UDC 512.552.7

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TORSION UNITS IN INTEGRAL GROUP RING OF SYMMETRIC GROUP OF DEGREE SEVEN

Using the Luthar-Passi method and results of Hertweck, we consider the famoust Zassenhaus conjecture for the normalized unit group of the integral group ring of the symmetric group of degree seven. As a consequence, we achieve the solution of the Kimmerle's conjecture about prime graphs for the group of units.

Використовуючи метод Лутера-Пассі і результати Гертвека, розглядається відому гіпотезу Цассенхауза для групи нормованих одиниць цілочислового групового кільця симетричної групи степеня сім. Як результат, отримано розв'язок гіпотези Кіммерле про головні графи для групи одиниць.

Let $V(\mathbb{Z}G)$ denotes the group of normalized units of the integral group ring $\mathbb{Z}G$ of a finite group G. One of the most interesting conjectures in the theory of integral group rings is the conjecture of *H. Zassenhaus*:

Conjecture 1 (ZC)**.** *Every torsion unit u in $V(\mathbb{Z}G)$ is conjugate to an element in G within the rational group algebra $\mathbb{Q}G$; i.e. there exist a group element g in G and a unit w in $\mathbb{Q}G$ such that $w^{-1}uw = g$.*

In parallel to the (ZC) and as a usefull technique that we have used is the cojecture of *W. Kimmerle*, which involves the concept of prime graph (see [21]): For a finite group G, let $pr(G)$ denotes the set of all prime divisors of the order of G. The *Gruenberg-Kegel* graph (or the prime graph) of G is a $\pi(G)$ with vertices labelled by primes from $pr(G)$, such that vertices p and q are adjacent if and only if there is an element of order pq in the group G.

Conjecture 2 (KC)**.** *If G is a finite group, then $\pi(G) = \pi(V(\mathbb{Z}G))$, where $\pi(G)$ is the prime graph of the group G.*

Obviously, the *Zassenhaus conjecture* (ZC) implies the *Kimmerle conjecture* (KC). In [21], it was shown that the (KC) holds for finite Frobenius and solvable groups. Note that with respect to the so-called p-version of the *Zassenhaus* conjecture the investigation of Frobenius groups was completed by *V. Bovdi* and *M. Hertweck* (see [2]). In the papers [3]— [13] and [15], the (KC) was studied for certain Mathieu, Conway, Janko, Held, O'Nan, Rudvalis, Suzuki, Higman-Sims and McLaughlin simple sporadic groups. In [25], we had a partial answer for the alternating group A_6 of degree six, then *M. Hertweck* complete the remaining case for A_6 in [19] (note that for larger alternating groups the problem is still open). In the present paper we confirm the (KC) for the symmetric group S_7 of degree 7.

In order to state the result, for a group G, let $\mathcal{C} = \{C_1, \dots, C_{nt}, \dots\}$ be the collection of all conjugacy classes of G, where the first index denotes the order of the elements of this conjugacy class and $C_1 = \{1\}$. For any unit $u = \sum \alpha_g g \in V(\mathbb{Z}G)$ of order k, let ν_{nt} denote the partial augmentation $\nu_{nt}(u) = \varepsilon_{C_{nt}}(u) = \sum_{g \in C_{nt}} \alpha_g$ of u with respect to C_{nt}. From *Berman's Theorem* (see [1]), we know that $\nu_1 = \alpha_1 = 0$ and $\nu_c = 0$ for any central element $c \in G$, and that

$$\sum_{C_{nt} \in \mathcal{C}} \nu_{nt} = 1. \qquad (1)$$

Hence, for any character χ of G, we have $\chi(u) = \sum \nu_{nt}\chi(h_{nt})$, where h_{nt} is a representative of a conjugacy class C_{nt}.

Our main results are the following:

Theorem 1. *Let G denote the symmetric group S_7 of degree seven. If u is a torsion unit in $V(\mathbb{Z}G)$ of order $|u|$, and $\mathfrak{PA}(u)$ denotes the tuple*

$$(\nu_{2a}, \nu_{2b}, \nu_{2c}, \nu_{3a}, \nu_{3b}, \nu_{4a}, \nu_{4b}, \nu_{5a}, \nu_{6a}, \nu_{6b}, \nu_{6c}, \nu_{7a}, \nu_{10a}, \nu_{12a}) \text{ in } \mathbb{Z}^{14}$$

of partial augmentations of u in $V(\mathbb{Z}G)$. Then the following statements hold:

(i) If $|u| \neq 20$, then $|u|$ coincides with the order of some $g \in G$.

(ii) If $|u| \in \{3, 5, 7, 10\}$, then u is rationally conjugate to some $g \in G$.

(iii) If $|u| = 2$, the tuple of the partial augmentations $(\nu_{2a}, \nu_{2b}, \nu_{2c})$ of u belongs to the set $\{(1,0,0), (0,1,0), (0,0,1), (0,-1,2), (1,-1,1), (1,1,-1)\}$ and $\nu_{kx} = 0$ whenever $kx \notin \{2a, 2b, 2c\}$.

And hence as a direct consequence, we accomplish the (KC) as follows.

Corollary 1. *If $G \cong S_7$, then $\pi(G) = \pi(V(\mathbb{Z}G))$.*

For a torsion u in $V(\mathbb{Z}G)$, the **(ZC)** provides that $\chi(u) = \chi(x_i)$ for some $x_i \in G$; and hence an equivalent statement for it was given in the following statement:

Fact 1. *(see [22]) If $u \in V(\mathbb{Z}G)$ is a torsion unit of order k. Then u is conjugate to an element g in G if and only if for each positive divisor d of k there is precisely one conjugacy class C with non-zero partial augmentation $\varepsilon_C(u^d) \neq 0$.*

In fact to establish our investigation, we consider the calculation, by *GAP*, of the indicated numbers $\mu_m(u,\chi)$ in what follow for each possible order k of a torsion unit u in $V(\mathbb{Z}G)$, taking in account the relation between $|u|$ and the partial augmentations $\nu_i = \varepsilon_{C_i}(u)$ given in the next three Facts.

Fact 2. *(see [18], Proposition 3 and [20], Lemma 5.6]) Let G be a finite group and let u be a torsion unit in $V(\mathbb{Z}G)$. If $x \in G$ whose p-part, for some prime p, has order strictly greater than the order of the p-part of u, then $\varepsilon_x(u) = 0$.*

Fact 3. *(see [20], [22]) Let either p be a prime divisor of $|G|$ or $p = 0$. Suppose that $u \in V(\mathbb{Z}G)$ has finite order k such that k and p are coprime if $p \neq 0$. If ζ is a primitive k-th root of unity and χ is either a classical character or a p-Brauer character of G then, for every integer m, the number*

$$\mu_l(u,\chi,p) = \frac{1}{k}\sum_{d|k} Tr_{(\zeta^d)/}\{\chi(u^d)\zeta^{-dm}\}$$

is a non-negative integer.

Note that if $p = 0$, we will use the notation $\mu_l(u,\chi,*)$ instead of $\mu_l(u,\chi,0)$.

Fact 4. *(see [16]) The order of a torsion unit $u \in V(\mathbb{Z}G)$ is a divisor of* $\exp(G)$.

Proof. In this section, the symmetric group of degree seven is denoted by S_7. It is known, by [17], that

$$|S_7| = 7! = 5040 = 2^4 \cdot 3^2 \cdot 5 \cdot 7 \qquad \text{and} \qquad exp(S_7) = 420 = 2^2 \cdot 3 \cdot 5 \cdot 7.$$

Obviously, the group S_7 has 15 conjugacy classes $1a$, $2a$, $2b$, $2c$, $3a$, $3b$, $4a$, $4b$, $5a$, $6a$, $6b$, $6c$, $7a$, $10a$ and $12a$, where j is the order of elements in conjugacy classes ja, jb and $jc, j \in \{1, 2, 3, 4, 5, 6, 7, 10\}$. Since conjugate group elements have same character, then for any normalized unit $u = \sum \alpha_i g_i \in V(\mathbb{Z}S_7)$, its character is $\chi(u) = \sum_{i=1}^{15} \nu_i \chi(x_i)$, where $\nu_i's (\in \mathbb{Z})$ are partial augmentations $\varepsilon_{C_i}(u)$ of u, and $x_i's$ are representatives of distinct conjugacy classes C_i in S_7.

If u is torsion in $V(\mathbb{Z}S_7)$ and $|u| = n$, then the **(ZC)** provides that $\chi(u) = \chi(x_i)$ for some $x_i \in G$; and hence an equivalent statement for the **(ZC)** was given in [22, 23]. The character table of S_7, as well as the Brauer character tables (denoted by $\mathfrak{BCT}(p)$, where $p \in \{2, 3, 5, 7\}$), can be found by the computational algebra system *GAP* in [17]. Throughout the paper we use the notation of *GAP* [17] for the indexation of the characters and conjugacy classes of S_7.

¿From the structure of the group S_7, its known that it possesses elements of orders 2, 3, 4, 5, 6, 7, 10 and 12. We begin our investigation with units of orders 2, 3, 5, 7 and 10. But, by Fact 4, the order of each torsion unit divides the exponent 420 of S_7, then it remains to consider only units of orders 14, 15, 20, 21 and 35. We prove that all units of these orders (except for 20) do not appear in $V(\mathbb{Z}S_7)$.

Now, we study each case according to Fact 2, to find the appropriate partial augmentations of those involved in (1). Then we apply Fact 3 to the apropriate character to get a system of inequalities. In all our computation we use the package LAGUNA [14] for the computational algebra system *GAP* [17].

• Let $|u| \in \{5, 7\}$. Then, by Fact 2, there is only one conjugacy class in S_7 consisting of elements of each order $|u|$. Thus for each order $|u|$ there is precisely one conjugacy class with non-zero partial augmentation. Then, by Fact 1, any unit u, where $|u| \in \{5, 7\}$, is rationally conjugate to some g in G.

• If $|u| = 2$, then by (1) and Fact 2, we have that $\nu_{2a} + \nu_{2b} + \nu_{2c} = 1$. Applying Fact 3 to the character χ_2, χ_3 and $1\chi_4$, we get the following system of inequalities

$$\begin{aligned}
\mu_0(u, \chi_2, *) &= \tfrac{1}{2}(\nu_{2a} - \nu_{2b} - \nu_{2c} + 1) \geq 0;\\
\mu_1(u, \chi_2, *) &= \tfrac{1}{2}(-(\nu_{2a} - \nu_{2b} - \nu_{2c}) + 1) \geq 0;\\
\mu_0(u, \chi_3, *) &= \tfrac{1}{2}(2\nu_{2a} + 4\nu_{2b} + 6) \geq 0;\\
\mu_1(u, \chi_3, *) &= \tfrac{1}{2}(-(2\nu_{2a} + 4\nu_{2b}) + 6) \geq 0;\\
\mu_1(u, \chi_4, *) &= \tfrac{1}{2}(-2\nu_{2a} + 4\nu_{2b} + 6) \geq 0.
\end{aligned}$$

¿From the requirement, in Fact 3, that all $\mu_i(u, \chi_j, p)$ must be non-negative integers, the system has only the solutions

$$(\nu_{2a}, \nu_{2b}, \nu_{2c}) \in \{(1,0,0), (0,1,0), (0,0,1), (0,-1,2), (1,-1,1), (1,1,-1)\}.$$

• Let $|u| = 3$. By (1) and Fact 2, we have that $\nu_{3a} + \nu_{3b} = 1$. Applying Fact 3 to the characters χ_2 and χ_3 and from Brauer character tables for $p = 2$ and 7, we get

$$\begin{gathered}
\mu_0(u, \chi_3, *) = \tfrac{1}{3}(6\nu_{3a} + 6) \geq 0; \quad \mu_1(u, \chi_3, *) = \tfrac{1}{3}(-3\nu_{3a} + 6) \geq 0;\\
\mu_0(u, \chi_2, 2) = \tfrac{1}{3}(-2(4\nu_{3a} - 2\nu_{3b}) + 8) \geq 0;\\
\mu_1(u, \chi_3, 7) = \tfrac{1}{3}(4\nu_{3a} - 2\nu_{3b} + 5) \geq 0,
\end{gathered}$$

that has only the two trivial integeral solutions $(1, 0)$ and $(0, 1)$ for (ν_{3a}, ν_{3b}). Then, by Fact 1, each unit u of order 3 is rationally conjugate to some g in G.

• Let u be a unit of order 10. By (1) and Fact 2, we have that

$$\nu_{2a}+\nu_{5a}+\nu_{2b}+\nu_{2c}+\nu_{10a}=1.$$

Since$|u^5|=2$, for any character χ of S_7 we need only to consider six cases for $(\nu_{2a},\nu_{2b},\nu_{2c})$ been found for involution units above. We consider each case separately and in the same order:

Case 1. Let $\chi(u^5)=\chi(2a)$. Using Fact 3, we get the system

$$\begin{aligned}
\mu_1(u,\chi_2,*)&=\tfrac{1}{10}(\nu_{2a}+\nu_{5a}-\nu_{2b}-\nu_{2c}-\nu_{10a}-1)\geq 0;\\
\mu_2(u,\chi_2,*)&=\tfrac{1}{10}(-(\nu_{2a}+\nu_{5a}-\nu_{2b}-\nu_{2c}-\nu_{10a})+1)\geq 0;\\
\mu_0(u,\chi_3,*)&=\tfrac{1}{10}(4(2\nu_{2a}+\nu_{5a}+4\nu_{2b}-\nu_{10a})+12)\geq 0;\\
\mu_1(u,\chi_3,*)&=\tfrac{1}{10}(2\nu_{2a}+\nu_{5a}+4\nu_{2b}-\nu_{10a}+3)\geq 0;\\
\mu_5(u,\chi_3,*)&=\tfrac{1}{10}(-4(2\nu_{2a}+\nu_{5a}+4\nu_{2b}-\nu_{10a})+8)\geq 0;\\
\mu_0(u,\chi_4,*)&=\tfrac{1}{10}(4(2\nu_{2a}+\nu_{5a}-4\nu_{2b}+\nu_{10a})+12)\geq 0;\\
\mu_1(u,\chi_4,*)&=\tfrac{1}{10}(2\nu_{2a}+\nu_{5a}-4\nu_{2b}+\nu_{10a}+3)\geq 0;\\
\mu_5(u,\chi_4,*)&=\tfrac{1}{10}(-4(2\nu_{2a}+\nu_{5a}-4\nu_{2b}+\nu_{10a})+8)\geq 0;\\
\mu_5(u,\chi_5,*)&=\tfrac{1}{10}(16\nu_{2a}+24)\geq 0;\\
\mu_0(u,\chi_{10},*)&=\tfrac{1}{10}(-4(\nu_{2a}-5\nu_{2b}+3\nu_{2c})+14)\geq 0;\\
\mu_1(u,\chi_{10},*)&=\tfrac{1}{10}(-(\nu_{2a}-5\nu_{2b}+3\nu_{2c})+16)\geq 0;\\
\mu_5(u,\chi_{10},*)&=\tfrac{1}{10}(4(\nu_{2a}-5\nu_{2b}+3\nu_{2c})+16)\geq 0,
\end{aligned}$$

which has no integral solution for $(\nu_{2a},\nu_{5a},\nu_{2b},\nu_{2c},\nu_{10a})$.

Case 2. Let $\chi(u^5)=\chi(2c)$. Using Fact 3, we get the system

$$\begin{aligned}
\mu_0(u,\chi_3,*)&=\tfrac{1}{10}(4(2\nu_{2a}+\nu_{5a}+4\nu_{2b}-\nu_{10a})+10)\geq 0;\\
\mu_5(u,\chi_3,*)&=\tfrac{1}{10}(-4(2\nu_{2a}+\nu_{5a}+4\nu_{2b}-\nu_{10a})+10)\geq 0;\\
\mu_1(u,\chi_3,*)&=\tfrac{1}{10}(2(\nu_{2a}+\nu_{5a}+4\nu_{2b}-\nu_{10a})+5)\geq 0,
\end{aligned}$$

which has no integral solution for $(\nu_{2a},\nu_{5a},\nu_{2b},\nu_{2c},\nu_{10a})$.

Case 3. Let $\chi(u^5)=\chi(2b)$. Using Fact 3, we get the system

$$\begin{aligned}
\mu_0(u,\chi_2,*)&=\tfrac{1}{10}(4(\nu_{2a}+\nu_{5a}-\nu_{2b}-\nu_{2c}-\nu_{10a})+4)\geq 0;\\
\mu_2(u,\chi_2,*)&=\tfrac{1}{10}(-(\nu_{2a}+\nu_{5a}-\nu_{2b}-\nu_{2c}-\nu_{10a})-1)\geq 0;\\
\mu_1(u,\chi_3,*)&=\tfrac{1}{10}(2\nu_{2a}+\nu_{5a}+4\nu_{2b}-\nu_{10a}+1)\geq 0;\\
\mu_5(u,\chi_3,*)&=\tfrac{1}{10}(-4(2\nu_{2a}+\nu_{5a}+4\nu_{2b}-\nu_{10a})+6)\geq 0;\\
\mu_0(u,\chi_4,*)&=\tfrac{1}{10}(4(2\nu_{2a}+\nu_{5a}-4\nu_{2b}+\nu_{10a})+6)\geq 0;\\
\mu_2(u,\chi_4,*)&=\tfrac{1}{10}(-(2\nu_{2a}+\nu_{5a}-4\nu_{2b}+\nu_{10a})+1)\geq 0;\\
\mu_0(u,\chi_{10},*)&=\tfrac{1}{10}(-4(\nu_{2a}-5\nu_{2b}+3\nu_{2c})+20)\geq 0;\\
\mu_1(u,\chi_{10},*)&=\tfrac{1}{10}(-(\nu_{2a}-5\nu_{2b}+3\nu_{2c})+10)\geq 0;\\
\mu_5(u,\chi_{10},*)&=\tfrac{1}{10}(4(\nu_{2a}-5\nu_{2b}+3\nu_{2c})+10)\geq 0,
\end{aligned}$$

that has only the following trivial solution: $(0,0,0,0,1)$.

Case 4. Let $\chi(u^5)=-\chi(2b)+2\chi(2c)$. Using Fact 3, we obtain

$$\begin{aligned}
\mu_0(u,\chi_2,*) &= \tfrac{1}{10}(4(\nu_{2a}+\nu_{5a}-\nu_{2b}-\nu_{2c}-\nu_{10a})+4) \geq 0;\\
\mu_2(u,\chi_2,*) &= \tfrac{1}{10}(-(\nu_{2a}+\nu_{5a}-\nu_{2b}-\nu_{2c}-\nu_{10a})-1) \geq 0;\\
\mu_0(u,\chi_3,*) &= \tfrac{1}{10}(4(2\nu_{2a}+\nu_{5a}+4\nu_{2b}-\nu_{10a})+6) \geq 0;\\
\mu_2(u,\chi_3,*) &= \tfrac{1}{10}(-(2\nu_{2a}+\nu_{5a}+4\nu_{2b}-\nu_{10a})+1) \geq 0;\\
\mu_1(u,\chi_4,*) &= \tfrac{1}{10}(2\nu_{2a}+\nu_{5a}-4\nu_{2b}+\nu_{10a}+1) \geq 0;\\
\mu_5(u,\chi_4,*) &= \tfrac{1}{10}(-4(2\nu_{2a}+\nu_{5a}-4\nu_{2b}+\nu_{10a})+6) \geq 0;\\
\mu_0(u,\chi_{10},*) &= \tfrac{1}{10}(-4(\nu_{2a}-5\nu_{2b}+3\nu_{2c})+4) \geq 0;\\
\mu_2(u,\chi_{10},*) &= \tfrac{1}{10}(\nu_{2a}-5\nu_{2b}+3\nu_{2c}+4) \geq 0,
\end{aligned}$$

which has no integrral solution for $(\nu_{2a},\nu_{5a},\nu_{2b},\nu_{2c},\nu_{10a})$.

Case 5. Let $\chi(u^5)=\chi(2a)-\chi(2b)+\chi(2c)$. Using Fact 3, we get that

$$\begin{aligned}
\mu_1(u,\chi_2,*) &= \tfrac{1}{10}(\nu_{2a}+\nu_{5a}-\nu_{2b}-\nu_{2c}-\nu_{10a}-1) \geq 0;\\
\mu_2(u,\chi_2,*) &= \tfrac{1}{10}(-(\nu_{2a}+\nu_{5a}-\nu_{2b}-\nu_{2c}-\nu_{10a})+1) \geq 0;\\
\mu_0(u,\chi_3,*) &= \tfrac{1}{10}(4(2\nu_{2a}+\nu_{5a}+4\nu_{2b}-\nu_{10a})+8) \geq 0;\\
\mu_2(u,\chi_3,*) &= \tfrac{1}{10}(-(2\nu_{2a}+\nu_{5a}+4\nu_{2b}-\nu_{10a})+3) \geq 0;\\
\mu_1(u,\chi_4,*) &= \tfrac{1}{10}(2\nu_{2a}+\nu_{5a}-4\nu_{2b}+\nu_{10a}-1) \geq 0;\\
\mu_5(u,\chi_4,*) &= \tfrac{1}{10}(-4(2\nu_{2a}+\nu_{5a}-4\nu_{2b}+\nu_{10a})+4) \geq 0;\\
\mu_0(u,\chi_{10},*) &= \tfrac{1}{10}(-4(\nu_{2a}-5\nu_{2b}+3\nu_{2c})+6) \geq 0;\\
\mu_2(u,\chi_{10},*) &= \tfrac{1}{10}(\nu_{2a}-5\nu_{2b}+3\nu_{2c}+6) \geq 0,
\end{aligned}$$

which has no integral solutions for $(\nu_{2a},\nu_{5a},\nu_{2b},\nu_{2c},\nu_{10a})$.

Case 6. Let $\chi(u^5)=\chi(2a)+\chi(2b)-\chi(2c)$. Using Fact 3, we get that

$$\begin{aligned}
\mu_1(u,\chi_2,*) &= \tfrac{1}{10}(\nu_{2a}+\nu_{5a}-\nu_{2b}-\nu_{2c}-\nu_{10a}-1) \geq 0;\\
\mu_2(u,\chi_2,*) &= \tfrac{1}{10}(-(\nu_{2a}+\nu_{5a}-\nu_{2b}-\nu_{2c}-\nu_{10a})+1) \geq 0;\\
\mu_1(u,\chi_3,*) &= \tfrac{1}{10}(2\nu_{2a}+\nu_{5a}+4\nu_{2b}-\nu_{10a}-1) \geq 0;\\
\mu_5(u,\chi_3,*) &= \tfrac{1}{10}(-4(2\nu_{2a}+\nu_{5a}+4\nu_{2b}-\nu_{10a})+4) \geq 0;\\
\mu_0(u,\chi_4,*) &= \tfrac{1}{10}(4(2\nu_{2a}+\nu_{5a}-4\nu_{2b}+\nu_{10a})+8) \geq 0;\\
\mu_2(u,\chi_4,*) &= \tfrac{1}{10}(-2\nu_{2a}-\nu_{5a}+4\nu_{2b}-\nu_{10a}+3) \geq 0;\\
\mu_0(u,\chi_{10},*) &= \tfrac{1}{10}(-4(\nu_{2a}-5\nu_{2b}+3\nu_{2c})+22) \geq 0;\\
\mu_1(u,\chi_{10},*) &= \tfrac{1}{10}(-(\nu_{2a}-5\nu_{2b}+3\nu_{2c})+8) \geq 0;\\
\mu_5(u,\chi_{10},*) &= \tfrac{1}{10}(4(\nu_{2a}-5\nu_{2b}+3\nu_{2c})+8) \geq 0;\\
\mu_0(u,\chi_{11},*) &= \tfrac{1}{10}(-4(\nu_{2a}+5\nu_{2b}-3\nu_{2c})+6) \geq 0,
\end{aligned}$$

which has no solution for $(\nu_{2a},\nu_{5a},\nu_{2b},\nu_{2c},\nu_{10a})$. Thus, for units of orders 10, there is precisely one conjugacy class with non-zero partial augmentation. Then, by Fact 1, each unit of order 10 is rationally conjugate to some $g\in G$, so part (ii) of the Theorem is complete.

• Let $|u|=14$. By (1) and Fact 2, we have $\nu_{2a}+\nu_{2b}+\nu_{2c}+\nu_{7a}=1$. Since $|u^7|=2$ for any character χ of S_7 we need to consider six cases for $(\nu_{2a},\nu_{2b},\nu_{2c})$ been found for involution units above. We consider each case separately and in the same order:

Case 1. Let $\chi(u^7) = \chi(2a)$. Applying Fact 3 to the character χ_3, we get

$$\mu_0(u, \chi_3, *) = -\mu_7(u, \chi_3, *) = \frac{1}{14}(3(2\nu_{2a} - \nu_{7a} + 4\nu_{2b}) + 1) = 0,$$

which has no integral solution for $(\nu_{2a}, \nu_{7a}, \nu_{2b})$.

Case 2. Let $\chi(u^7) = \chi(2b)$. Then, by Fact 3, we get the system

$$\mu_0(u, \chi_3, *) = -\mu_7(u, \chi_3, *) = \frac{1}{14}(3(2\nu_{2a} - \nu_{7a} + 4\nu_{2b}) + 2) = 0,$$

which has no integral solution for $(\nu_{2a}, \nu_{7a}, \nu_{2b})$.

Case 3. Let $\chi(u^7) = -\chi(2b) + 2\chi(2c)$. Then,by Fact 3, we obtain

$$\mu_0(u, \chi_3, *) = -\mu_7(u, \chi_3, *) = \frac{1}{14}(3(2\nu_{2a} - \nu_{7a} + 4\nu_{2b}) - 2) = 0,$$

which has no integral solution for $(\nu_{2a}, \nu_{7a}, \nu_{2b})$.

Case 4. Let $\chi(u^7) = \chi(2a) - \chi(2b) + \chi(2c)$. Then,by Fact 3, we get

$$\mu_0(u, \chi_3, *) = -\mu_7(u, \chi_3, *) = \frac{1}{14}(3(2\nu_{2a} - \nu_{7a} + 4\nu_{2b}) - 1) = 0,$$

which has no integral solution for $(\nu_{2a}, \nu_{7a}, \nu_{2b})$.

Case 5. Let $\chi(u^7) = \chi(2a) + \chi(2b) - \chi(2c)$. Then,by Fact 3, we get

$$\mu_0(u, \chi_4, *) = -\mu_7(u, \chi_4, *) = \frac{1}{14}(3(2\nu_{2a} - \nu_{7a} + 4\nu_{2b}) - 1) = 0,$$

which has no integral solution for $(\nu_{2a}, \nu_{7a}, \nu_{2b})$.

Case 6. Let $\chi(u^7) = \chi(2c)$. Applying Fact 3 to the characters χ_2 and χ_3, we obtain the following system of inequalities

$$\begin{aligned}
\mu_0(u, \chi_2, *) &= \tfrac{1}{14}(6(\nu_{2a} + 6\nu_{7a} - 6\nu_{2b} - 6\nu_{2c}) + 6) \geq 0;\\
\mu_2(u, \chi_2, *) &= \tfrac{1}{14}(-(\nu_{2a} + \nu_{7a} - \nu_{2b} - \nu_{2c}) - 1) \geq 0;\\
\mu_0(u, \chi_3, *) &= \tfrac{1}{14}(6(2\nu_{2a} - \nu_{7a} + 4\nu_{2b}) \geq 0;\\
\mu_7(u, \chi_3, *) &= \tfrac{1}{14}(-6(2\nu_{2a} - \nu_{7a} + 4\nu_{2b})) \geq 0;\\
\mu_1(u, \chi_3, *) &= \tfrac{1}{14}(2\nu_{2a} - \nu_{7a} + 4\nu_{2b} + 7) \geq 0,
\end{aligned}$$

which has no integral solution for $(\nu_{2a}, \nu_{7a}, \nu_{2b})$.

Hence there is no unit in $V(\mathbb{Z}S_7)$ of order 14.

• Let $|u| = 15$. By (1) and Fact 2, we have $\nu_{3a} + \nu_{3b} + \nu_{5a} = 1$. Since $|u^5| = 3$, for any character χ of G, we need only to consider the two trivial integeral solutions for (ν_{3a}, ν_{3b}) apper for units of order 3.

Case 1. Let $\chi(u^5) = \chi(3a)$. Applying Fact 3 for the character χ_5 of G, we get

$$\begin{aligned}
\mu_0(u, \chi_5, *) &= \tfrac{1}{15}(16(\nu_{3a} + \nu_{3b}) + 24) \geq 0;\\
\mu_5(u, \chi_5, *) &= \tfrac{1}{15}(-8(\nu_{3a} + \nu_{3b}) + 18) \geq 0,
\end{aligned}$$

and this system has no integral solutions (ν_{3a}, ν_{3b}).

Case 2. Let $\chi(u^5) = \chi(3b)$. Applying Fact 3 for the character χ_3 of G, we get

$$\begin{aligned}
\mu_0(u, \chi_3, *) &= \tfrac{1}{15}(8(3\nu_{3a} + \nu_{5a}) + 10) \geq 0;\\
\mu_3(u, \chi_3, *) &= \tfrac{1}{15}(-2(3\nu_{3a} + \nu_{5a} + 5) \geq 0,
\end{aligned}$$

and this system has no integral solutions (ν_{3a}, ν_{5a}). Hence there is no unit in $V(\mathbb{Z}S_7)$ of order 15.

• Let $|u| = 21$. By (1) and Fact 2, we have $\nu_{3a} + \nu_{3b} + \nu_{7a} = 1$. Since $|u^7| = 3$, for any character χ of G, we need only to consider the two trivial integeral solutions for (ν_{3a}, ν_{3b}) appear for units of order 3.

Case 1. Let $\chi(u^7) = \chi(3a)$. Applying Fact 3 for the character χ_3 of G, we get

$$\mu_0(u, \chi_3, *) = \tfrac{1}{21}(12(3\nu_{3a} - \nu_{7a}) + 6) \geq 0;$$
$$\mu_7(u, \chi_3, *) = \tfrac{1}{21}(-6(3\nu_{3a} - \nu_{7a}) - 3) \geq 0,$$

and this system has no integral solution for (ν_{3a}, ν_{7a}).

Case 2. Let $\chi(u^7) = \chi(3b)$. Applying Fact 3 for the character χ_3 of G, we get

$$\mu_0(u, \chi_3, *) = \tfrac{1}{21}(12(3\nu_{3a} - \nu_{7a})) \geq 0;$$
$$\mu_7(u, \chi_3, *) = \tfrac{1}{21}(-6(3\nu_{3a} - \nu_{7a})) \geq 0;$$
$$\mu_1(u, \chi_3, *) = \tfrac{1}{21}((3\nu_{3a} - \nu_{7a}) + 7) \geq 0,$$

and this system has no integral solution for (ν_{3a}, ν_{7a}). Hence there is no unit in $V(\mathbb{Z}S_7)$ of order 21.

• Let $|u| = 35$. By (1) and Fact 2, we have $\nu_{5a} + \nu_{7a} = 1$. Applying Fact 3 for the character χ_3 of G, we obtain the following system of inequalities

$$\mu_0(u, \chi_3, *) = \tfrac{1}{35}(24(\nu_{5a} - \nu_{7a}) + 4) \geq 0;$$
$$\mu_7(u, \chi_3, *) = \tfrac{1}{35}(-6(\nu_{5a} - \nu_{7a}) - 1) \geq 0,$$

that leads to a contradiction, and hence there is no unit in $V(\mathbb{Z}S_7)$ of order 35.

Therefore the proof is complete.

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