V_{\text{Coul}}(r)$ and the long-range Lorentz-scalar and Lorentz-vector linear potentials $S_{\text{l.r.}}(r)$ and $V_{\text{l.r.}}(r)$. In the semiclassical approximation, we obtain simple asymptotic formulas for the energy and mass spectra and for the mean radii of D, D_s, B, and B_s mesons, which ensure a high accuracy of calculations even for states with the radial quantum number $n_r \sim 1$. We show that the fine structure of P-wave states in heavy–light mesons is primarily sensitive to the choice of two parameters: the strong-coupling constant α_s and the coefficient λ of mixing of the long-range scalar and vector potentials $S_{\text{l.r.}}(r)$ and $V_{\text{l.r.}}(r)$.

Keywords: Dirac equation, Lorentz structure of interaction potentials, heavy–light quark–antiquark system

1. Introduction

As demonstrated by numerous experiments, the majority of presently known particles have an internal structure, i.e., are composite objects. First, this pertains to hadrons, which, according to contemporary ideas, are composite states of colored quarks and gluons. Describing the mass spectra and decay probabilities of composite objects requires constructing a consistent theory of bound states, which should be based on the fundamental principles of local quantum field theory and use its apparatus [1]. But calculating these characteristics of composite systems directly in the local quantum field theory is not always possible, because the only known calculation method in this theory is still based on the perturbation theory, while the nature of creating a bound state of interacting particles must undoubtedly be determined by nonperturbative effects.

The most effective calculation method beyond the perturbation theory for constructing the theory of bound states is to use the dynamical equations. The point is that even if we can construct kernels of dynamical equations only in the lower orders of the perturbation theory, developing methods for solving them exactly or approximately (but without using the perturbation theory) allows taking nonperturbative effects of interaction into account when evaluating observable characteristics of the bound states. In a nonrelativistic case, such a theory is formulated in the language of the classical potential using the dynamical Schrödinger equation. But at large bond energies, the corresponding theory must be essentially relativistic. In this regard, the way to solve this problem was indicated about half a century ago based on using the dynamical equations in the local quantum field theory, examples of which are the Bethe–Salpeter equation [2], the quasipotential equation [3], and other equations [4].

*Uzhgorod National University, Uzhgorod, Ukraine, e-mail: lazur@univ.uzhgorod.ua.

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The Dirac equation with a mixed scalar–vector interaction plays an important role in the contemporary development of the relativistic theory of bound states. It is valuable because it provides an adequate mathematical model for a wide circle of problems in hadronic physics in which it is possible to pass consistently from a two-particle problem to the external field approximation. This equation indicates the presence of the spin and spin moment for the quark and antiquark, and the problems of describing fine and superfine structures in the energy spectra of heavy–light ($\mathrm{Q}\bar{\mathrm{q}}$) mesons, which are the QCD analogues of hydrogen-like atoms, arise naturally from this equation. Treating the Dirac equation in the limit of an infinitely heavy quark Q as an equation for a single light antiquark $\bar{\mathrm{q}}$ (similarly to the case of hydrogen-like atoms), we can study several important aspects of the theory of heavy–light quark–antiquark systems, in particular, the relativistic dynamics of the light antiquark $\bar{\mathrm{q}}$ in the external field of the heavy quark Q, the Lorentz structure of the long-range component of the $\mathrm{Q}\bar{\mathrm{q}}$ interaction, the fine structure of the spectrum of heavy–light mesons, and the influence of the spontaneous breaking of chiral symmetry on the spectrum.

The mathematical theory of the Dirac equation with a scalar–vector interaction was developed in [5] (see [6]–[8] for a detailed bibliography). Certain progress was achieved in constructing exact solutions of equations of this type with potentials corresponding to different types of interaction [5]. But in most cases, attempts to construct exact solutions of this equation for more or less realistic potentials encounter difficulties that have not yet been overcome. The known methods for investigating this equation approximately (the perturbation theory in the coupling constant, etc.) do not provide complete knowledge about the behavior of the wave functions and mass spectrum in the most interesting domain of values of the coupling constant for hadronic systems containing one light quark together with one (D and B mesons) or two (twice heavy Ξ and Ω barions) heavy (anti)quarks; relativistic and nonperturbative effects evidently play an important role in such systems. Therefore, when constructing approximate methods for investigating bound states of the Dirac equation, nonperturbative methods, in which the expansion parameter in the potential is not considered small, are especially important. Among these methods, one of the most widely used is the method of asymptotic expansion in the Planck constant $\hbar$, which is called the semiclassical approximation.

The rigorous theory of semiclassical asymptotic expansions including the scattering problem together with spectral problems, was constructed in Maslov's fundamental monograph [9] and subsequent papers [10]. The WKB method for fermions satisfying the Dirac equation with a purely vector interaction (including states lying near the boundary of the lower continuum) was developed in detail in [11]–[13]. Namely the semiclassical methods resulted in the majority of "memorable" results in the known theory of superheavy atoms [14]. The construction of semiclassical solutions of the spinor equation with a scalar–vector interaction was recently reported in [15], [16]. In [16], we used the WKB method to study the behavior of a relativistic spin-1/2 particle in the presence of both the scalar and the vector external fields with potentials of the confining type. For the Cornell model of interquark interaction, we obtained simple asymptotic formulas for the energy and mass spectra and for the mean radii of heavy–light (D, $\mathrm{D_s}$, B, and $\mathrm{B_s}$) mesons. These formulas ensure a high accuracy of calculations even for states with the radial quantum number $n_{\mathrm{r}} \sim 1$.

2. Semiclassical approximation for the Dirac equation with a vector and scalar interaction potential

The problem of describing the motion of a relativistic spin-1/2 particle in a central field composed of scalar and vector external fields after the separation of variables reduces to solving the system of radial Dirac equations ($c = 1$)

$$\begin{aligned} &\hbar\frac{dF}{dr} + \frac{\tilde{k}}{r}F - \big[\big(E - V(r)\big) + \big(m + S(r)\big)\big]G = 0, \\ &\hbar\frac{dG}{dr} - \frac{\tilde{k}}{r}G + \big[\big(E - V(r)\big) - \big(m + S(r)\big)\big]F = 0. \end{aligned} \tag{1}$$

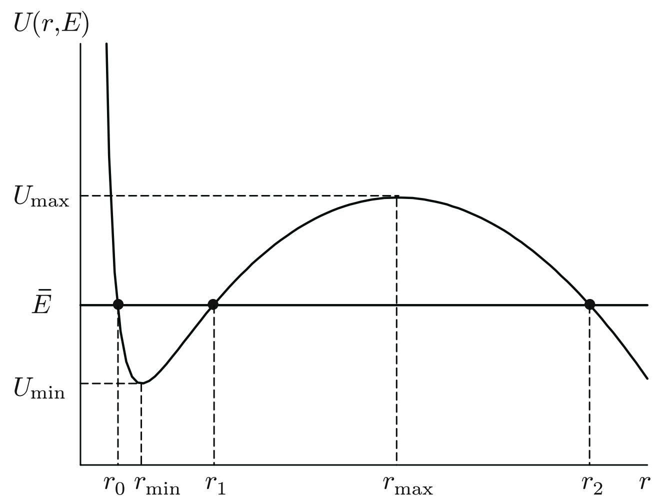

Fig. 1. The form of the EP $U(r,E)$ of the barrier type; r_0, r_1, and r_2 are roots of the equation $p^2(r) = 0$.

Here and hereafter, we use the notation $F(r) = rf(r)$ and $G(r) = rg(r)$, where $f(r)$ and $g(r)$ are the radial functions for the respective upper and lower components of the Dirac bispinor [17], E and m are the total energy and rest mass of the particle, $S(r)$ is the Lorentz-scalar potential, and the potential $V(r)$ up to a multiplier coincides with the zeroth (temporal) component of the four-vector potential $A_\mu = (A_0, \mathbf{A})$, where $\mathbf{A} = 0$, $V(r) = -eA_0(r)$, and $e > 0$. In system (1), $\tilde{k} = \hbar k$, where the quantum number

$$k = \begin{cases} -(l+1) & \text{for } j = l + 1/2 \quad (l = 0, 1, \dots), \\ l & \text{for } j = l - 1/2 \quad (l = 1, 2, \dots), \end{cases}$$

j is the total angular moment of the fermion, and l is the orbital moment (for the upper component of $F(r)$), and hence $|k| = j + 1/2 = 1, 2, \dots$.

The systematic study of the theory of the semiclassical approximation (as $\hbar \to 0$) for the Dirac equation with a scalar–vector interaction was started in [16]. Formal asymptotic expansions in powers of $\hbar$ in initial Dirac system (1) for the radial functions $F(r)$ and $G(r)$ result in a chain of matrix differential equations, which can be solved consecutively using the known technique of left and right eigenvectors of the homogeneous system. For the effective potential (EP) of the barrier type (see Fig. 1)

$$U(r, E) = \frac{E}{m} V + S + \frac{S^2 - V^2}{2m} + \frac{k^2}{2mr^2}, \tag{2}$$

semiclassical expressions were obtained for the wave functions in the classically forbidden and permitted bands and also the quantization condition determining the energy (position) of the bound state E in the mixture of the scalar and vector potentials:

$$\int_{r_0}^{r_1} \left(p + \frac{kw}{pr} \right) dr = \left(n_{\mathrm{r}} + \frac{1}{2} \right) \pi, \qquad w = \frac{1}{2} \left(\frac{V' - S'}{m + S + E - V} - \frac{1}{r} \right). \tag{3}$$

Here, $n_{\mathrm{r}} = 0, 1, 2, \dots$ is the radial quantum number, and

$$p(r) = \left[\big(E - V(r)\big)^2 - \big(m + S(r)\big)^2 - \left(\frac{k}{r} \right)^2 \right]^{1/2} \tag{4}$$

is the semiclassical momentum for the radial motion of the particle in the potential well $r_0 < r < r_1$, where r_0 and r_1 are the turning points, i.e., the roots of the equation $p^2(r) = 0$.

The new quantization rule (3) differs from the standard Bohr–Sommerfeld quantization condition [18] by the relativistic expression for the momentum $p(r)$ and by the correction proportional to w, which takes the spin–orbital interaction into account and results in the splitting of levels with different signs of the quantum number k.

The spectral problem for the Dirac equation with the potentials $S(r)$ and $V(r)$ of the confining type considered in subsequent sections illustrates applying these methods to problems in hadronic physics. Other types of the potentials $S(r)$ and $V(r)$ and also a more detailed mathematical description of the WKB method for the Dirac equation with a scalar–vector interaction can be found in [16].

3. The dependence of the EP $U(r, E)$ on the Lorentz structure of the external field

The simplest model of the interaction of a relativistic spin-1/2 particle simultaneously with both scalar and vector external fields, which we meet below when calculating the semiclassical spectrum of relativistic bound states (see Sec. 4), is governed by the potentials

$$V(r) \equiv V_{\text{Coul}}(r) + V_{\text{l.r.}}(r) = -\frac{\xi}{r} + \lambda v(r), \tag{5a}$$

$$S(r) \equiv S_{\text{l.r.}}(r) = (1-\lambda)v(r), \qquad v(r) = \sigma r + V_0, \tag{5b}$$

where V_0 is a real constant, ξ is the Coulomb coefficient, and λ is the parameter of mixing between the vector and scalar long-range potentials $V_{\text{l.r.}}(r)$ and $S_{\text{l.r.}}(r)$ with $0 \le \lambda \le 1$. Below in this section, we do not restrict the value or even the sign of the parameter σ.

The relation between the EP $U(r, E)$ and initial potentials (5) directly entering the Dirac equation is rather complicated: $U(r, E)$ depends not only on r and model parameters (5) but also on the level energy E and on the total moment j. What is especially important for us here is that the EP $U(r, E)$ takes essentially different forms for $\lambda < 1/2$, $\lambda > 1/2$, and $\lambda = 1/2$.

Our goal is to investigate the behavior of the EP $U(r, E)$ at large and small r. Substituting $V(r)$ and $S(r)$ of form (5) in (2) and keeping only the most singular terms as $r \to 0$ and only the leading terms (in r) as $r \to \infty$, we obtain

$$U(r,E) \sim \begin{cases} \dfrac{(1-2\lambda)\sigma^2}{2m} r^2 + \ldots, & r \to \infty, \quad \lambda \ne \dfrac{1}{2}, & \text{(6a)} \\[2ex] \dfrac{E+m}{2m} \sigma r + \ldots, & r \to \infty, \quad \lambda = \dfrac{1}{2}, & \text{(6b)} \\[2ex] \dfrac{\gamma^2}{2mr^2}, & r \to 0, \quad \gamma^2 = k^2 - \xi^2. & \text{(6c)} \end{cases}$$

We first note that only the quadratic (in S and V) term $(S^2 - V^2)/2m$ is essential in the asymptotic domain in formula (2) for $\lambda \ne 1/2$; this term has the behavior $(1-2\lambda)\sigma^2 r^2/2m$ as $r \to \infty$. It is hence obvious that for any sign of the parameter σ, the EP $U(r, E)$ of model (5) under consideration (at sufficiently large distances) is an attractive potential for $\lambda > 1/2$ and a repulsive potential for $\lambda < 1/2$. Both types of behavior (i.e., attraction for $\lambda > 1/2$ and repulsion for $\lambda < 1/2$) are purely relativistic effects related to the fact that the interaction of the fermion with the scalar external field $S(r)$ is added to the scalar quantity

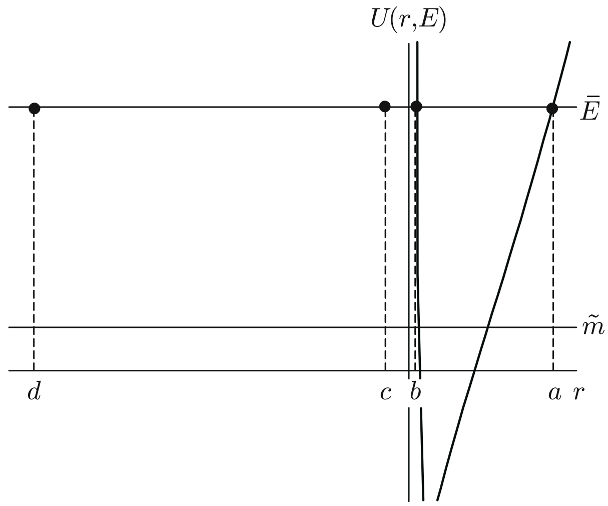

Fig. 2. The EP $U(r,E)$ of Dirac system (1) with potential (9) in the case where $\lambda < 1/2$, $\sigma > 0$, and $\widetilde{E} > \tilde{m}$; a, b, c, and d are the quasimomentum roots in (12).

m, the particle mass, while the vector potential $V(r)$ is introduced into the free Dirac equation minimally as the temporal component of the Lorentz-vector A_μ.

It is clear from what was said above that for $\lambda < 1/2$, the EP $U(r,E)$ of model (5) is an unboundedly increasing (as r increases) confining potential with only a discrete spectrum of energy levels; it is then essential that the quadratic dependence of the EP $U(r,E)$ on r (and hence the confinement property) appears because of the relativistic terms $(S^2 - V^2)/2m$. An example form of the EP $U(r,E)$ for $\lambda < 1/2$ is shown in Fig. 2. It is amazing that bound states are present in composite field (5) under consideration for $\lambda < 1/2$ even in the case where the initial long-range potential $v(r) = \sigma r + V_0$ corresponds to attraction ($\sigma < 0$, $V_0 < 0$).

But for $\lambda > 1/2$ and an arbitrary value of $\sigma \neq 0$, the effective Hamiltonian H of the squared Dirac equation in external field (5) has complex eigenvalues of energy because the EP $U(r,E)$ becomes negative in this case (at sufficiently large distances) and less than the effective particle energy $\overline{E} = (E^2 - m^2)/2m$, which corresponds to attraction. Therefore, for $\lambda > 1/2$, the EP $U(r,E)$ of model (5) has the form of a well separated from the external domain by a wide potential barrier (for $|\sigma| \ll 1$; see Fig. 1). It is obvious that the leading contribution to forming the barrier of the EP $U(r,E)$ comes from the Lorentz-vector component $V_{\text{l.r.}}(r)$ of the long-range potential $v(r)$. Furthermore, as follows from (2) and (6a), in the presence of only a vector field ($\lambda = 1$), the EP $U(r,E)$ does not have the confining property even when the initial long-range potential $v(r) = \sigma r + V_0$ corresponds either to attraction ($\sigma < 0$, $V_0 < 0$) or to repulsion ($\sigma > 0$, $V_0 > 0$). This is the principal difference between relativistic potential model (5) under consideration and the analogous nonrelativistic model in which the EP $U_{\text{eff}}^{\text{n.r.}}(r) = -\xi/r + \sigma r + V_0 + l(l+1)/2r^2$ in the radial Schrödinger equation has the barrier for negative values of the parameters σ and V_0, which results in quasistationary states with complex energies appearing instead of discrete levels. On the contrary, if $\sigma > 0$, then the EP $U_{\text{eff}}^{\text{n.r.}}(r)$ becomes an unboundedly increasing confining potential with only the discrete spectrum of energy levels. The absence of bound states in the Dirac equation with a linearly increasing vector potential $V(r)$ was first noted in [19].

The semiclassical formulas for the wave functions in the domain $r > r_2$ [16] imply the asymptotic form of the radial functions $F(r)$ and $G(r)$ as $r \to \infty$. It then happens that the wave functions decrease exponentially at large distances for $0 \leq \lambda < 1/2$ and oscillate if $1/2 < \lambda \leq 1$. As an illustration, we present this asymptotic behavior for the radial function corresponding to the upper component of the Dirac bispinor

$(r \to \infty)$:

$$F \sim \begin{cases} \exp\left(-\dfrac{\sqrt{1-2\lambda}|\sigma|}{2} r^2\right) & \text{for } 0 \le \lambda < 1/2, \\ \exp\left(i \dfrac{\sqrt{2\lambda - 1}|\sigma|}{2} r^2\right) & \text{for } 1/2 < \lambda \le 1. \end{cases} \tag{7}$$

It hence follows that the relativistic solutions for potential model (5) (depending on the value of the mixing parameter λ) constitute stationary or quasistationary systems satisfying different boundary conditions (7) for $\lambda < 1/2$ or $\lambda > 1/2$.

We point out one more important particular case realized at $\lambda = 1/2$. Substituting potentials (5) with the value $\lambda = 1/2$ in expression (2), we see that the quadratic dependence of the "tail" of $U(r, E)$ on r disappears and the long-range components $V_{\text{l.r.}}(r)$ and $S_{\text{l.r.}}(r)$ of the first two terms dominate EP (2) at large r, which results in a practically linear dependence of $U(r, E)$ on r (see (6b)). We note that we again obtain a linear confining potential, which has only the discrete spectrum, at positive values of σ, while for negative (sufficiently small) values of σ, the EP $U(r, E)$ of model (5) has a wide barrier. Because of this, level decay by percolation through the potential barrier becomes possible, i.e., the bound level becomes a quasistationary exponentially decaying state with the complex energy $E = E_r - i\Gamma/2$. From the analyticity standpoint, the above behavior of the EP $U(r, E)$ for $\sigma < 0$ and $\sigma > 0$ allows studying how the discrete spectrum continues from the real axis to the complex plane.

Summarizing, we can say that varying one of the parameters of interaction model (5), the coefficient λ of mixing the scalar and vector long-range potentials $S_{\text{l.r.}}(r)$ and $V_{\text{l.r.}}(r)$, in the interval $0 \le \lambda \le 1$, we obtain qualitatively different forms of the EP $U(r, E)$: from the confining potential with only the discrete spectrum for $\lambda < 1/2$ to the potential with the potential barrier and quasistationary energy levels for $\lambda > 1/2$ through the physically important intermediate case $\lambda = 1/2$, where the asymptotic behavior (as $r \to \infty$) of the "tail" of the EP $U(r, E)$ switches from quadratic (6a) to linear (6b) (see above).

For the squared Dirac equation in composite field (5), the form of the EP becomes more complicated: expression (2) for $U(r, E)$ acquires small corrections due to the particle spin and the related spin–orbital interaction. It is clear from the nature of the conclusions about the behavior of the EP $U(r, E)$ for $\lambda < 1/2$, $\lambda > 1/2$, and $\lambda = 1/2$ that the indicated changes of the form of $U(r, E)$ do not change the results qualitatively.

Everything said above remains valid for the spherically symmetric potentials $S(r)$ and $V(r)$ with the powerlike or logarithmic behavior ($v(r) \sim \sigma r^\beta$, $\beta > 0$, or $v(r) \sim g \log r$) of the long-range part $v(r)$ at infinity.

Having clarified the qualitative aspects, we now concentrate on a practical application of the above apparatus of semiclassical asymptotic behavior to heavy–light mesons.

4. Semiclassical description of the energy spectrum of heavy–light quark–antiquark systems

To use the potential approach to describe properties of heavy–light mesons, we must construct the quark–antiquark interaction potential. As is known from QCD, because of the asymptotic freedom property, the Coulomb-type potential of the one-gluon exchange gives the leading contribution at small distances ($r < 0.25\,\text{Fm}$).

As the distance increases, the long-range confining interaction (the confinement), whose actual form has not yet been established in the QCD framework, prevails. The confining potential may have a complicated Lorentz structure. For example, it was shown in [20], [21] that the interaction of the quark–antiquark pair with a fluctuating gluon vacuum field at a finite correlation length results in a linearly increasing potential. The spin-dependent potential obtained with that approach has a structure that is characteristic of scalar confinement. On the other hand, the infrared asymptotic behavior of the gluon propagator of the form $D(\mathbf{k}^2) \sim 1/(\mathbf{k}^2)^2$ was obtained in [22] by analyzing the system of the Schwinger–Dyson equations. Such an asymptotic behavior in the static limit results in a linearly growing vector confining potential. It is therefore most plausible that the confining potential comprises a mixture of vector and scalar parts. Moreover, lattice calculations [23] based on the first principles of QCD support a linear confinement proportional to $r/4\pi\alpha'(0)$ (where $\alpha'(0)$ is the slope of the hadronic Regge trajectory). From the above considerations, we assume that the $\mathrm{Q}\bar{\mathrm{q}}$ interaction is a combination of the following potentials:

a. the one-gluon exchange potential $V_{\mathrm{Coul}}(r) = -\xi/r$, where $\xi = 4/3\alpha_{\mathrm{s}}$, α_{s} is the strong coupling constant

$$\alpha_{\mathrm{s}}(Q) = \frac{12\pi}{(33 - 2N_f)\log(Q^2/\Lambda^2)}, \tag{8}$$

N_f is the number of quark flavors, and $\Lambda = 360\,\mathrm{MeV}$ is the QCD parameter,

b. the long-range linear scalar confining potential $S_{\mathrm{conf}}(r) = (1-\lambda)v(r)$, where $v(r)$ is determined by expression (5b), and

c. the long-range linear vector potential $V_{\mathrm{conf}}(r) = \lambda v(r)$.

The total effective quark–antiquark interaction is then described by a combination of the perturbative one-gluon exchange potential $V_{\mathrm{Coul}}(r)$ and the scalar and vector long-range confining potentials $S_{\mathrm{conf}}(r)$ and $V_{\mathrm{conf}}(r)$,

$$\begin{aligned} V(r) &= V_{\mathrm{Coul}}(r) + V_{\mathrm{conf}}(r) = -\frac{\xi}{r} + \lambda(\sigma r + V_0), \\ S(r) &= S_{\mathrm{conf}}(r) = (1-\lambda)(\sigma r + V_0), \quad 0 \le \lambda < 1/2. \end{aligned} \tag{9}$$

Here, $\sigma = 0.18\,\mathrm{GeV}^2$ is the string tension, V_0 is the constant of the additive shift of the bond energy, and the coefficient λ of mixing between the vector and scalar confining potentials is the adjustable parameter. We can consider that the value of α_{s} is approximately the same for each family of heavy–light mesons and twice-heavy barions and changes in accordance with (8) only when we pass from one family to another.

We cannot solve Dirac system (1) with potentials (9) exactly; we hence use the semiclassical approximation method, which provides a high accuracy even for low-lying quantum numbers in the case of scalar and vector fields of the Coulomb and oscillatory types [16].

Choosing the mixing coefficient in the range $0 \le \lambda < 1/2$ corresponds to the scalar confinement prevailing. In this case, the EP $U(r, E)$ of our model has the form of a standard oscillator well with a single minimum (at the point $r_{\min} \approx \gamma^2/\widetilde{E}\xi$) and no maximums (see Fig. 2). The equation $p^2 = 2m\big(\overline{E} - U(r,E)\big) = 0$ determining the turning points then results in the complete fourth-degree algebraic equation $r^4 + fr^3 + gr^2 + hr + l = 0$ with the coefficients

$$\begin{aligned} f &= \frac{2\big[\widetilde{m}(1-\lambda) + \widetilde{E}\lambda\big]}{(1-2\lambda)\sigma}, \qquad & g &= -\frac{\widetilde{E}^2 - \widetilde{m}^2 - 2\xi\sigma\lambda}{(1-2\lambda)\sigma^2}, \\ h &= -\frac{2\widetilde{E}\xi}{(1-2\lambda)\sigma^2}, & l &= \frac{\gamma^2}{(1-2\lambda)\sigma^2}, \end{aligned} \tag{10}$$

where

$$\widetilde{E} = E - \lambda V_0, \qquad \widetilde{m} = m + (1-\lambda)V_0 \tag{11}$$

are the characteristic parameters with the respective meanings of the "shifted" energy and the "shifted" mass. This equation has four real roots $d < c < b < a$ determined by the equalities

$$
\begin{aligned}
a &= -\frac{f}{4} + \frac{1}{2}(\Xi + \Delta_+), \qquad c = -\frac{f}{4} - \frac{1}{2}(\Xi - \Delta_-), \\
b &= -\frac{f}{4} + \frac{1}{2}(\Xi - \Delta_+), \qquad d = -\frac{f}{4} - \frac{1}{2}(\Xi + \Delta_-).
\end{aligned} \tag{12}
$$

Here, we use the notation

$$
\begin{aligned}
&\Xi = \left[\frac{f^2}{4} - \frac{2g}{3} + \frac{u}{3}\left(\frac{2}{Z}\right)^{1/3} + \frac{1}{3}\left(\frac{Z}{2}\right)^{1/3}\right]^{1/2}, \qquad \Delta_\pm = \sqrt{F \pm \frac{D}{4\Xi}}, \\
&F = \frac{f^2}{2} - \frac{4g}{3} - \frac{u}{3}\left(\frac{2}{Z}\right)^{1/3} - \frac{1}{3}\left(\frac{Z}{2}\right)^{1/3}, \qquad Z = v + \sqrt{-4u^3 + v^2}\,, \\
&D = -f^3 + 4fg - 8h, \qquad v = 2g^3 - 9fgh + 27h^2 + 27f^2 l - 72gl, \\
&u = g^2 - 3fh + 12l.
\end{aligned}
$$

For the potentials under consideration, the semiclassical momentum is determined by equalities (4) and (9). Using formulas (12), we represent it in the form convenient for what follows ($\sigma > 0$ and $\sigma < 0$)

$$
p(r) = |\sigma|\sqrt{1-2\lambda}\,\frac{R(r)}{r} = |\sigma|\sqrt{1-2\lambda}\,\frac{\big[(a-r)(r-b)(r-c)(r-d)\big]^{1/2}}{r}. \tag{13}
$$

We integrate in quantization condition (3) over the classically allowed domain between the two positive turning points $b = r_0 < r_1 = a$, while the other two turning points ($d < c < 0$) are in the nonphysical domain $r < 0$. Using formula (13), we transform quantization integrals (3) into the sum of the integrals

$$
\begin{aligned}
J_1 &= \int_b^a p(r)\,dr = -|\sigma|\sqrt{1-2\lambda}\int_b^a \frac{(r^3 + fr^2 + gr + h + lr^{-1})}{R}\,dr, \\
J_2 &= \int_b^a \frac{kw}{p(r)r}\,dr = -\frac{k}{2|\sigma|\sqrt{1-2\lambda}}\left[\int_b^a \frac{dr}{(r-\lambda_+)R} + \int_b^a \frac{dr}{(r-\lambda_-)R}\right],
\end{aligned} \tag{14}
$$

where we introduce the notation

$$
\lambda_\pm = -\frac{\widetilde{E} + \widetilde{m} \mp \sqrt{(\widetilde{E} + \widetilde{m})^2 - 4\sigma\xi(1-2\lambda)}}{2\sigma(1-2\lambda)}.
$$

Writing condition (3) in terms of J_1 and J_2 is advantageous compared with the initial representation because the integrals contained in J_1 and J_2 can be expressed in terms of complete elliptic integrals.

The particle energy spectrum is determined by quantization condition (3), which, after quantization integrals (14) are evaluated (see the appendix), becomes the transcendental equation

$$
\begin{aligned}
&\frac{-2\sqrt{1-2\lambda}}{\sqrt{(a-c)(b-d)}}\left[\frac{|\sigma|(b-c)^2}{\Re}\left[N_1F(\chi) + N_2E(\chi) + N_3\Pi(\nu,\chi) + N_4\Pi\left(\frac{c\nu}{b},\chi\right)\right] + \right. \\
&\quad \left. + \frac{k}{2(1-2\lambda)|\sigma|}\big[(b-c)\big(N_5\Pi(\nu_+,\chi) + N_6\Pi(\nu_-,\chi)\big) + N_7F(\chi)\big]\right] = \left(n_{\mathrm{r}} + \frac{1}{2}\right)\pi,
\end{aligned} \tag{15}
$$

where $F(\chi)$, $E(\chi)$, and $\Pi(\nu,\chi)$ are the complete elliptic integrals of the respective first, second, and third kind (see formulas (A.1)). The mathematical details of calculating integrals of type (14) can be found in [24], [25], and the expressions for ν, χ, $\nu_\pm$, $\Re$, and N_i ($i=1,\dots,7$) are collected in the appendix because they are rather cumbersome.

Finding an "exact" solution of Eq. (15) in the general case is, of course, impossible, but the situation is simplified with the increase in the energy E or in the approximation of "weak" long-range field (as compared with the Coulomb field). The first case corresponds to the fact that for not too large (i.e., for "intermediate") values of the parameters ξ and σ (namely, for $\sigma \lesssim 0.2\,\mathrm{GeV}^2$ and $0.3 < \xi < 0.8$), the condition $\widetilde{E}^2 \gg \sigma\gamma$ is well satisfied for all possible values $E_{n_r k}$ of the heavy–light meson energy levels, and the second case is realized when the condition $\sigma \ll \xi\widetilde{m}^2$ is satisfied. In the framework of our consideration (i.e., for the physics of heavy–light mesons), only the first case is interesting, while the second case is most often encountered in approximate calculations of those properties of low-lying hadronic states that do not depend directly on the presence or absence of confinement.

A simple and often effective method for deriving asymptotic expansions of integrals of form (14) is to expand a quasimomentum $p(r)$ in a small parameter, the interaction, and integrate the obtained series term by term. We then indicate two special features of this procedure for calculating the integrals J_1 and J_2 containing the small parameter. First, it is obvious from analyzing expressions (12) that in addition to the level $E=m$, we must introduce one more characteristic energy level $\widetilde{E}=\widetilde{m}$, which divides the domains of applicability of the asymptotic approximations for the quantization integrals J_1 and J_2 obtained below. Using the relations $\widetilde{E}>\widetilde{m}$ and $\widetilde{E}<\widetilde{m}$, we can show that in these two domains of the spectrum, the motion is semiclassical if the condition $\sigma/\xi\widetilde{m}^2 \ll 1$ is satisfied for $\widetilde{E}<\widetilde{m}$ and the condition $\sigma\gamma/\widetilde{E}^2 \ll 1$ is satisfied for $\widetilde{E}>\widetilde{m}$. This gives the possibility of obtaining expressions for J_1 and J_2 in elementary functions using the formal expansion of the quasimomentum in a power series in a small dimensionless parameter (which is $\sigma\gamma/\widetilde{E}^2 \ll 1$ or $\sigma/\xi\widetilde{m}^2 \ll 1$). Second, the further analysis depends essentially on the mutual positions of the turning points a, b, c, and d. Then, depending on the relative values of $\widetilde{E}$ and the level $\widetilde{m}$, we consider several typical situations.

Case A: Let $\sigma>0$ and the conditions $\sigma \ll \xi\widetilde{m}^2$ and $\widetilde{E}<\widetilde{m}$ be satisfied. This situation describes deep levels whose energy is close to the bottom of the scalar–vector well $U(r,E)$. Estimating expressions (12) for the turning points in the approximation $\sigma/\xi\widetilde{m}^2 \ll 1$ and preserving only the two first terms in the small parameter expansion, we can easily obtain

$$\begin{aligned}
a &\approx \frac{\widetilde{E}\xi+\theta}{\mu^2}\left[1-\frac{\widetilde{E}\xi+\theta}{\mu^4}\left(\eta_1+\frac{\widetilde{m}\xi\eta_2}{\mu}\right)\sigma\right],\\
b &\approx \frac{\widetilde{E}\xi-\theta}{\mu^2}\left[1-\frac{\widetilde{E}\xi-\theta}{\mu^4}\left(\eta_1-\frac{\widetilde{m}\xi\eta_2}{\mu}\right)\sigma\right],\\
c &\approx -\frac{\widetilde{m}-\widetilde{E}}{\sigma}-\frac{\xi}{\widetilde{m}-\widetilde{E}},\qquad d \approx -\frac{\widetilde{m}+\widetilde{E}}{\sigma(1-2\lambda)}+\frac{\xi}{\widetilde{m}+\widetilde{E}}.
\end{aligned}\tag{16}$$

Here and hereafter, we use the notation

$$\begin{aligned}
\theta &= \sqrt{(\widetilde{E}k)^2-(\widetilde{m}\gamma)^2}\,,\qquad \mu=\sqrt{\widetilde{m}^2-\widetilde{E}^2}\,,\\
\eta_1 &= (1-\lambda)\widetilde{m}+\lambda\widetilde{E},\qquad \eta_2=\lambda\widetilde{m}+(1-\lambda)\widetilde{E}.
\end{aligned}\tag{17}$$

It follows from asymptotic expressions (16) that the positive turning points a and b depend weakly on σ and are determined only by the Coulomb field. The other two (negative) turning points c and d

depend mainly on the linear part $v(r)$ of interaction (9), but their values are "corrected" by the quantities $\mp\xi/(\widetilde{m}\mp\widetilde{E})$, which are due to the Coulomb long-range interaction. It is also obvious from (16) that for small positive values of σ, the turning points c and d are sufficiently far from the two points a and b and tend to $-\infty$ in the limit as $\sigma\to 0$.

The properties of deeply lying levels for massive quarks ($\widetilde{m}^2\xi \gg \sigma$) are mainly determined by the Coulomb potential. Treating the long-range potential $v(r)$ as a small perturbation, we can expand the semiclassical momentum $p(r)$ in the domain of the potential well $b<r<a$ in a series in increasing powers of $r/|c|\ll 1$ and $r/|d|\ll 1$. Calculating the table integrals in (3), whose sum gives the value of the quantization integrals J_1 and J_2 up to terms of the order $O\big((\sigma/\xi\widetilde{m}^2)^2\big)$, we then obtain the equation, which can be easily solved for the level energies,

$$E_{n_r k}=\widetilde{E}_0+\lambda V_0+\frac{\sigma}{2\xi\widetilde{m}^2}\left[\left(\frac{\xi^2\widetilde{m}^2}{\mu_0^2}-k^2\right)\eta_{10}+\left(\frac{2\xi^2\widetilde{m}\widetilde{E}_0}{\mu_0^2}-k\right)\eta_{20}\right]+O\left(\left(\frac{\sigma}{\xi\widetilde{m}^2}\right)^2\right), \tag{18}$$

where $\widetilde{E}_0=\widetilde{m}l[1+\xi^2/(n_\mathrm{r}'+\gamma)^2]^{-1/2}$ is the Dirac level of the energy of the fermion (with the effective mass $\widetilde{m}=m+(1-\lambda)V_0$) in the Coulomb field, $n_\mathrm{r}'=n_\mathrm{r}+(1+\operatorname{sgn}k)/2$, and the quantities μ_0, η_{10}, and η_{20} are obtained from μ, η_1, and η_2 by substituting $\widetilde{E}_0$ for $\widetilde{E}$. The previously accepted condition $\sigma>0$ is unnecessary here because this result remains applicable also in the case of negative values of the parameter σ.

Formula (18) can also be found using the standard perturbation theory, but this involves rather cumbersome calculations. Using semiclassical formulas (3) and (14) dramatically simplifies calculations. As is shown by comparing with the result obtained by numerically integrating Eq. (1), formula (18) ensures a good accuracy for calculating the spectra of bound systems of heavy quarks (for example, $Q\overline{Q}$ mesons; see [26]).

Calculations, which we omit here, demonstrate that in the case $\lambda<1/2$ and for (sufficiently small) negative values of σ, the EP $U(r,E)$ has the shape of a double well. If we neglect the barrier penetrability in the region $c<r<b$ between the two wells, then the semiclassical quantization conditions in this well can be written merely as the conditions on the phase integrals over the domain of the semiclassical motion in each of the wells. Quantizing in the left well by formula (3) then results in formula (18) above.

Case B: In the domain $\widetilde{E}>\widetilde{m}$ and $\sigma>0$, which is of actual importance for the physics of heavy–light mesons, a small dimensionless parameter $\sigma\gamma/\widetilde{E}^2$ appears in the spectral problem. Imposing the condition $\sigma\gamma/\widetilde{E}^2\ll 1$, we can easily obtain the approximate expressions for the turning points from exact formulas (12):

$$\begin{aligned} a&\approx\frac{\widetilde{E}-\widetilde{m}}{\sigma}+\frac{\xi}{\widetilde{E}-\widetilde{m}}, & b&\approx\frac{-\widetilde{E}\xi+\theta}{\widetilde{E}^2-\widetilde{m}^2},\\ c&\approx\frac{-\widetilde{E}\xi-\theta}{\widetilde{E}^2-\widetilde{m}^2}, & d&\approx-\frac{\widetilde{E}+\widetilde{m}}{\sigma(1-2\lambda)}+\frac{\xi}{\widetilde{E}+\widetilde{m}}. \end{aligned} \tag{19}$$

As can be seen from these formulas, the turning points a and b are rather distant from each other, and the above expansion for the quasimomentum $p(r)$ is not applicable in the whole integration domain. Nevertheless, using the condition $\sigma\gamma/\widetilde{E}^2\ll 1$, we can use the approximation method to evaluate the quantization integrals based on the idea of splitting the whole integration domain $[b,a]$ into the intervals $[b,\tilde{r}]$ and $[\tilde{r},a]$ in each of which only the dominating interaction type is taken into account exactly while the other integration types are treated as perturbations.

We now find a point $\tilde{r}$ that divides the integration domain $b\le r\le a$ into the domain $b\le r\le\tilde{r}$ where the Coulomb potential prevails and the domain $\tilde{r}\le r\le a$ where the long-range potential $v(r)$ prevails.

The method for choosing such a point is not unique. The most natural seems to find a point $\tilde{r}$ where the long-range potential $v(r)$ is equal to the Coulomb potential. From this requirement, we have $\tilde{r} \approx \sqrt{\widetilde{E}\xi/\eta_1\sigma}$.

We can calculate the quantization integrals (for $\sigma\gamma/\widetilde{E}^2 \ll 1$) as follows. We calculate integrals (14) by expanding the quasimomentum $p(r)$ in a power series in the parameters $r/a \ll 1$ and $r/|d| \ll 1$ in the domain $b \le r \le \tilde{r}$ and in the small parameters $b/r \ll 1$ and $|c|/r \ll 1$ in the domain $\tilde{r} \le r \le a$. Splitting the integration interval at the point $\tilde{r} \approx \sqrt{\widetilde{E}\xi/\eta_1\sigma}$ therefore gives the representation for J_1,

$$J_1 = \sigma\sqrt{1-2\lambda}\,(j_1 + j_2), \tag{20}$$

where the integrals j_1 and j_2 can be written as follows up to terms of the first order in the corresponding small parameters r/a, $r/|d|$ and b/r, $|c|/r$ in the expansions for the quasimomentum $p(r)$:

$$\begin{aligned} j_1 &= \sqrt{-ad}\int_b^{\tilde{r}} \frac{\sqrt{(r-b)(r-c)}}{r}\left[1 - \frac{a+d}{2ad}r + \cdots\right] dr, \\ j_2 &= \int_{\tilde{r}}^a \sqrt{(a-r)(r-d)}\left[1 - \frac{b+c}{2r} + \cdots\right] dr. \end{aligned} \tag{21}$$

Calculating the table integrals in (21) and collecting terms with like dependence on σ, we obtain

$$\begin{aligned} J_1 = \sigma\sqrt{-ad(1-2\lambda)}\,\Bigg[&\frac{b+c}{2}\log\left(\frac{(a-d)(c-b)}{16ad}\right) - \sqrt{-bc}\,\arccos\left(\frac{b+c}{b-c}\right) + \\ &+ \frac{a+d}{4} + \frac{1}{4\sqrt{-ad}}\left(\frac{(a-d)^2}{2} - (a+d)(b+c)\right)\arccos\left(\frac{d+a}{d-a}\right)\Bigg] + O\left(\frac{\sigma\gamma}{\widetilde{E}^2}\right). \end{aligned} \tag{22}$$

We note that when the asymptotic expressions for j_1 and j_2 are added, the result does not contain the parameter $\tilde{r}$.

To expand the integral J_2 in the small parameter $\sigma\gamma/\widetilde{E}^2$, we represent it as a sum of two terms,

$$J_2 = -\frac{k}{2\sigma\sqrt{1-2\lambda}}(\tilde{j}_1 + \tilde{j}_2), \tag{23}$$

where the integrals $\tilde{j}_1$ and $\tilde{j}_2$ can be written in the forms

$$\begin{aligned} \tilde{j}_1 &\approx \frac{1}{\sqrt{-ad}}\int_b^{\tilde{r}} \frac{dr}{(r+\tilde{p})\sqrt{(r-b)(r-c)}}, & \tilde{p} &= \frac{\xi}{\widetilde{E}+\widetilde{m}}, \\ \tilde{j}_2 &\approx \int_{\tilde{r}}^a \frac{1}{\sqrt{(a-r)(r-d)}}\left[\frac{1}{r^2} + \frac{1}{r(r+\tilde{q})}\right] dr, & \tilde{q} &= \frac{\widetilde{E}+\widetilde{m}}{\sigma(1-2\lambda)}. \end{aligned} \tag{24}$$

An elementary calculation of the integrals in (24) results in

$$J_2 = -\frac{k}{2|\sigma|\sqrt{1-2\lambda}}\frac{\arccos\big((b+c+2\xi/(\widetilde{E}+\widetilde{m}))/(b-c)\big)}{\sqrt{ad\big(b+\xi/(\widetilde{E}+\widetilde{m})\big)\big(c+\xi/(\widetilde{E}+\widetilde{m})\big)}} + O\left(\frac{\sigma\gamma}{\widetilde{E}^2}\right). \tag{25}$$

Adding expansions (22) and (25) and combining terms of like orders in σ, we obtain the transcendental equation determining the energy spectrum from (3),

$$\begin{aligned} &\frac{\eta_1\sqrt{\widetilde{E}^2-\widetilde{m}^2}}{2\sigma(2\lambda-1)} - \eta\left(\frac{\eta_2^2}{2\sigma(2\lambda-1)} + \lambda\xi\right) - \gamma\arccos\left(\frac{-\widetilde{E}\xi}{\theta}\right) - \\ &\quad - \frac{\widetilde{E}\xi}{\sqrt{\widetilde{E}^2-\widetilde{m}^2}}\log\left(\frac{\sigma\eta_2\theta}{4e(\widetilde{E}^2-\widetilde{m}^2)^2}\right) - \frac{\operatorname{sgn}k}{2}\arccos\left(\frac{-\widetilde{m}\xi}{\theta}\right) = \left(n_{\mathrm{r}} + \frac{1}{2}\right)\pi, \end{aligned} \tag{26}$$

where

$$\eta = (1 - 2\lambda)^{-1/2} \arccos(\eta_1/\eta_2). \tag{27}$$

Although Eq. (26) is much simpler than "exact" semiclassical equation (15) for the energy levels, solving it still requires numerical calculations. Below, we consider several limiting cases where Eq. (26) is simplified and can be investigated analytically.

For the parameter values $\sigma \lesssim 0.2\,\mathrm{GeV}^2$ and $0.3 < \xi < 0.8$, the condition $\widetilde{E} \gg \widetilde{m}$ is well satisfied for all possible values of the level energies $E_{n_\mathrm{r}k}$ of heavy–light mesons. If we expand the left-hand side of (26) in $\widetilde{m}/\widetilde{E} \ll 1$ up to third-degree terms, we obtain the transcendental equation for $E_{n_\mathrm{r}k}$:

$$\begin{aligned}\big[(1-\lambda)A - \lambda\big]\widetilde{E}^2 + 2\widetilde{m}\widetilde{E}(1-\lambda)(\lambda A - 1) - 2\sigma(1-2\lambda)\left(\pi N + \xi \log \frac{\sigma|k|(1-\lambda)}{4\widetilde{E}^2}\right) - \\ - \lambda\widetilde{m}^2 + \lambda\big[\lambda\widetilde{m}^2 - 2\sigma\xi(1-\lambda)\big]A = 0,\end{aligned} \tag{28}$$

where

$$A = \frac{\arccos\big(\lambda/(1-\lambda)\big)}{\sqrt{1-2\lambda}}, \qquad N = n_\mathrm{r} + \frac{1}{2} + \frac{\operatorname{sgn} k}{4} + \frac{1}{\pi}\left(\gamma \arccos\left(-\frac{\xi}{|k|}\right) - \xi\right). \tag{29}$$

Solving this equation by the method of consecutive iterations, we obtain the desired expression for the eigenvalues $E_{n_\mathrm{r}k}$ in the first approximation (up to terms of the order $O(\sigma\gamma/\widetilde{E}^2)$):

$$\begin{aligned}E_{n_\mathrm{r}k}^{\mathrm{WKB(as)}} = \zeta^{-1}\Bigg\{B + \Bigg(B^2 + \zeta\bigg[2\sigma(1-2\lambda)\bigg(\xi\log\frac{\sigma|k|(1-\lambda)}{4(\widetilde{E}^{(0)})^2} + 3\xi + \lambda\xi A + \pi N\bigg) + \\ + \lambda\widetilde{m}^2(1-\lambda A)\bigg]\Bigg)^{1/2}\Bigg\} + \lambda V_0,\end{aligned} \tag{30}$$

where

$$\zeta = (1-\lambda)^2 A - \lambda - \frac{2\sigma\xi(1-2\lambda)}{(\widetilde{E}^{(0)})^2}, \qquad B = (1-\lambda)(1-\lambda A)\widetilde{m} - \frac{4\sigma\xi(1-2\lambda)}{\widetilde{E}^{(0)}},$$

and $\widetilde{E}^{(0)} = E^{(0)} - \lambda V_0$. Here, $E^{(0)}$ is the zeroth approximation for the energy on which the quantity $E_{n_\mathrm{r}k}$ depends rather weakly, and we can set $E^{(0)} \approx E_{n_\mathrm{r}k}(\xi)\big|_{\xi=0}$ in most cases.

We have obtained formula (30) for the energy levels $E_{n_\mathrm{r}k}$, which depend nonanalytically on the string tension σ and which therefore cannot be obtained in the perturbation theory framework. We mention that for a purely scalar confinement ($\lambda = 0$), formula (30) is simplified to

$$E_{n_\mathrm{r}k}^{\mathrm{WKB(as)}} = \frac{2}{\pi}\left[m + \sqrt{m^2 + \sigma\pi\left(\xi\log\frac{\sigma|k|}{\big(2E^{(0)}\big)^2} + \pi N\right)}\right]. \tag{31}$$

The results of calculating the energy levels $E_{n_\mathrm{r}k}^{\mathrm{WKB}}$ and $E_{n_\mathrm{r}k}^{\mathrm{WKB(as)}}$ based on transcendental equation (15) and asymptotic formula (30) together with the exact values of $E_{n_\mathrm{r}k}$ obtained by solving the Dirac equation numerically are presented in Table 1 for $n_\mathrm{r} = 0, 1, 2$ and $k = \pm1, \pm2$. In these calculations, we set the values of α_s, λ, V_0, $m_{\mathrm{u,d}}$, and m_s to those used in QCD to describe the states of B ($\mathrm{b\bar{u}}$ or $\mathrm{b\bar{d}}$) and $\mathrm{B_s}$ ($\mathrm{b\bar{s}}$) mesons. As can be seen in Table 1, the semiclassical values $E_{n_\mathrm{r}k}^{\mathrm{WKB}}$ and $E_{n_\mathrm{r}k}^{\mathrm{WKB(as)}}$ ensure the respective 1% and 2% accuracies (except the energy of states with the radial quantum number $n_\mathrm{r} = 0$, for which the accuracy of both formulas is about 8%). The accuracy of determining $E_{n_\mathrm{r}k}$ from semiclassical formula (30) is therefore such that the first-order approximation usually suffices for practical purposes.

Table 1

$L_j(n_r, k)$		$b\bar{u}$, $b\bar{d}$: $E_{n_r k}$	$E_{n_r k}^{\mathrm{WKB}}$	$E_{n_r k}^{\mathrm{WKB(as)}}$	$b\bar{s}$: $E_{n_r k}$	$E_{n_r k}^{\mathrm{WKB}}$	$E_{n_r k}^{\mathrm{WKB(as)}}$
$S_{1/2}$	$(0, -1)$	0.4327	0.4408	0.4729	0.5248	0.5322	0.5623
	$(1, -1)$	0.8796	0.8838	0.8943	0.9750	0.9791	0.9912
	$(2, -1)$	1.1978	1.2009	1.2066	1.2946	1.2976	1.3049
$P_{3/2}$	$(0, -2)$	0.7355	0.7373	0.7504	0.8376	0.8392	0.8460
	$(1, -2)$	1.0880	1.0892	1.0947	1.1879	1.1890	1.1927
	$(2, -2)$	1.3658	1.3667	1.3699	1.4650	1.4659	1.4685
$P_{1/2}$	$(0, 1)$	0.7249	0.7293	0.7030	0.8235	0.8278	0.7985
	$(1, 1)$	1.0701	1.0733	1.0594	1.1696	1.1728	1.1572
	$(2, 1)$	1.3470	1.3496	1.3405	1.4466	1.4492	1.4390
$D_{3/2}$	$(0, 2)$	0.9661	0.9671	0.9343	1.0655	1.0665	1.0315
	$(1, 2)$	1.2588	1.2596	1.2385	1.3583	1.3591	1.3369
	$(2, 2)$	1.5058	1.5066	1.4914	1.6052	1.6059	1.5901

The results of calculating the level energies $E_{n_r k}^{\mathrm{WKB}}$ (based on transcendental equation (15)) and $E_{n_r k}^{\mathrm{WKB(as)}}$ (based on semiclassical expression (30)) and also the exact values of $E_{n_r k}$ calculated at the parameter values $\alpha_s = 0.3$, $\lambda = 0.3$, $V_0 = -0.45\,\mathrm{GeV}$, $m_{u,d} = 0.33\,\mathrm{GeV}$, and $m_s = 0.5\,\mathrm{GeV}$ (the energies are measured in GeV).

To find the dependence of the energy eigenvalues $E_{n_r k}$ on the Coulomb coupling constant ξ, we solved transcendental equation (15) numerically with the following choice of parameters determining the form of initial interaction potentials (9): $\alpha_s = 0.3$, $\lambda = 0.3$, $V_0 = -0.45\,\mathrm{GeV}$, and $m_{u,d} = 0.33\,\mathrm{GeV}$. The graphs of dependences of energy levels on the ratio $\xi/|k|$ are shown in Fig. 3, where solid lines indicate the dependence of several lowest levels ($n_r = 0$) with the given value of k and dashed lines correspond to the excited states ($n_r = 1$). As could be expected, as the Coulomb parameter ξ increases, energy levels decrease monotonically and develop a square-root singularity as $\xi \to |k|$. This is a manifestation of the “falling to the center” phenomenon for the Dirac equation in composite field (9) with the vector potential $V(r)$, which has the Coulomb singularity at zero, $V(r) \approx V_{\mathrm{Coul}}(r) = -\xi/r$ as $r \to 0$. As is known [27], every truncation of the potential $V(r)$ at small distances removes the square-root singularity in the energies $E_{n_r k}$, and the curve of the level of $E_{n_r k}(\xi)$ can then be smoothly continued to the domain $E < 0$.

It can be seen from Fig. 3 that for the states with the same n_r, the level energies with $k > 0$ lie much above the levels with $k < 0$. This is the influence of the centrifugal barrier (for instance, this barrier is absent for states with $k = -1$, while it suppresses the probability of the presence of the quark at large distances for states with $k = +1$). These conclusions are completely confirmed by numerically solving Dirac system (1) with potentials (9); the results of this were presented in [27].

We also note that the energies of the lowest levels ($n_r = 0$) with $k < 0$ reach the zero level ($E = 0$) at the maximum possible value of the Coulomb coupling constant $\xi = -k$ (see Figs. 3*a*–3*c*). All other states also have the singularity of the square-root type at $\xi = |k|$, but their energies remain positive.

The above study of the spectrum of Dirac equation in composite field (9) using the WKB approximation is of practical interest because calculating integrals in quantization condition (3) is much easier in many

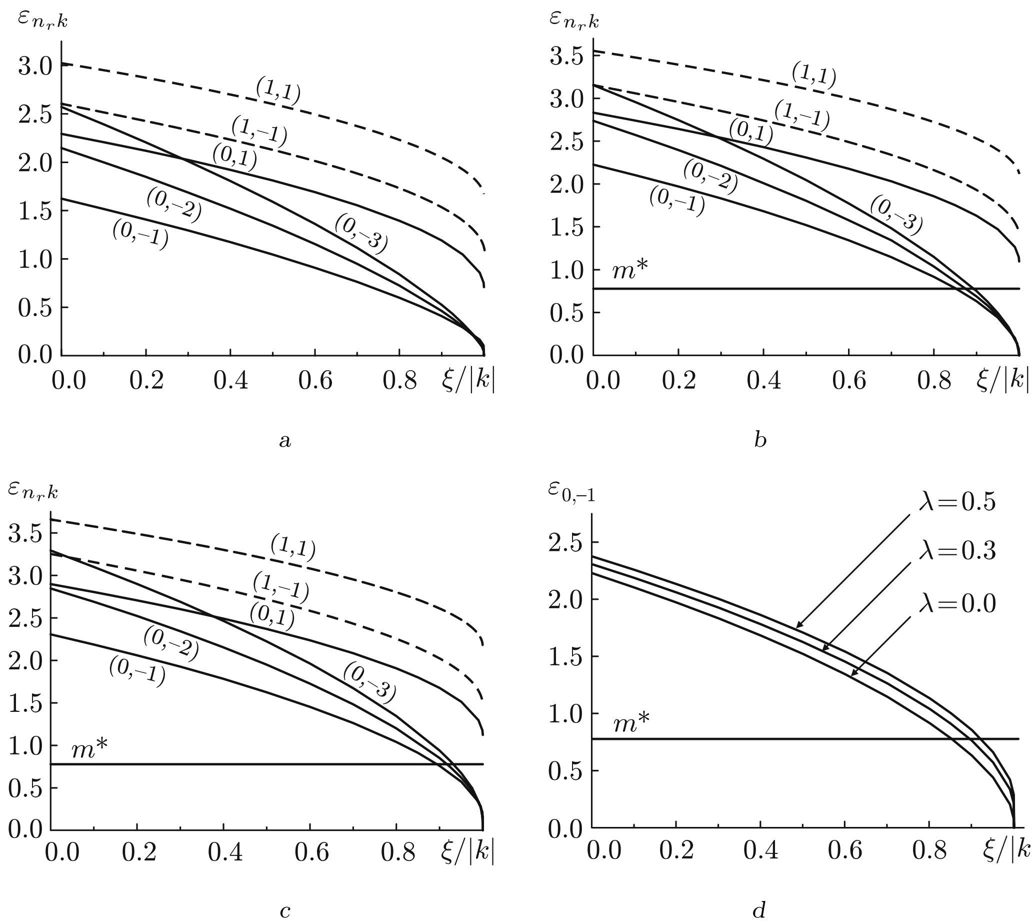

Fig. 3. (*a–c*) The dependence of the level energies $\varepsilon_{n_r k} = E_{n_r k}/\sqrt{\sigma}$ on $\xi/|k|$. Solid lines correspond to the lowest levels ($n_r = 0$) with the given value of k, and dashed lines correspond to the excited states ($n_r = 1$). The parameter values are (*a*) $m = 0$ and $\lambda = 0$, (*b*) $m = 0.33\,\mathrm{GeV}$ and $\lambda = 0$, (*c*) $m = 0.33\,\mathrm{GeV}$, $\sigma = 0.18\,\mathrm{GeV}^2$, and $\lambda = 0.3$. Here, $m^* = m/\sqrt{\sigma}$. (*d*) The dependence of the level energy $\varepsilon_{0,-1}$ on $\xi/|k|$ for different values of the parameter λ at $m = 0.33\,\mathrm{GeV}$.

cases than finding exact values of energy levels by numerically solving system of radial Dirac equations (1).

5. The mass spectrum of heavy–light quark systems

The qualitative picture of forming bound states in a $\mathrm{Q\bar{q}}$ system is determined by the presence of the scale parameter Λ_{QCD} of the confinement of the light antiquark $\bar{\mathrm{q}}$: $\Lambda_{\mathrm{QCD}} \ll m_{\mathrm{Q}}$, where m_{Q} is the mass of the heavy quark Q. Under this condition, the heavy quark Q affects the light quark $\bar{\mathrm{q}}$ as a local static source of the color (gluon) QCD field. The presence of a small parameter $\Lambda_{\mathrm{QCD}}/m_{\mathrm{Q}} \ll 1$ allowed developing powerful means for studying QCD in interactions between heavy and light quarks. For example, a consistent scheme of the effective theory of heavy quarks for hadronic systems with one heavy quark ($\mathrm{Q\bar{q}}$, Qqq) was developed (see, e.g., [6] and the references therein). In the leading term of this theory (i.e., in the static limit as $m_{\mathrm{Q}} \to \infty$), first, the spin of the heavy quark Q splits from the interaction with weakly virtual gluons, second, the effective Hamiltonian exactly corresponds to the Dirac Hamiltonian of one-particle problem (1), and the energy of the spin–orbital interaction of the light antiquark $\bar{\mathrm{q}}$ becomes the leading term of spin interactions. This is manifested in the approximate Isgur–Wise spin symmetry [28] for the heavy quark.

In the leading order in $1/m_{\mathrm{Q}}$, the mass spectrum of meson states with one heavy quark is given by the expression [6], [29]–[31]

$$M_{n_{\mathrm{r}}k}^{\mathrm{theor}}(\mathrm{Q\bar{q}}) = E_{n_{\mathrm{r}}k} + \sqrt{E_{n_{\mathrm{r}}k}^2 - m_{\mathrm{q}}^2 + m_{\mathrm{Q}}^2}\,, \tag{32}$$

where m_{Q} and m_{q} are the masses of the heavy quark Q and the light quark $\bar{\mathrm{q}}$ constituting the $\mathrm{Q\bar{q}}$ meson. Calculating the mass spectrum of $\mathrm{Q\bar{q}}$ mesons therefore reduces to consistently calculating the energy eigenvalues of Dirac equation (1) in composite field (9) whose source here is the heavy quark Q.

The symmetry properties of Dirac equation (1) drastically simplify the problem of classifying states of heavy–light mesons. Because the Hamiltonian of Eq. (1) does not contain terms describing the interaction of the spin of the Q quark with the orbital and spin moments $\vec{l}$ and $\vec{s}_{\mathrm{q}}$ of the light antiquark, both the spin moment $\vec{S}_{\mathrm{Q}}$ of the heavy quark Q and the total moment $\vec{j} = \vec{s}_{\mathrm{q}} + \vec{l}$ of the light antiquark $\bar{\mathrm{q}}$ are two separate integrals of motion. This allows classifying the states by the quantum numbers $j = 1/2, 3/2, \ldots$ of the operator of the total moment of the light antiquark $\bar{\mathrm{q}}$, while the states of the total moment of the composite $\mathrm{Q\bar{q}}$ system $\vec{J} = \vec{j} + \vec{S}_{\mathrm{Q}}$ are degenerate with respect to the orientation of the spin $\vec{S}_{\mathrm{Q}}$ of the heavy quark Q. Two almost degenerate states of the composite $\mathrm{Q\bar{q}}$-system with $J = j \pm 1/2$ in the spin symmetry approximation [28] therefore correspond to each state of the Dirac equation with the given j and with the spatial parity $P = (-1)^{l+1}$. Masses of the j^P-states of the $\mathrm{Q\bar{q}}$ meson are also degenerate with respect to J, and these states therefore have identical wave functions.

The values $l = 0$ (s states in the quark–antiquark model) and $j = 1/2^-$ correspond to the ground state of the $\mathrm{Q\bar{q}}$ meson. This doublet consists of two states $J^P = (0^-, 1^-)$. In the case $l = 1$ (the p state in the quark model), we have two states with $j = 1/2^+$ and $j = 3/2^+$ and two corresponding doublets $J^P = (0^+, 1^+)$ and $J^P = (1^+, 2^+)$.

As usual, we introduce a concise notation for the families of D and $\mathrm{D_s}$ mesons: (D_0^*, D_1') are the components of the charmed doublet $J^P = (0^+, 1^+)$ with $j = 1/2^+$ for nonstrange states (the $\mathrm{c\bar{u}}$ system), ($D_{\mathrm{s}0}^*$, $D_{\mathrm{s}1}'$) are the components of the same doublet for strange states (the $\mathrm{c\bar{s}}$ system), and (D_1, D_2^*) and ($D_{\mathrm{s}1}$, $D_{\mathrm{s}2}^*$) are the components of the doublet $J^P = (1^+, 2^+)$ with $j = 3/2^+$ for the respective nonstrange and strange states. We also use the analogous notation system for B and $\mathrm{B_s}$ families.

Above, we did not take the level superfine structure (SFS) into account, and the proposed potential model can predict only the position of the center of masses of the SFS multiplet comprising sublevels with different moments $\vec{J} = \vec{j} + \vec{S}_{\mathrm{Q}}$. In actual $\mathrm{Q\bar{q}}$ systems, the degeneracy of doublet states corresponding to different moments $J = j \pm 1/2$ at the given j is broken primarily because of the $\vec{s}_{\mathrm{q}}\vec{S}_{\mathrm{Q}}$ interaction. Therefore, to be able to compare our theoretical predictions with experimental data, we present the observation values for the centers of masses of the SFS multiplets in Tables 2–5; these centers of masses are calculated by the known formula

$$M_{\mathrm{exp}} = \frac{\sum_J \big((2J+1)M_J\big)}{\sum_J (2J+1)}, \tag{33}$$

where M_J is the experimental value of the mass of state with the given J.

Based on these observations, we have tried to describe the spectra of masses of low-lying states of the heavy–light B ($\mathrm{b\bar{u}}$ or $\mathrm{b\bar{d}}$), $\mathrm{B_s}$ ($\mathrm{b\bar{s}}$), D ($\mathrm{c\bar{u}}$ or $\mathrm{c\bar{d}}$), and $\mathrm{D_s}$ ($\mathrm{c\bar{s}}$) mesons considering σ and λ to be universal quantities and setting the values of the parameters α_{s} and V_0 constant in every family of heavy–light mesons allowing them to vary slightly only when passing from one family to another. All the parameters σ, λ, α_{s}, and V_0 of potential model (9) were determined by fitting the known data for the mass spectra of pseudoscalar D and B mesons. The found values of the parameters are consequently used below in other applications in the framework of our approach, for example, when describing the spectra of the strange $\mathrm{D_s}$ and $\mathrm{B_s}$ mesons.

We use only one a priori restriction: the value of the coefficient λ of mixing between the vector and scalar long-range potentials $V_{\mathrm{conf}}(r)$ and $S_{\mathrm{conf}}(r)$ must lie in the interval $0 \le \lambda < 1/2$ for the EP $U(r, E)$ of interaction model (9) to be a confining-type potential. The value of the parameter λ was obtained by fitting experimental data [32], [33] on the fine structure of P-wave levels in D and B mesons. It was established that the fine structure of the P-wave states in the heavy–light (D, D_s, B, and B_s) mesons is primarily sensitive to the choice of the mixing coefficient λ and to the value of the strong coupling constant α_s. Comparing the results of calculations based on formulas (15) and (32) with the experimental data [32], [33], we find that the best agreement is reached at $\lambda = 0.3$ and for the parameter choices

$$\sigma = 0.18\,\mathrm{GeV}^2, \qquad \alpha_s(\mathrm{c\bar{u}\ or\ c\bar{d}}) = 0.386, \qquad \alpha_s(\mathrm{b\bar{u}\ or\ b\bar{d}}) = 0.3,$$

$$\alpha_s(\mathrm{s\bar{u}\ or\ s\bar{d}}) = 0.421, \qquad V_0(\mathrm{c\bar{u}\ or\ c\bar{d}}) = -375\,\mathrm{MeV}, \qquad V_0(\mathrm{b\bar{u}\ or\ b\bar{d}}) = -450\,\mathrm{MeV}.$$

For the masses of u, d, s, c, and b quarks, we used their constituent masses $m_{u,d} = 330\,\mathrm{MeV}$, $m_s = 500\,\mathrm{MeV}$, $m_c = 1550\,\mathrm{MeV}$, and $m_b = 4880\,\mathrm{MeV}$. When calculating the mass spectrum, we neglected electromagnetic interaction and the difference of the masses of u and d quarks, therefore considering the particles D^+, D^-, D^0, and $\overline{D}^0$, for example, to be the same state of the $Q\bar{q}$ system, $J^P = 0^-$. Correspondingly, we do not distinguish between the interaction parameters σ, λ, α_s, and V_0 for these particles. The mass spectra of D and D_s mesons calculated in this approximation are presented in Tables 2 and 3.

Table 2

L_j	(n_r, k)	M_{theor}	M_{exp}	$\langle r\rangle_{\text{numer}}$	$\langle r\rangle$ (35)
$S_{1/2}$	$(0, -1)$	2001.5	1971.1	0.472	0.402
	$(1, -1)$	2632.3	< 2637	0.684	0.664
$P_{3/2}$	$(0, -2)$	2443.2	2447.3	0.678	0.632
	$(1, -2)$	2981.9	–	0.856	0.833
$P_{1/2}$	$(0, 1)$	2403.7	2407.8	0.513	0.568
	$(1, 1)$	2933.4	–	0.770	0.788

The mass spectrum and the mean radii of D mesons obtained in the WKB approximation for potentials (9) (masses are expressed in MeV and the mean radii are expressed in Fm).

Table 3

L_j	(n_r, k)	M_{theor}	M_{exp}		$\langle r\rangle_{\text{numer}}$	$\langle r\rangle$ (35)
$S_{1/2}$	$(0, -1)$	2069.0	2072		0.416	0.359
	$(1, -1)$	2737.4	–		0.646	0.628
$P_{3/2}$	$(0, -2)$	2552.1	2559.2 (I)	2530.7 (II)	0.625	0.588
	$(1, -2)$	3107.2	–	–	0.814	0.795
$P_{1/2}$	$(0, 1)$	2508.5	2423.8 (I)	2480.9 (II)	0.504	0.536
	$(1, 1)$	3058.5	–	–	0.739	0.756

The mass spectrum and the mean radii of D_s mesons obtained in the WKB approximation for potentials (9) (masses are expressed in MeV and the mean radii are expressed in Fm).

The agreement between the model and experiment is in the 3–5% range, except for the masses of states $P_{3/2}$ and $P_{1/2}$ of the $c\bar{s}$ system for which the mismatch depends on the interpretation of the $D_{s1}(2536)^{\pm}$ meson and is 10% if we consider it to be the vector state $J^P = 1^+$ belonging to the doublet $j = 3/2^+$ or 4% if we consider it to be the state $J^P = 1^+$ of the doublet $j = 1/2^+$. There is a rather broad spectrum of opinions concerning the identification of the states $P_{3/2}$ and $P_{1/2}$ of the meson with the quark content $c\bar{s}$ (see, e.g., [34]–[44]). For example, the state $J^P = 2^+$ of a relatively narrow doublet $j = 3/2^+$ is related to $D_{s2}(2573)$, while the vector state $J^P = 1^+$ belonging to the doublet $j = 3/2^+$ is generally related to the isotopic singlet $D_{s1}(2536)^{\pm}$ meson with the mass $2535.35 \pm 0.34 \pm 0.5\,\mathrm{MeV}$ (values (I) for M_{exp} in Table 3) [34]–[37], [40]–[44]. On the other hand, the state $D_{s1}(2536)^{\pm}$ was associated with the state $J^P = 1^+$ of the wide doublet $j = 1/2^+$ in [38], [39] (values (II) in Table 3). Therefore, a reliable experimental identification of this state is still lacking. We note that our calculations agree better with the second possibility.

For $b\bar{u}$ and $b\bar{s}$ systems, we obtained a good agreement of our results with the experimental data for the ground state with $j = 1/2^-$ and for the p state with $j = 3/2^+$ (see Tables 4 and 5). For states in the doublet $j = 1/2^+$, we have only theoretical predictions of other authors. For the $b\bar{u}$ system, our results agree with the data obtained in [45], and a remarkable agreement with the results in [34], [35], [46] was obtained for the $b\bar{s}$ system.

Table 4

L_j	(n_{r}, k)	M_{theor}	M_{exp}	$\langle r\rangle_{\mathrm{numer}}$	$\langle r\rangle$ (35)
$S_{1/2}$	$(0,-1)$	5329.5	5313.5	0.516	0.448
	$(1,-1)$	5832.2	–	0.728	0.708
$P_{3/2}$	$(0,-2)$	5661.6	< 5698	0.711	0.666
	$(1,-2)$	6078.4	–	0.888	0.865
$P_{1/2}$	$(0,1)$	5652.4	5751.6 [34] 5624 [45]	0.577	0.612
	$(1,1)$	6059.0	–	0.812	0.829

The mass spectrum and the mean radii of B mesons obtained in the WKB approximation for potentials (9) (masses are expressed in MeV and the mean radii are expressed in Fm).

Table 5

L_j	(n_{r}, k)	M_{theor}	M_{exp}	$\langle r\rangle_{\mathrm{numer}}$	$\langle r\rangle$ (35)
$S_{1/2}$	$(0,-1)$	5415.6	5404.8	0.457	0.404
	$(1,-1)$	5931.2	–	0.688	0.671
$P_{3/2}$	$(0,-2)$	5765.6	< 5853	0.656	0.619
	$(1,-2)$	6186.8	–	0.845	0.826
$P_{1/2}$	$(0,1)$	5752.2	5751.8 [34] 5753.3 [35] 5700.5 [45] 5755.0 [46] 5790.3 [47]	0.547	0.575
	$(1,1)$	6166.8	–	0.779	0.795

The mass spectrum and the mean radii of B_s mesons obtained in the WKB approximation for potentials (9) (masses are expressed in MeV and the mean radii are expressed in Fm).

In the leading approximation (in $1/m_{\mathrm{Q}}$), the wave functions and excitation energies of the strange quark in the field of a heavy c or b quark reproduce the corresponding characteristics of heavy–light mesons with light u and d quarks with high accuracy. Therefore, up to an additive upward shift of masses on the value of the current mass of the strange quark

$$m_{\rm s} \approx M[{\rm D_s}] - M[{\rm D}] \approx M[{\rm B_s}] - M[{\rm B}] \approx 0.1\,{\rm GeV},$$

the level systems for $\rm D_s$ and $\rm B_s$ mesons coincides with the respective level systems for D and B mesons if we do not take the level splitting depending on the spin of the heavy quark into account. Further, the spin–orbital splitting of lower states of $\rm D_s$ and $\rm B_s$ mesons for the levels $P_{3/2}$ and $P_{1/2}$ is 35% larger than that of the D and B mesons.

Not only the spectrum of bound systems, all other observable characteristics of heavy–light mesons can be calculated in the framework of the semiclassical approach under consideration. For example, an important meson characteristics is the mean radius $\langle r\rangle$, which determines the radius of the light quark orbit in a definite state $|n_{\rm r}k\rangle$ in the case of hydrogen-like quark systems. We first obtain general formulas expressing the means of type $\langle r^m\rangle$ (i.e., the moments of the probability distribution density) in terms of semiclassical asymptotic expressions for solutions of the Dirac equation. Using the standard procedure, we obtain the known semiclassical formula

$$\langle r^m\rangle = \int_0^\infty \chi^+ r^m \chi\, dr = \int_0^\infty \bigl(|F(r)|^2 + \bigl|G(r)\bigr|^2\bigr) r^m\, dr \approx \frac{2}{T}\int_{r_0}^{r_1} \frac{E - V(r)}{p(r)} r^m\, dr, \tag{34}$$

where the period T of radial oscillations of the classical relativistic particle is given by the formula $T = 2\int_{r_0}^{r_1}\bigl(E - V(r)\bigr)/p(r)\, dr$ [16].

All the integrals in (34) can be expressed in terms of complete elliptic integrals (A.1). In particular, the mean radius of the bound state is

$$\langle r\rangle = \frac{4\bigl[n_1 F(\chi) + n_2 E(\chi) + n_3 \Pi(\nu,\chi)\bigr]}{T\sqrt{(a-c)(b-d)(1-2\lambda)}\,|\sigma|}, \tag{35}$$

where

$$T = \frac{4\bigl[n_4 F(\chi) + n_5 E(\chi) + n_6 \Pi(\nu,\chi)\bigr]}{\sqrt{(a-c)(b-d)(1-2\lambda)}\,|\sigma|} \tag{36}$$

and the quantities n_i $(i = 1,\dots,6)$ are defined in the appendix. The calculation results for $\langle r\rangle$, according to formulas (35) and (36) for different states of D, $\rm D_s$, B, and $\rm B_s$ mesons are presented in the last columns in Tables 2–5. We see that the semiclassical approximation well describes the numerical simulation results $\langle r\rangle_{\rm numer}$ and ensures an accuracy up to 3% (except the ground state). Calculations demonstrate that the mean radius of the $\rm Q\bar{q}$ system increases monotonically as the energy increases.

In addition to the “exact” semiclassical formulas (35) and (36), it is desirable to find approximate analytic expressions for the quantities $\langle r\rangle$ and T. We already addressed an analogous problem in the preceding section when constructing asymptotic approximations for the quantization integrals.

If the condition $\sigma/\xi\widetilde{m}^2 \ll 1$ is satisfied in the spectral domain $\widetilde{E} < \widetilde{m}$, then the only essential contribution to the integral determining the mean radius $\langle r\rangle$ comes from the domain of the integration variable r where the long-range potential $v(r)$ can be considered a small perturbation. Neglecting this potential, we obtain the expressions for the mean radius and the period in the zeroth approximation:

$$\langle r\rangle \approx \frac{\pi\widetilde{E}_0}{T\mu_0^3}\biggl(\frac{3\xi^2\widetilde{m}^2}{\mu_0^2} - k^2\biggr), \qquad T \approx \frac{2\pi\xi\widetilde{m}^2}{\mu_0^3}. \tag{37}$$

A more accurate expression (than (37)) for the mean radius can be obtained if we use the exact solutions of Dirac system (1) in the Coulomb field in the integral $\int_0^\infty\bigl(\bigl|F(r)\bigr|^2 + \bigl|G(r)\bigr|^2\bigr) r\, dr \equiv \langle r\rangle$ [17]. The resulting expression for the mean radius of the hydrogen-like system becomes

$$\langle r\rangle_{\mathrm{Coul}} = \frac{\widetilde{E}_0}{2\xi\widetilde{m}^2}\left(\frac{3\xi^2\widetilde{m}^2}{\mu_0^2} - k^2 - \frac{k\widetilde{m}}{\widetilde{E}_0}\right), \tag{38}$$

and it coincides with (37) at large values of the radial quantum number n_{r}.

This simple approximation ensures an amazingly good accuracy for deeply lying levels (but, of course, not at $E = 0$). For example, for the first three terms $1S_{1/2}$, $1P_{1/2}$, and $2S_{1/2}$ of the b quark ($m_{\mathrm{b}} = 4.88\,\mathrm{GeV}$), we obtain the respective values $\langle r\rangle = 0.153\,\mathrm{Fm}$, $0.501\,\mathrm{Fm}$, and $0.609\,\mathrm{Fm}$ from (38), while the exact calculation (the numerical solution of the Dirac equation with potentials (9) at $\xi = 0.4$, $\lambda = 0.3$, $V_0 = -0.45\,\mathrm{GeV}$, and $\sigma = 0.18\,\mathrm{GeV}^2$) yields the respective values $\langle r\rangle = 0.153\,\mathrm{Fm}$, $0.493\,\mathrm{Fm}$, and $0.600\,\mathrm{Fm}$. Our approximation therefore ensures a high accuracy in the case of heavy quarks.

Unfortunately, the domain of applicability of such an approximation is restricted by the condition $\sigma/\xi\widetilde{m}^2 \ll 1$. Because the problem of a size of a bound state of the $\mathrm{Q}\bar{\mathrm{q}}$ system is important, we consider it from the quantitative standpoint. We use the fact that the condition $\sigma\gamma/\widetilde{E}^2 \ll 1$ is satisfied for all typical values of the parameters ξ and σ of heavy–light quarks in the spectrum domain $\widetilde{E} > \widetilde{m}$ under investigation. In this case, the light quark motion is mainly determined by the linear potential, and the Coulomb interaction can be considered a perturbation. In some cases, the zeroth approximation suffices for calculating $\langle r\rangle$ and T,

$$\langle r\rangle \approx \frac{2}{T\sigma^2(1-2\lambda)}\left\{\left(\frac{3\lambda\eta_1}{2(1-2\lambda)} + \widetilde{E}\right)\sqrt{\widetilde{E}^2 - \widetilde{m}^2} - \right.$$
$$\left. - \frac{1}{\sqrt{1-2\lambda}}\left[\widetilde{E}\eta_1 + \frac{\lambda}{2}\left(\frac{3\eta_1^2}{1-2\lambda} + \widetilde{E}^2 - \widetilde{m}^2\right)\right]\arccos\frac{\eta_1}{\eta_2}\right\}, \tag{39}$$

$$T \approx \frac{2}{\sigma(1-2\lambda)}\left[-\lambda\sqrt{\widetilde{E}^2 - \widetilde{m}^2} + (1-\lambda)\eta\eta_2\right], \tag{40}$$

where the quantity $\widetilde{E}$ is determined in (11), the quantities η_1 and η_2 are determined in (17), and the quantity η is determined in (27). For example, for the first three terms $1S_{1/2}$, $1P_{1/2}$, and $2S_{1/2}$ of the B meson ($m_{\mathrm{b}} = 4.88\,\mathrm{GeV}$ and $m_{\mathrm{u}} = 0.33\,\mathrm{GeV}$), we obtain the respective quantities $\langle r\rangle = 0.381\,\mathrm{Fm}$, $0.576\,\mathrm{Fm}$, and $0.681\,\mathrm{Fm}$ in approximation (39), (40), and the calculation using "exact" formulas (35) and (36) (at $\alpha_{\mathrm{s}} = 0.3$, $\lambda = 0.3$, $V_0 = -0.45\,\mathrm{GeV}$, and $\sigma = 0.18\,\mathrm{GeV}^2$) yields the respective values $0.448\,\mathrm{Fm}$, $0.612\,\mathrm{Fm}$, and $0.708\,\mathrm{Fm}$. This approximation therefore ensures an acceptable accuracy for calculating the mean radii of the $\mathrm{Q}\bar{\mathrm{q}}$ mesons.

Appendix

We consider the quantization integral J_1. We rewrite the expression for J_1 in (14) as the sum of integrals

$$J_1 = -|\sigma|\sqrt{1-2\lambda}\,(l\Im_{-1} + h\Im_0 + g\Im_1 + f\Im_2 + \Im_3), \qquad \Im_n = \int_b^a \frac{r^n}{R(r)}\,dr,$$

where the quantities f, g, h, and l are determined in (10) and the quantity $R(r)$ is determined in (13). After the standard change of the integration variable [24]

$$r = \frac{b(a-c) - c(a-b)\sin^2\varphi}{a - c - (a-b)\sin^2\varphi},$$

the integrals $\Im_n$ are expressed in terms of the complete elliptic integrals of the first, second, and third kind, which are written in the conventional notation [25] as

$$F(\chi) = \int_0^{\pi/2} \frac{d\varphi}{\Delta}, \qquad E(\chi) = \int_0^{\pi/2} \Delta\, d\varphi, \qquad \Pi(\nu,\chi) = \int_0^{\pi/2} \frac{d\varphi}{(1-\nu\sin^2\varphi)\Delta}, \tag{A.1}$$
$$\Delta = \sqrt{1-\chi^2\sin^2\varphi}\,, \qquad \nu = \frac{a-b}{a-c}, \qquad \chi = \sqrt{\nu\frac{(c-d)}{(b-d)}}.$$

We thus obtain the representations for $\Im_{-1},\ldots,\Im_3$:

$$\Im_{-1} = \int_b^a \frac{dr}{rR} = \frac{2}{\sqrt{(a-c)(b-d)}bc}\left[bF(\chi) - (b-c)\Pi\left(\frac{c}{b}\nu,\chi\right)\right], \tag{A.2}$$

$$\Im_0 = \int_b^a \frac{dr}{R} = \frac{2}{\sqrt{(a-c)(b-d)}}F(\chi), \tag{A.3}$$

$$\Im_1 = \int_b^a \frac{r\,dr}{R} = \frac{2}{\sqrt{(a-c)(b-d)}}\big[cF(\chi) + (b-c)\Pi(\nu,\chi)\big], \tag{A.4}$$

$$\Im_2 = \int_b^a \frac{r^2\,dr}{R} = \frac{2}{\sqrt{(a-c)(b-d)}}\left[c^2F(\chi) + c(b-c)\Pi(\nu,\chi) + \right.$$
$$\left. + (b-c)^2T_2\left(\frac{\pi}{2},\nu,\chi\right)\right], \tag{A.5}$$

$$\Im_3 = \int_b^a \frac{r^3\,dr}{R} = \frac{2}{\sqrt{(a-c)(b-d)}}\left[c^3F(\chi) + 3c^2(b-c)\Pi(\nu,\chi) + \right.$$
$$\left. + 3c(b-c)^2T_2\left(\frac{\pi}{2},\nu,\chi\right) + (b-c)^3T_3\left(\frac{\pi}{2},\nu,\chi\right)\right]. \tag{A.6}$$

The integrals of the form

$$T_n(\varphi,\nu,\chi) = \int_0^{\varphi} \frac{d\varphi}{(1-\nu\sin^2\varphi)^n\Delta}$$

are calculated using the recurrence relation

$$T_{n-3} = \frac{1}{(2n-5)\chi^2}\left\{\frac{-\nu^2\Delta\sin\varphi\cos\varphi}{(1-\nu\sin^2\varphi)^{n-1}} + 2(n-2)\big[3\chi^2 - \nu(1+\chi^2)\big]T_{n-2} - \right.$$
$$\left. - (2n-3)\big[\chi^2(3-2\nu) + \nu(\nu-2)\big]T_{n-1} + 2(n-1)(\chi^2-\nu)(1-\nu)T_n\right\}.$$

We analogously find the integrals in the expression for J_2 in (14):

$$\int_b^a \frac{dr}{(r-\lambda_\pm)R} = \frac{2}{\sqrt{(a-c)(b-d)}\,(b-\lambda_\pm)(\lambda_\pm - c)} \times$$
$$\times\left[(\lambda_\pm - b)F(\chi) - (b-c)\Pi\left(\frac{(\lambda_\pm - c)}{(\lambda_\pm - b)}\nu,\chi\right)\right]. \tag{A.7}$$

After expressions (A.2)–(A.7) are substituted in integrals (14), quantization condition (3) becomes transcendental equation (15), where

$$\nu_\pm = \frac{\lambda_\pm - c}{\lambda_\pm - b}\nu, \qquad \Re = (1-\nu)(\chi^2-\nu), \qquad \aleph = \chi^2(3-2\nu) + \nu(\nu-2),$$

$$N_1 = \frac{\chi^2(b-c)}{4} - \frac{3\aleph(b-c)}{8(1-\nu)} - \frac{(\chi^2-\nu)}{2}(f+3c) + \\ + \frac{\Re}{(b-c)^2}\left(c^3 + c^2 f + cg + h + \frac{l}{c}\right),$$

$$N_2 = -\frac{\nu}{2}\left[f + 3c + \frac{3}{4}\frac{(b-c)\aleph}{\Re}\right],$$

$$N_3 = \frac{1}{2}\left[\frac{3}{4}\frac{(b-c)\aleph^2}{\Re} + \frac{2\Re}{(b-c)}(3c^2+2cf+g) + (b-c)\big((1+\chi^2)\nu - 3\chi^2\big) + \\ + \aleph(f+3c)\right], \qquad N_4 = -\frac{\Re}{(b-c)}\frac{l}{bc},$$

$$N_5 = \big[(b-\lambda_+)(\lambda_+-c)\big]^{-1}, \qquad N_6 = \big[(b-\lambda_-)(\lambda_--c)\big]^{-1},$$

$$N_7 = \frac{2}{(\lambda_+-c)(\lambda_--c)}\left(c + \frac{\widetilde{E}+\widetilde{m}}{2(1-2\lambda)\sigma}\right).$$

We analogously find the integrals appearing when calculating the mean radii by formula (34). We present the quantities n_i $(i = 1, \ldots, 6)$ in formulas (35) and (36):

$$n_1 = \widetilde{E}\left(c^2 - \frac{(b-c)^2}{2(1-\nu)}\right) - \lambda\sigma\left(c^3 - \frac{3c(b-c)^2}{2(1-\nu)} + \\ + \frac{(b-c)^3}{4\Re}\left(\chi^2 - \frac{3\aleph}{2(1-\nu)}\right)\right) + \xi c,$$

$$n_2 = -\frac{\nu(b-c)^2}{2\Re}\left[\widetilde{E} - 3\lambda\sigma\left(c + \frac{(b-c)\aleph}{4\Re}\right)\right],$$

$$n_3 = (b-c)\left[\widetilde{E}\left(2c + \frac{(b-c)\aleph}{2\Re}\right) - \lambda\sigma\left(3c^2 + \frac{(b-c)\aleph}{2\Re}\times \\ \times\left(3c - \frac{(b-c)\big(3\chi^2 - \nu(1+\chi^2)\big)}{\aleph} + \frac{3(b-c)\aleph}{4\Re}\right)\right) + \xi\right],$$

$$n_4 = c\widetilde{E} - \lambda\sigma\left(c^2 - \frac{(b-c)^2}{2(1-\nu)}\right) + \xi,$$

$$n_5 = \frac{\lambda\sigma\nu(b-c)^2}{2\Re}, \qquad n_6 = (b-c)\left[\widetilde{E} - \lambda\sigma\left(2c + \frac{(b-c)\aleph}{2\Re}\right)\right].$$

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