CRITICAL CHARGE IN MODIFIED QUANTUM ELECTRODYNAMICS

V.YU. LAZUR, O.K. REITY, V.V. BONDARCHUK, V.K. REITY

Uzhgorod National University
(54, Voloshin Str., Uzhgorod 88000, Ukraine; e-mail: lazur@univ.uzhgorod.ua; reiti@univ.uzhgorod.ua)

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For a simple model of extended source (a nucleus), we have got the exact normed solutions of the Dirac equation with a scalar-vector potential of the Coulomb type and a transcendental equation which determines the levels of the ground and excited electron states in the subcritical region $Z < Z_{\rm cr}$. We have constructed the equation for the critical charge of a nucleus, at which the level descends into the lower energy continuum. A strong influence of the Lorentz structure of interaction potentials on the critical charge and the discrete spectrum of a fermion in scalar and vector Coulomb-like fields is revealed.

In the recent years, a significant interest is attracted to the study of the behavior of the quantum systems of fermions in the joint presence of electromagnetic (vector) and scalar external fields. Such systems possess a number of unordinary features which significantly differ from those inherent in fermions in the presence of only the electromagnetic field. For example, a scalar field acts identically on particles and antiparticles, as distinct from the electromagnetic field. Therefore, the pattern of the energy levels of fermions, which interact with scalar and vector (for example, Coulomb) fields simultaneously, can significantly differ from the customary spectrum of the relativistic Coulomb problem. This is manifested, in particular, in that the discrete spectra of particles and antiparticles are symmetric relative to the zero level ($E = 0$) in the case of the interaction of massive fermions with a purely scalar external field.

We note also that the spin-orbital interactions have opposite signs for the scalar and vector fields. In the vector field, spins are oriented in the direction $[\vec{F}, \vec{p}]$, where $\vec{F}$ is the force acting on a particle, and $\vec{p}$ is its momentum, whereas they are oriented in the direction $-[\vec{F}, \vec{p}]$ in the scalar field. This reasoning gives a clear explanation to the fact that the level $j = 3/2$, $l = 1$ lies under the level $j = 1/2$, $l = 1$ in the scalar field. In the vector field, the situation is opposite.

To get agreement with the physical situation, the mathematical studies of phenomena which occur in strong vector (for example, electric) and scalar fields should be performed on the basis of the exact solutions of relativistic wave equations under conditions of a nonzero external field. Contrary to the case of the interaction with the electric field which is introduced into the Dirac free equation minimally as the time component of the 4-potential A_μ (vector coupling), the account of the interaction of a massive fermion with a scalar external field S is realized with the help of the change $m_0c^2 \to m_0c^2 + S$ (scalar coupling). Then, in the presence of static scalar $S(\vec{r})$ and electrostatic $V(\vec{r})$ external fields, the Dirac equation takes the form

$$\left[c\vec{\alpha}\hat{\vec{p}} + \beta\left(m_0c^2 + S(\vec{r})\right) - \left(E - V(\vec{r})\right)\right]\Psi(\vec{r}) = 0, \qquad (1)$$

where $\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ and β – the standard Dirac matrices, $\hat{\vec{p}}$ – the operator of momentum, and E and m_0 – total energy and the rest mass of a particle, respectively. We emphasize that $S(\vec{r})$ is a Lorentz-scalar, and $V(\vec{r})$ is the zero component of a Lorentz-vector.

Within a simple model of the interaction of a fermion with scalar and vector external fields of the Coulomb type,

$$V(r) = -\hbar c\frac{\alpha_V}{r}, \qquad S(r) = -\hbar c\frac{\alpha_S}{r}, \qquad (2)$$

where α_S and α_V stand for, respectively, the scalar and electrostatic coupling constants, the solutions and the spectrum of the Dirac equation were determined in [1, 2]. This model is frequently used as the initial approximation on the relativistic description of the spectra of such “exotic” hydrogen-like (HL) systems as leptoatoms (see, e.g., [3, 4]), whose components interact through the exchange of quanta of the fields of two different types. Whereas the Coulomb interaction is conditioned by the exchange of a virtual photon (a quantum of the electromagnetic field), the lepton-nucleus interaction which is responsible for the scalar coupling can be realized by the exchange of a virtual neutral particle with spin 0. The main candidate for this role is a scalar σ-meson. We note that the theory gives serious arguments in the favor of its existence (see, e.g.„ [5] and references therein). In addition, two experimental groups [6,7] have recently reported on the observation of an anomalously wide scalar resonance in the cascades of nonleptonic decays of the heavy (D, B, and J/Ψ) mesons. We pay attention also to the fact that a scalar meson observed in the mentioned experiments has a quite great mass ($M_\sigma = 390$ MeV [6–8]) and, therefore, the scalar potential $S(r)$ which corresponds to the exchange by such a particle (named below as the one-meson exchange potential) is, in fact, a short-range one (of the Yukawa-type). Nevertheless, many interesting peculiarities of the energy spectrum of leptoatoms can be revealed already in the frame of the comparatively simple model (2) with the scalar-vector interaction (see [1–4]). These peculiarities are conserved also on a more realistic consideration. One of the important advantages of the given model consists in that it admits the exact solution of the Dirac equation (1) in terms of the known special functions (confluent hypergeometric functions) [2–4]. These solutions can be used as a basis for the construction of various corrections which take the "realistic" (Yukawa) form of the one-meson exchange potential $S(r)$, the effects of the motion and the structure of the nucleus, and the radiation-related corrections into account. This explains, possibly, the great attention (see, e.g., [9, 10]) given recently to the relativistic problem of the motion of a fermion in an external scalar field and an external electric one of the Coulomb type (2).

The additional stimuli to study similar problems appear recently in the theory of strong interactions – quantum chromodynamics (QCD) and QED of superstrong Coulomb fields. In the first case, we will consider the models of the structure of mixed mesons (QCD-analogs of HL-atoms) which are composed, for example, of one light antiquark $\bar{q}$ and one heavy quark Q (Q$\bar{q}$-mesons (see, e.g., [11–13]). Let us consider the Dirac equation in the approximation of an infinitely heavy quark Q as the equation of motion for one light antiquark $\bar{q}$. We can study (like the case of HL-atoms) a number of important aspects of the theory of (heavy quark)-(light antiquark) systems: the relativistic dynamics of a light antiquark $\bar{q}$ in the external field formed by a heavy quark Q, the Lorentz-structure of the long-range (confining) part of the Q$\bar{q}$-interaction, the fine structure of the spectrum of mixed mesons, the influence of a spontaneous breaking of the chiral symmetry on the spectrum, etc. As known from QCD [11–13], the ordinary Coulomb potential of the one-gluon exchange $V(r) = -4\alpha_s \hbar c/(3r)$, where α_s is the strong interaction constant, gives the main contribution at small distances to the Q$\bar{q}$-interaction due to the phenomenon of asymptotic freedom. With increase in the distance, the scalar confining interaction becomes the basic one (confinement). But its "exact" form has not been else established. The first-principles QCD-calculations on lattices [14] separate the linear (scalar) confinement $S(r) = \hbar c\sigma r$ at great distances, where σ is the tension of a string. It is clear that all other interactions are important on a more comprehensive description of the properties of mesons, but they are low-intensity interactions as compared with the scalar potential which binds quarks into mesons. We will not consider these questions further, because they are sufficiently completely clarified in review [14].

The particular interest in the above-considered circle of problems arises recently in QED. As well known (see [3, 15]), the essential theoretical parameters in the electrodynamics of superstrong Coulomb fields are the critical charge of a nucleus $Z_{\rm cr}$ and the critical distance $R_{\rm cr}$ in the system of two colliding heavy nuclei. If these parameters are attained, the ground level of the electron spectrum descends to the boundary of the lower continuum. Then (i.e., at $Z = Z_1 + Z_2 > Z_{\rm cr}$ or $R < R_{\rm cr}$) the spontaneous generation of positrons from vacuum becomes possible. The experimental observation of this effect would mean the verification of the status of QED and the Dirac equation in the new region of superstrong fields, rather than in the traditional direction of superhigh energies and small distances. However, the experiments started almost a quarter of the century at GSI (Darmstadt, Germany) on the accelerator of heavy ions UNILAC gave no positive result in the search for this fundamental process. In view of such a situation, a number of theorists (e.g., [2, 16]) has considered different modifications of QED and their influence on the spontaneous generation of positrons. In particular, let us accept the viewpoint of the authors of the well-known book [3] (see also [2]) who believe that, under conditions of the experiments at GSI (i.e., in superstrong Coulomb fields), an additional Yukawa scalar one-meson exchange potential $S(r)$ appears in the interaction of an electron with the nucleus (together with the Coulomb potential). If so, of primary importance becomes the question about the influence of this scalar potential on the critical charge $Z_{\rm cr}$ and the critical distance $R_{\rm cr}$. In this case, the qualitative aspect of the question can be elucidated within a model with Coulomb-like scalar interaction (2).

Finally, we note that the spinor equation (1) with mixed scalar-vector coupling is of interest from the viewpoint of its possible application in the theory of hadronic atoms [17]. In principle, it cannot be excluded that the same equation can be also useful for the description of some effects in solid-state physics (for example, in two-band semiconductors [18]). Because the interest in the above-mentioned physical applications of model (2) will increase undoubtedly in the future, the statement of a relativistic Coulomb problem for the Dirac equation with scalar and vector potentials of the Coulomb type seems to be expedient.

The structure of the present work is as follows. The second section has auxiliary character. It contains the statement of the problem and the brief analysis of peculiarities of the motion of a relativistic electron in the external scalar-vector field (2) of a point-like source (a nucleus). In the third section, we consider the solution and the spectrum of the Dirac equation with the mixed Lorentz-structure of the interaction potential for an electron in the field of the nucleus with charge $Z > 137$, when the "drop to the center" occurs in the approximation of of a point charge. The account for the finite size of the nucleus, which leads to the regularization of the scalar and vector Coulomb-like potentials (2) as $r \to 0$, allows us to pass through the point $Z = 137$ up to the critical value $Z_{\rm cr}$, at which the energy level reaches the boundary of the lower continuum $E = -m_0c^2$. In particular, within a simple model of extended source, we will obtain a transcendental equation which determines implicitly the energy levels of the ground and excited electron states in the region $Z < Z_{\rm cr}$. In detail, we will consider the case of the limitedly small cut-off radius if the Coulomb-like vector and scalar fields, where the parameter $\Lambda = \ln(\lambda_c/r_N) \gg 1$, where $\lambda_c = \hbar/m_0c$ is the Compton wavelength. In this case, the small parameter Λ^{-1} appears in the problem. This allows us to determine the asymptotic formulas for the critical charge of a nucleus and the ground-state energy in the region $r_N \ll \lambda_c$. In the fourth section, we will get the equation for the determination of the critical charge $Z_{\rm cr}$, at which the ground-state level of the electron spectrum descends to the boundary of the lower continuum, and the spontaneous generation of positrons from vacuum becomes possible. We study the dependence of $Z_{\rm cr}$ on the scalar coupling constant α_S and reveal the strong influence of the Lorentz structure of interaction potentials on the critical charge of a nucleus and the energy spectrum of a spinor particle in the external scalar-vector field.

1. Exact Solution of the Dirac Equation with Scalar and Vector Potentials of the Coulomb Type

In this section, we consider the given problem in the approximation, in which the size and the structure of a nucleus can be neglected. We assume that the scalar-vector interaction potential is determined by formula (2) for all values $0 \leqslant r < \infty$. Taking the central symmetry of the potential energy of a fermion in such a field into account, it is convenient to pass into the spherical coordinate system with the origin located at the nucleus. Respectively, we will seek the wave function $\Psi(\vec{r})$ of the stationary state (in the standard representation) in the form

$$\Psi(\vec{r}) = \frac{1}{r}\begin{pmatrix} F(r)\Omega_{jlm}(\vec{n}) \\ (-1)^{l-l'+1}G(r)\Omega_{jl'm}(\vec{n}) \end{pmatrix}, \ \vec{n} = \vec{r}/r, \qquad (3)$$

where $\Omega_{jlm}(\vec{n})$ – spherical spinor [19], $j = 1/2, 3/2, ...$ – total angular momentum, $l = j \pm 1/2$ – orbital angular momentum, $l' = 2j - l$, $m = -j, -j+1, ..., j$ – projections of the total angular momentum onto the quantization axis.

Substituting (3) in Eq. (1), we get the system of equations for radial functions

$$\left.\begin{aligned} &\frac{dF}{dr} + \frac{k}{r}F - \frac{1}{\hbar c}\left[E - V(r) + m_0c^2 + S(r)\right]G = 0, \\ &\frac{dG}{dr} - \frac{k}{r}G + \frac{1}{\hbar c}\left[E - V(r) - m_0c^2 - S(r)\right]F = 0, \end{aligned}\right\} \qquad (4)$$

where $k = \pm(j + 1/2)$.

According to the character of the behavior of the radial functions $F(r)$ and $G(r)$ in the asymptotic regions of large and small r, we seek the solutions of the system of equations (4) with the scalar-vector interaction (2) in the form [3, 4]

$$F = \sqrt{m_0c^2 + E}\, e^{-\rho/2}\rho^{\gamma}(Q_1 + Q_2),$$

$$G = -\sqrt{m_0c^2 - E}\, e^{-\rho/2}\rho^{\gamma}(Q_1 - Q_2), \qquad (5)$$

where we introduce the notation

$$\rho = 2\lambda r, \ \lambda = \sqrt{m_0^2c^4 - E^2}/(\hbar c), \ \gamma = \sqrt{k^2 - \alpha_V^2 + \alpha_S^2}.$$

The solution, which is finite as $\rho \to 0$, of the system of equations (4) can be represented in terms of a confluent hypergeometric function $F(a,b;z)$ by using relation (5) and the equalities

$$Q_1 = AF\left(\gamma-\chi, 2\gamma+1;\rho\right),$$

$$Q_2 = -BF(\gamma+1-\chi, 2\gamma+1;\rho), \tag{6}$$

where $\chi = (\alpha_V E + \alpha_S m_0 c^2)/(\hbar c\lambda)$. Setting $\rho = 0$ in one of the equations for the functions Q_1 and Q_2, we find the connection between the constants A and B:

$$B = \frac{\hbar c\gamma\lambda - \alpha_V E - \alpha_S m_0 c^2}{\hbar c k\lambda - \alpha_V m_0 c^2 - \alpha_S E}\,A. \tag{7}$$

The condition of finiteness of the radial wave functions $F(r)$ and $G(r)$ as $r\to\infty$ yields the equation for the possible values of energy in the form

$$\frac{\alpha_V E + \alpha_S m_0 c^2}{\hbar c\lambda} = n_r + \gamma, \tag{8}$$

where n_r – nonnegative integer, and

$$n_r = \begin{cases} 0,1,2,..., & k<0; \\ 1,2,3,..., & k>0. \end{cases}$$

For bound states ($E < m_0c^2$), the wave function (3) must be normed by the condition $\int|\Psi|^2 d\vec{r} = 1$; this yields the condition for the intermediate normalization for radial functions:

$$\int\limits_0^\infty \left(F^2+G^2\right)dr = 1. \tag{9}$$

By integrating, we get the formula for the common normalizing constant

$$A = \frac{1}{\Gamma(2\gamma+1)}\sqrt{\frac{\lambda\Gamma(2\gamma+n_r+1)(N-k)}{2m_0c^2N\,n_r!}}, \tag{10}$$

where $N = (\alpha_V m_0 c^2 + \alpha_S E)/(\hbar c\lambda)$.

Collecting the obtained formulas, we can write the final expressions for the normed radial wave functions:

$$\left.\begin{matrix} F \\ G \end{matrix}\right\} = \pm A\sqrt{m_0c^2 \pm E}\,\rho^\gamma e^{-\rho/2}\left[F(-n_r, 2\gamma+1;\rho)\mp\right.$$

$$\left.\mp\, n_r(N-k)^{-1}F(-n_r+1, 2\gamma+1;\rho)\right] \tag{11}$$

(the upper and lower signs are referred, respectively, to F and G).

Solving Eq. (8) for E, we obtain the formula for the discrete energy levels [3,4]:

$$E = m_0c^2\frac{\pm(n_r+\gamma)\sqrt{n_r(n_r+2\gamma)+k^2} - \alpha_V\alpha_S}{\alpha_V^2 + (n_r+\gamma)^2}. \tag{12}$$

Formula (12) determines the so-called fine structure of the energy levels of a relativistic HL-atom and is a generalization of the well-known formula of Dirac–Sommerfeld [19] to the case of a scalar-vector interaction of the Coulomb type. In what follows, we will take the positive sign of the root in (12) and consider only electron levels; for $\alpha_S = 0$, the second branch of the energy spectrum leads to a side solution of Eq. (8).

Setting $n_r = 0$ and $k = -1$ in (12), we get the energy of the electron on the lowest level:

$$E_0 = m_0c^2\frac{\sqrt{1-\alpha_V^2+\alpha_S^2} - \alpha_V\alpha_S}{1+\alpha_S^2}. \tag{13}$$

Let us discuss the content of formula (13). As seen, with increase in the vector coupling constant $\alpha_V = Z\alpha$ (where Z – charge of the nucleus of an atom, and $\alpha \approx 1/137$ – fine structure constant of a relativistic HL-atom), the ground-state energy E_0 decreases, passes through zero at $\alpha_V = 1$, and terminates at $\alpha_V = \sqrt{1+\alpha_S^2}$ (for the excited states, this occurs at $\alpha_V = \sqrt{k^2+\alpha_S^2}$). The continuation of formula (13) into the region $\alpha_V > \sqrt{1+\alpha_S^2}$ leads to complex values of the energy and to the oscillation of wave functions as $r\to 0$, which corresponds to the situation of the "drop to the center", which is inadmissible in relativistic theory. The appearance of this difficulty is related to the idealization of the problem, namely to the neglect of a finite size of the nucleus. At small values of the charge Z, the nucleus can be considered as point-like. That is, the account for its radius r_N gives very small corrections to the energies of levels. However, when α_V approaches $\sqrt{1+\alpha_S^2}$, the situation changes basically.

The account for the finite size of a nucleus removes the mentioned anomaly in the behavior of levels. Pomeranchuk and Smorodinsky were the first who noticed this fact [20] in 1945. Introducing the finite radius of a nucleus r_N, they showed that the solution of the ordinary Dirac equation (with the vector type of interaction) with the Coulomb potential cut-off at small distances exists in the whole region from $Z = 0$ ($E = m_0c^2$) to $Z = Z_{\rm cr}(r_N)$ ($E = -m_0c^2$) and estimated the critical charge $Z_{\rm cr}(r_N)$, at which the energy of the level $1s_{1/2}$ reaches $E = -m_0c^2$. Moreover, it turned out that, with increase in Z in the region $Z > 137$, the energy levels become negative and continue to descend down to the boundary of the lower continuum $E = -m_0c^2$. The analogous behavior of the energy levels of an electron is also observed in the case considered here of the scalar-vector interaction at $\alpha_V > \sqrt{1+\alpha_S^2}$. However, the intersection of the boundary of the lower continuum $E = -m_0c^2$ occurs at significantly greater values of the critical charge $Z_{\rm cr}$, than those in the purely vector case. The further discussion of the questions related to the movement of levels near $E = -m_0c^2$ and the analysis of methods of the determination of the critical charge will be performed in the next sections.

2. Discrete Spectrum at $\alpha_V > \sqrt{k^2+\alpha_S^2}$

In order to find the energy spectrum of an electron in the Coulomb field of the nucleus with $\alpha_V > \sqrt{k^2+\alpha_S^2}$, it is necessary to set some boundary condition at zero (which is equivalent to the determination of a self-adjoint extension of the energy operator [21]). Only after that the problem becomes mathematically correct [22,23]. Physically, the setting of the boundary condition at zero means the cut-off of potentials (2) at small distances, i.e., the account for the finite size of the nucleus.

We assume that $V(r)$ and $S(r)$ are Coulomb-like down to the surface of the nucleus. Inside the nucleus, let they look as

$$V(r) = \begin{cases} -\hbar c\frac{\alpha_V}{r}, & r > r_N, \\ -\hbar c\frac{\alpha_V}{r_N} f\left(\frac{r}{r_N}\right), & 0 \leqslant r \leqslant r_N, \end{cases} \quad (14)$$

$$S(r) = \begin{cases} -\hbar c\frac{\alpha_S}{r}, & r > r_N, \\ -\hbar c\frac{\alpha_S}{r_N} f\left(\frac{r}{r_N}\right), & 0 \leqslant r \leqslant r_N. \end{cases} \quad (15)$$

Here, $f(x)$ – cut-off function which takes the finite size of the nucleus into account, and $0 \leqslant x = r/r_N \leqslant 1$. Most frequently are used two simple models of cut-off [20, 24–26]:

Model I. $f(x) = 1$, i.e., the rectangular cut-off. For a vector potential, this corresponds to the concentration of the whole electric charge on the surface of the nucleus.

Model II. $f(x) = (3 - x^2)/2$, which corresponds to a uniform distribution of the charge over the bulk of the nucleus in the vector case.

In order to obtain the spectrum of the Dirac equation with potentials (14) and (15) and to determine the critical charge $Z_{\rm cr}$, it is necessary to solve this equation inside $(0 < r < r_N)$ and outside $(r > r_N)$ of the nucleus, which requires to carry out the numerical calculations for model II at $0 < r < r_N$. We restrict ourselves by the model of rectangular cut-off of both potentials, for which the Dirac equation can be solved in analytic form. The more realistic choice of the potential form inside of the nucleus is mainly reduced to the increase of the maximum values of $V(0)$ and $S(0)$ by a factor of 1.5, which affects slightly the final results (see, e.g., [26] in the purely vector case). We now pass to the description of the procedure of solution of the system of Dirac equations (4) at $0 \leqslant r \leqslant r_N$.

Excluding the function $G(r)$ from system (4), we get the equation for $F(r)$ in the form

$$\frac{d^2F(r)}{dr^2} + \left[K^2 - \frac{k(k+1)}{r^2}\right] F(r) = 0. \quad (16)$$

Here, we took into account that $V, S = const$ in the entire interior region $0 \leqslant r \leqslant r_N$, and the constant

$$K = \frac{\sqrt{(E-V)^2 - (m_0c^2+S)^2}}{\hbar c}.$$

The general solution of Eq. (16) looks as

$$F(r) = \sqrt{r}\left[\widetilde{A} J_{|k+1/2|}(Kr) + \widetilde{B} N_{|k+1/2|}(Kr)\right], \quad (17)$$

where $J_n(x)$ and $N_n(x)$ are, respectively, the Bessel and Neumann functions of the integer order n [27]. Writing $G(r)$ in terms of $F(r)$ with the help of (4) and using the recurrence relations for the functions $J_n(x)$ and $N_n(x)$, we get the corresponding formula for the lower component:

$$G(r) = {\rm sgn}k \frac{\hbar c K\sqrt{r}}{E - V + m_0c^2 + S}\left[\widetilde{A} J_{|k-1/2|}(Kr) + \right.$$

$$\left. + \widetilde{B} N_{|k-1/2|}(Kr)\right]. \quad (18)$$

The condition of the finiteness of $F(r)$ at $r = 0$ implies that $\widetilde{B} = 0$. Then the final formulas for the radial functions in the interior region $0 \leqslant r \leqslant r_N$ take a simpler form

$$F(r) = \widetilde{A}\sqrt{r} J_{|k+1/2|}(Kr), \quad (19)$$

$$G(r) = \widetilde{A}\, {\rm sgn}k \frac{\hbar c K\sqrt{r}}{E - V + m_0c^2 + S} J_{|k-1/2|}(Kr). \quad (20)$$

In the exterior region $r > r_N$, the potentials $V(r)$ and $S(r)$ are Coulomb-like, and the solution of the Dirac system (4), which exponentially decreases at infinity, is determined by formulas of the type (5), and the functions Q_1 and Q_2 are expressed through confluent hypergeometric functions analogously to (6). However, it is necessary now to take both signs of the quantity $\gamma = \sqrt{k^2 - \alpha_V^2 + \alpha_S^2}$ into account. Therefore, instead of (6), we have the representation

$$Q_j = C_j \Psi(\chi_j, 2\gamma + 1; \rho),\ j = 1, 2, \tag{21}$$

where $\chi_1 = \gamma - \chi$, $\chi_2 = \chi_1 + 1$, C_1 and C_2 – some constants, and $\Psi(a, b; z)$ – irregular solution of the confluent hypergeometric equation. The regular solution $F(a, b; z)$ of this equation is not suitable because of its growth at infinity. Substituting (21) and (5) in (4) and using the recurrence relations between confluent hypergeometric functions [27], we get the connection between the constants C_1 and C_2:

$$\frac{C_2}{C_1} = k + \frac{\alpha_V m_0 c^2 + \alpha_S E}{\hbar c \lambda} = k + N. \tag{22}$$

The relation between the constants $\widetilde{A}$ and C_1 will be established by sewing the formulas for $F(r)$ obtained for the interior (see (19)) and exterior (formulas (5) and (21)) regions at the point $r = r_N$:

$$\frac{\widetilde{A}}{C_1} = \sqrt{\frac{m_0 c^2 + E}{r_N}} \frac{\rho_N^\gamma e^{-\rho_N/2}}{J_{|k+1/2|}(K r_N)} [\Psi(\chi_1, 2\gamma + 1; \rho_N) +$$

$$+ (k + N)\Psi(\chi_2, 2\gamma + 1; \rho_N)] \quad (\rho_N = 2\lambda r_N). \tag{23}$$

We will determine the constant C_1 (to within the phase factor) from the condition of intermediate normalization (9):

$$|C_1| = \left[\frac{2m_0 c^2 \nu \Gamma(2\gamma + 1)\Gamma(-2\gamma)}{\lambda \Gamma(\chi_2)\Gamma(\chi_2 - 2\gamma)}\right]^{-1/2}, \tag{24}$$

where $\nu = \xi C(C - k) - \gamma(2\varepsilon C + 1)$, $\xi = \psi(\chi_2 - 2\gamma) - \psi(\chi_2)$, $\varepsilon = E/m_0 c^2$, and $C = k + N$ (the mathematical details of the calculation of this integral are given in Appendix).

The sewing of the ratio G/F for the inner and outer solutions at the edge of the nucleus (at $r = r_N$) gives the equation

$$-\frac{\sqrt{m_0 c^2 - E}}{\sqrt{m_0 c^2 + E}} \frac{Q_1 - Q_2}{Q_1 + Q_2}\bigg|_{r = r_N} = A_k \tag{25}$$

which determines the spectrum of the Dirac equation in the region $-m_0 c^2 \le E \le m_0 c^2$. Here, A_k is the ratio of functions (19), (20) at $r = r_N$:

$$A_k = \mathrm{sgn} k \frac{\hbar c K}{E - V + m_0 c^2 + S} \frac{J_{|k-1/2|}(K r_N)}{J_{|k+1/2|}(K r_N)}. \tag{26}$$

Using the recurrence relation [27]

$$\frac{d\Psi(a, b; z)}{dz} = \frac{a}{z}\left[(a - b + 1)\Psi(a + 1, b; z) - \Psi(a, b; z)\right],$$

we write Eq. (25) in a more compact form

$$\rho_N \frac{\Psi'(\chi_1, 2\gamma + 1; \rho_N)}{\Psi(\chi_1, 2\gamma + 1; \rho_N)} = \frac{A_k(k + t\alpha_-) - kt - \alpha_+}{A_k + t} - \gamma, \tag{27}$$

where $t = -\sqrt{(m_0 c^2 - E)/(m_0 c^2 + E)}$, $\alpha_\pm = \alpha_V \pm \alpha_S$, the prime means the derivative with respect to ρ_N, and the parameters λ and χ_1 are the same as those in (21). The obtained exact equation (27) for the energy levels has a quite nontrivial analytic structure and would be not suitable for direct calculations. Therefore, it is expedient to make attempt to simplify relation (27), at least in some limit cases. It follows from the consideration of the significantly more completely studied relativistic Coulomb problem in the purely vector case [26], the simplifications are possible in the approximation of a small cut-off radius of the Coulomb field. Let us see as this approximation works in the case of Eq. (27).

Let us extrapolate the relation $r_N = R_0 A^{1/3}$ onto the region $Z > 137$, by taking (like for heavy nuclei) $A = 2.5Z$ and $R_0 = 1.1$ fm. Then the nucleus radius r_N turns out small as compared with the Compton wavelength of an electron (for example, we have $r_N \approx 0.02$ in units of $\hbar/m_0 c$ at $Z = 170$), and we can use the approximation $K \approx \sqrt{\alpha_+ \alpha_-}/r_N$. In this case, relation (26) losses the dependence on the energy E and takes a simpler form

$$A_k = \mathrm{sgn} k \sqrt{\frac{\alpha_+}{\alpha_-}} \frac{J_{|k-1/2|}(\sqrt{\alpha_+ \alpha_-})}{J_{|k+1/2|}(\sqrt{\alpha_+ \alpha_-})}. \tag{28}$$

The further simplifications of Eq. (27) are possible on the use of the expansion of the function $\Psi(a, b; z)$ near zero:

$$\Psi(a, b; z) = \frac{\Gamma(1 - b)}{\Gamma(a - b + 1)} + \frac{\Gamma(b - 1)}{\Gamma(a)} z^{1-b} + \ldots. \tag{29}$$

Calculating the logarithmic derivative of function (29), using the properties of Γ-functions, and substituting the result in (27), we arrive at the equation

$$(2\lambda r_N)^{2\gamma} = \frac{2\gamma \sin(2\pi\gamma)\Gamma^2(2\gamma)\Gamma(1 + \chi - \gamma)}{\pi \Gamma(1 + \chi + \gamma)} \times$$

$$\times \frac{\sin[\pi(\chi - \gamma)]}{\sin[\pi(\chi + \gamma)]} \frac{(A_k \alpha_- - k + \gamma)(t\alpha_- + k + \gamma)}{(A_k \alpha_- - k - \gamma)(t\alpha_- + k - \gamma)} \tag{30}$$

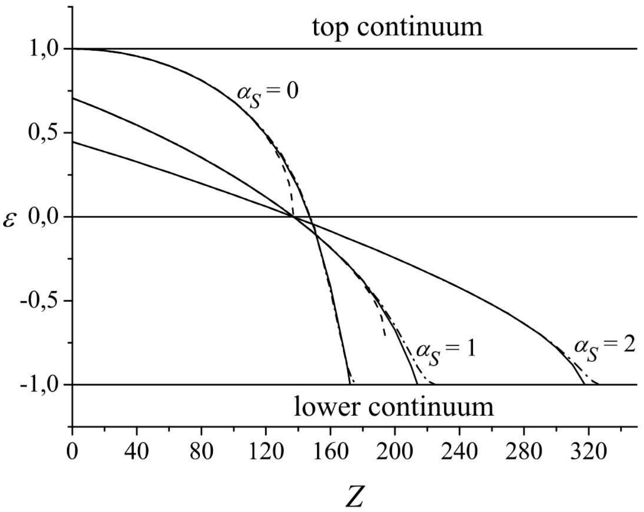

Fig. 1. Ground-state energy ($\varepsilon = E/mc^2$) versus the charge Z of the nucleus of an atom for the scalar coupling constant $\alpha_S = 0, 1, 2$: solid lines – numerical solutions of Eqs. (30) and (31); dash and dash-dotted lines are the results of calculations by formulas (13) and (40), respectively

which is convenient for the analysis of states of the discrete spectrum at $\alpha_V < \sqrt{k^2 + \alpha_S^2}$, when the quantity γ is real. It is easy to see that formula (30) is transformed into (8) as $r_N \to 0$, $\gamma \neq 0$, and the energy spectrum is described by equality (12).

In the case $\alpha_V > \sqrt{k^2 + \alpha_S^2}$ where $\gamma = i\theta$ becomes a purely imaginary quantity, we obtain the transcendental equation

$$\mathrm{ctg}\{\theta \ln(2\lambda r_N) - \arg[\Gamma(1 + 2i\theta)/\Gamma(-\chi + i\theta)]\} =$$

$$= \frac{(A_k \alpha_- - k)(t\alpha_- + k) - \theta^2}{\theta \alpha_- (A_k + t)} \tag{31}$$

which together with (30) determine implicitly the energy levels of an electron for the ground and excited states with regard for the finiteness of the size of the nucleus according to model I.

Equations (30), (31) are more simple than Eq. (26). However, their solution requires numerical calculations. In Fig. 1, we show the numerical solutions of Eqs. (30) and (31) for the ground state $1s_{1/2}$ for various values of the scalar coupling constant α_S. Let us analyze the movement of energy levels as functions of the charge $Z = 137\alpha_V$. First, we consider the purely vector case ($\alpha_S = 0$). Starting from $Z = 0$, the energy level descends to the Z axis and crosses it at $Z = 137$. Then the solutions of the Dirac equation for a point-like nucleus loss sense (dash line). The energy level of an electron in a HL-atom with the finite-size nucleus crosses the threshold $E = 0$ at $Z \approx 147$ (solid line) and descends into the lower continuum at $Z_{\mathrm{cr}} \approx 172$, where it becomes quasistationary. For $\alpha_S = 1$ and $\alpha_S = 2$, the level reaches zero at $Z \approx 137$ and descends to the boundary of the lower continuum at $Z_{\mathrm{cr}} \approx 214$ and $Z_{\mathrm{cr}} \approx 318$, respectively.

3. Logarithmic Approximation

Though we have already used the approximation $r_N \ll \lambda_c$ ($\lambda_c = \hbar/m_0c = 3.86 \cdot 10^{-11}$ cm is the Compton wavelength) in the derivation of Eqs. (30) and (31), we can get some analytic estimates for the energy levels and the critical charge Z_{cr}. To this end, following works [25], we impose an additional condition $|\ln(r_N/\lambda_c)| \gg 1$, which leads to the appearance of a large parameter $\Lambda = -\ln(r_N/\lambda_c) \gg 1$. For the size of nuclei $r_N \sim 10^{-12}$ cm, this parameter is not very large ($\Lambda \approx 3.5$), but such an approximation will give, as we will see below, a proper general pattern of the movement of the levels with change in Z.

Thus, we now pass to the practical use of the approximation $\Lambda \gg 1$. As $r_N \to 0$, the value of α_V is close to $\sqrt{k^2 + \alpha_S^2}$, and $\theta \to 0$. Therefore, we can set $\alpha_V = \sqrt{k^2 + \alpha_S^2}$ on the right-hand side of Eq. (31), and, according to (28),

$$A_k = \frac{k}{\tilde{\alpha}} \frac{J_{|k-1/2|}(|k|)}{J_{|k+1/2|}(|k|)}, \quad \tilde{\alpha} = \sqrt{k^2 + \alpha_S^2} - \alpha_S. \tag{32}$$

Since, as $\theta \to 0$,

$$\varphi = 2\theta\psi(1) - \arg\Gamma(-\chi + i\theta) + O(\theta^2),$$

where $\psi(z)$ is the logarithmic derivative of the Γ-function, Eq. (31) is reduced to

$$\frac{\pi n'}{\theta} = \frac{1}{A_k\tilde{\alpha} - k} + \frac{1}{t\tilde{\alpha} + k} - \frac{1}{\theta}\arg\Gamma(-\chi + i\theta) +$$

$$+ 2\psi(1) - \ln(2\lambda r_N), \tag{33}$$

where $n' = n_r + (1 - \mathrm{sgn}k)/2$ is an integer which enumerates the energy levels. As $\theta \to +0$, the function $\omega(x, \theta) = \arg\Gamma(x + i\theta)$ has discontinuities near the points $x = -n$, in which the poles of the Γ-function are positioned. Indeed, if $|x + n| \gg \theta$, then, to within terms of the order of θ^2,

$$\omega(x, \theta) = \begin{cases} \theta\psi(x), & x > 0, \\ -(n+1)\pi + \theta\psi(x), & -(n+1) < x < -n, \end{cases} \tag{34}$$

where $\psi(x)$ is the logarithmic derivative of the Γ-function. In the immediate neighborhood of the pole $x=-n$,

$$\omega(x,\theta)=-\left(n\pi+\operatorname{arcctg}\frac{x+n}{\theta}\right). \quad (35)$$

In the region $\theta\ll|x+n|\ll 1$, formulas (34) and (35) sew with each other, and both give

$$\omega(x,\theta)=-\left[(n+\nu)\pi+\theta/(x+n)+...\right], \quad (36)$$

where $\nu=0$ for $x>-n$ and $\nu=1$ for $x<-n$.

First, we set $E=-m_0c^2$. Then $\chi\to-\infty$. In view of relations (33) and (34) and the asymptotics of the digamma-function $\psi(z)$ [27]

$$\psi(z)=\ln z-\frac{1}{2z}+... \quad (z\to\infty,\ |\arg z|<\pi), \quad (37)$$

we get the relation

$$\frac{\pi n'}{\theta_{\rm cr}}=\Lambda+\frac{1}{A_k\tilde{\alpha}-k}+2\psi(1)-\ln(2\tilde{\alpha}) \quad (38)$$

for the determination of $\theta_{\rm cr}=\sqrt{\alpha_{V\rm cr}^2-\alpha_S^2-k^2}$ ($\alpha_{V\rm cr}=Z_{\rm cr}\alpha$). This yields

$$\alpha_{V cr}=\sqrt{k^2+\alpha_S^2}+\frac{\pi^2n'^2}{2\sqrt{k^2+\alpha_S^2}\,\Lambda^2}+O(\Lambda^{-3}). \quad (39)$$

It follows from (39) that, as $\Lambda\gg1$, the principal term of the asymptotics depends only on the cut-off radius r_N, and the dependence on the specific form of the cut-off function $f(x)$ arises only in the terms of the expansion (39) higher by order.

From formulas (33) and (35), we can get the explicit formula

$$E=m_0c^2\frac{(n_r+g)\sqrt{\alpha_V^2-\alpha_S^2+(n_r+g)^2}-\alpha_V\alpha_S}{\alpha_V^2+(n_r+g)^2}, \quad (40)$$

where $g=\theta\operatorname{ctg}(\theta\Lambda)$.

Possessing the analytic formula for the energy level (40), we will trace how the account for the finite size of the nucleus removes the singularity of formula (13) for $\alpha_V=\sqrt{k^2+\alpha_S^2}$. After the change $\theta\to-i\gamma$, $g\to\gamma\operatorname{cth}(\gamma\Lambda)$, formula (40) remains true also for $\alpha_V<\sqrt{k^2+\alpha_S^2}$. In the region $\alpha_V<\sqrt{k^2+\alpha_S^2}$ (under the condition $\Lambda\gamma\gg1$), $\operatorname{cth}(\gamma\Lambda)$ tends rapidly to 1, and formula (40) is transformed into (13). On the other hand, the point $\alpha_V=\sqrt{k^2+\alpha_S^2}$, $\gamma=0$, is not already singular for the function $E(Z)$. The less the nucleus radius, the steeper the level curve enters into the lower continuum.

4. Critical Charge of a Nucleus. Efficient Size of the System for $Z>137$

We now consider the solutions of the system of Dirac equations (4) for $E=-m_0c^2$ and will determine the corresponding critical values of the charge, $Z_{\rm cr}$. Excluding the function $G(r)$ from system (4), we get the equation

$$F''-\frac{V'-S'}{V-S}F'-\left[\frac{k(k+1)}{r^2}+\frac{k}{r}\frac{V'-S'}{V-S}-\right.$$

$$\left.-\frac{V-S}{\hbar^2c^2}(2m_0c^2+V+S)\right]F=0. \quad (41)$$

In the region $r>r_N$, the solution convergent at infinity is expressed (to within a constant) through the McDonald function of imaginary index [27]:

$$F(r)=K_{2i\theta}(\sqrt{8\alpha_-r/\lambda_c}),\ G(r)=(rF'+kF)/\alpha_-. \quad (42)$$

In the interior region $0\leqslant r\leqslant r_N$ (for model I of the cut-off function of the vector and scalar potentials (14) and (15)), the solution of system (4) is given by formulas (19) and (20), where we should change $E\to-m_0c^2$.

Sewing the obtained solutions at the point $r=r_N$,

$$\left(\frac{G(r)}{F(r)}\right)_{r=r_N-0}=\left(\frac{G(r)}{F(r)}\right)_{r=r_N+0}, \quad (43)$$

we get the transcendental equation for the critical charge (at a fixed r_N)

$$\operatorname{sgn}k\,Kr_N\frac{J_{|k-1/2|}(Kr_N)}{J_{|k+1/2|}(Kr_N)}=\frac{x}{2}\frac{K'_{2i\theta}(x)}{K_{2i\theta}(x)}+k, \quad (44)$$

where $x=\sqrt{8\alpha_-r_N/\lambda_c}$. Since the parameter r_N is small as compared with the Compton wavelength of an electron λ_c, we use the asymptotic representation for the McDonald function at small values of the argument:

$$K_{2i\theta}(x)=\sqrt{\frac{\pi}{2\theta\,\operatorname{sh}(2\pi\theta)}}\sin P+...,$$

$$P=\arg\Gamma(1+2i\theta)-\theta\ln(2\alpha_-r_N/\lambda_c).$$

Considering only the main terms in the expansions of the McDonald function and its derivative and taking the

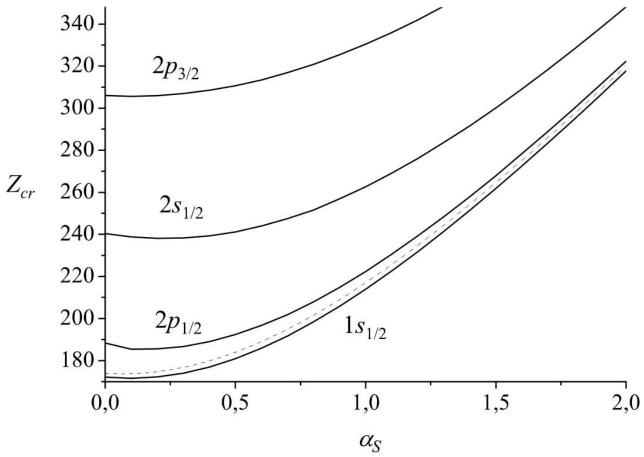

Fig. 2. Critical charge $Z_{\rm cr}$ versus the scalar coupling constant α_S for several low levels: solid lines – numerical solutions of Eqs. (45); dotted lines – the results of calculations by formula (38)

approximation $K \approx \sqrt{\alpha_+\alpha_-}/r_N$, we get Eq. (44) in the form

$$\theta \operatorname{ctg} P = k - \operatorname{sgn} k \sqrt{\alpha_V^2 - \alpha_S^2} \frac{J_{|k-1/2|}(\sqrt{\alpha_V^2 - \alpha_S^2})}{J_{|k+1/2|}(\sqrt{\alpha_V^2 - \alpha_S^2})}. \tag{45}$$

Equation (45) is transcendental relative to the critical value of the vector coupling constant $\alpha_V = \alpha_{Vcr} = Z_{\rm cr}\alpha$, where $Z_{\rm cr}$ – critical charge of the nucleus of an atom. The numerical solutions of Eq. (45) for several low states are shown in Fig. 2. It is seen that, for each of the levels, the function $Z_{\rm cr}(\alpha_S)$ has minimum at $\alpha_S \sim 0.1 \div 0.2$ and then grows sharply with increase in α_S. This means that the vacuum of quantum electrodynamics in a strong scalar-vector field of the Coulomb type should reveal the instability relative to the creation of electron-positron pairs at essentially higher values of the critical charge, than that in the purely vector case. For example, let $\alpha_S = 1.1$. In order that the process of spontaneous generation of positrons be started, it is necessary to bring nuclei with the total charge $Z_1 + Z_2 \geqslant Z_{\rm cr} = 222$ together. Thus, if the scalar interaction will turn out to be rather significant, this will make experiments aimed at the detection of this process to be practically impossible.

The dotted line in Fig. 2 shows the values of the critical charge for the ground state by formula (38) (the nucleus radius r_N was taken $0.02\,\hbar/m_0c$) which gives the more exact results than (39). As seen, the logarithmic approximation reproduces the results of numerical calculations for the lowest energy level quite well.

Of interest is the question on the localization of an electron, whose energy lies on the boundary of the band of the continuous spectrum $E \to -m_0c^2$. In the frame of the standard (purely vector) model, it was firstly assumed [24] that, as $Z \to Z_{\rm cr}$, there occurs the delocalization of the polarization of vacuum, i.e., the polarization charge goes away from the nucleus at arbitrary large distances. In this case, the main argument consisted in that the wave function of the bound state $F(r) \sim e^{-\lambda r}$ as $r \to \infty$, and the electron cloud will be seemingly delocalized as $E \to -m_0c^2$. However, the subsequent analysis showed that it is not the case (see, e.g., [25]). The analogous situation is observed also in our case. In formula (42), we use the asymptotics of the McDonald function at great values of the argument [27]:

$$K_{2i\theta}(x) = \sqrt{\frac{\pi}{2x}}\, e^{-x} \left(1 - \frac{4\theta^2 + 1/4}{2x} + ...\right). \tag{46}$$

It seen that $F(r) \sim \exp(-\sqrt{8\alpha_- r/\lambda_c})$ as $r \to \infty$ and $E = -m_0c^2$. The reason for the so sharp difference in the behaviors of the wave function of an electron at $E = \pm m_0c^2$ is the dependence of the effective potential on the sign of E. In our problem, the effective potential behaves itself as $-\hbar c(\alpha_V E/m_0c^2 + \alpha_S)/r$ for $r \to \infty$. That is, it is the attractive potential for $E = m_0c^2$ and the repulsive one for $E = -m_0c^2$ (at sufficiently large distances from the nucleus) for $\alpha_S = 0$. For $\alpha_S \neq 0$, none of the levels reaches the boundary of the upper continuous spectrum, and the comparison of these two limiting cases ($E = +m_0c^2$ and $E = -m_0c^2$) has no sense.

Thus, since the state of an electron for $E \to -m_0c^2$ remains bound and is not "distended", the arguments advanced in [24] in favor of the vacuum polarization delocalization are not valid. In view of the complexity of the question about the size of the bound state in the scalar-vector case, we consider it quantitatively. For this purpose, we will determine the mean radius of the system:

$$\bar{r} = \int\limits_0^\infty (F^2 + G^2) r dr. \tag{47}$$

Integral (47) is calculated in Appendix (see formulas (D1), (D2), (D8), and (D9)) and has form (in units of $\hbar/m_0c$)

$$\bar{r} = \{2C(C-k)(\chi\xi - 2\gamma) - \gamma(1 + \varepsilon C)(2\chi + 1) +$$

$$+\xi C[\varepsilon C(C-2k)-k]\}/(2\lambda\nu). \qquad (48)$$

The dependence of $\bar{r}$ on the charge, according to (48), for the ground state at $\alpha_S = 0,1,2$ is presented in Fig. 3. It is seen that, with increase in the charge, the effective size of the system decreases and remains finite on the boundary of the lower energy continuum $E = -m_0c^2$. Indeed, formula (48) at $E \to -m_0c^2$ gives the mean radius on the boundary of the lower continuum in the form

$$\bar{r} = \frac{4\theta^2+1}{10(\alpha_V-\alpha_S)}\,\frac{3\alpha_V-2\alpha_S+\tau(3-2k)}{2(2\alpha_V-\alpha_S)+\tau(1-2k)}, \qquad (49)$$

where $\tau = (1-k)/(\alpha_V-\alpha_S)$. For the ground state at $\alpha_S = 0,1,2$, respectively, we have $\bar{r} = 0.32$, 0.67, 1.12. That is, as was mentioned above, the bound state is not delocalized.

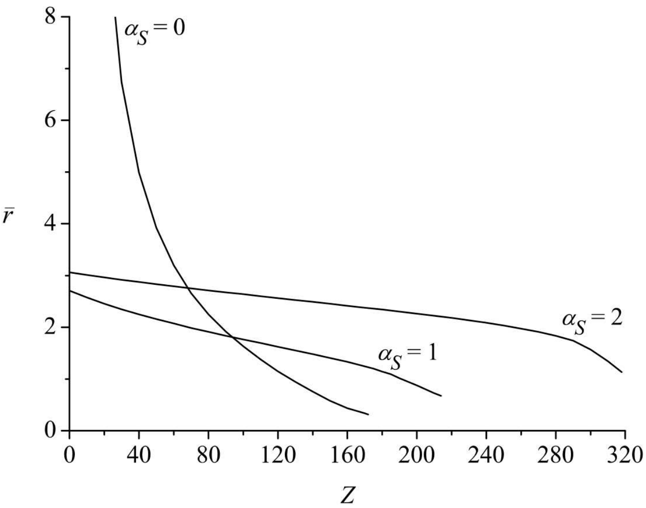

Fig. 3. Mean radius of the ground state as a function of the nucleus charge Z at $\alpha_S = 0,1,2$

APPENDIX

In order that to determine the normalization constant C_1 for the radial wave functions F and G (see Section 2) and the mean radius of the system $\bar{r}$ (see Section 4), it is necessary to calculate the integral

$$I_\mu = \int_0^\infty (F^2+G^2)r^\mu dr, \qquad \mu = 0,1. \qquad (D1)$$

Since the nucleus radius $r_N \ll \lambda_c$, and the region $0 \leqslant r \leqslant r_N$ gives a small contribution to integral (D1), we assume that the r adial wave functions F and G have the form (5) and (21) in the whole integration region $0 \leqslant r < \infty$. Then equality (D1) takes the form

$$I_\mu = \frac{2m_0c^2}{(2\lambda)^{\mu+1}}[X_\mu(\chi_1,\chi_1)+C^2X_\mu(\chi_2,\chi_2)+2\varepsilon CX_\mu(\chi_1,\chi_2)], \qquad (D2)$$

$$X_\mu(a,b) = \int_0^\infty \Psi(a,2\gamma+1;\rho)\Psi(b,2\gamma+1;\rho)\rho^{2\gamma+\mu}e^{-\rho}d\rho. \qquad (D3)$$

where $\varepsilon = E/(m_0c^2)$, $C = k+N$.

In order to calculate integral (D3), we replace each of the hypergeometric functions by its integral representation [27]:

$$\Psi(a,c;\rho) = \frac{1}{\Gamma(a)}\int_0^\infty e^{-\rho t}t^{a-1}(1+t)^{c-a-1}dt.$$

Then the integration with respect to ρ is reduced to the calculation of the integral

$$\int_0^\infty \rho^{2\gamma+\mu}e^{-\rho(t+t'+1)}d\rho = \frac{\Gamma(\zeta)}{(t+t'+1)^\zeta}, \qquad (D4)$$

where t and t' are the integration variables, and $\zeta = 2\gamma+\mu+1$. In view of (D4), the integration with respect to t will lead to the hypergeometric function ${}_2F_1(\zeta,a;a+\mu+1;t'/(t'+1))$. Expanding it in a series and integrating termwise, we get the relation

$$X_\mu(a,b) = \frac{(\mu!)^2\Gamma(\zeta)\,{}_3F_2(a,b,\zeta;a+\mu+1,b+\mu+1;1)}{\Gamma(a+\mu+1)\Gamma(b+\mu+1)}. \qquad (D5)$$

In the partial cases where $\mu = 0,1$ and $a,b = \chi_{1,2}$, formula (D5) takes a simpler form

$$X_0(\chi_1,\chi_1) = \frac{\Gamma(1+2\gamma)\Gamma(-2\gamma)}{\Gamma(\chi_1)\Gamma(\chi_1-2\gamma)}[\psi(\chi_1-2\gamma)-\psi(\chi_1)], \qquad (D6)$$

$$X_0(\chi_1,\chi_2) = \frac{\Gamma(1+2\gamma)\Gamma(1-2\gamma)}{\Gamma(\chi_2)\Gamma(\chi_2-2\gamma)}, \qquad (D7)$$

$$X_1(\chi_1,\chi_1) = \frac{\Gamma(1+2\gamma)\Gamma(-2\gamma)}{\Gamma(\chi_1)\Gamma(\chi_1-2\gamma)}\{(2\gamma-2\chi_1+1)\times$$
$$\times[\psi(\chi_1-2\gamma)-\psi(\chi_1)]-4\gamma\}, \qquad (D8)$$

$$X_1(\chi_1,\chi_2) = \frac{\Gamma(1+2\gamma)\Gamma(-2\gamma)}{\Gamma(\chi_2)\Gamma(\chi_2-2\gamma)}\{-\gamma(2\gamma-2\chi_1+1)+$$
$$+\chi_1(\chi_1-2\gamma)[\psi(\chi_2-2\gamma)-\psi(\chi_2)]\}, \qquad (D9)$$

where $\psi(x)$ is the digamma-function. $X_\mu(\chi_2,\chi_2)$ follows from $X_\mu(\chi_1,\chi_1)$ after the change $\chi_1 \to \chi_2$.

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КРИТИЧНИЙ ЗАРЯД У МОДИФІКОВАНІЙ КВАНТОВІЙ ЕЛЕКТРОДИНАМІЦІ

В.Ю. Лазур, О.К. Рейтій, В.В. Бондарчук, В.К. Рейтій

Р е з ю м е

Для однієї простої моделі протяжного джерела (ядра) отримано точні нормовані розв'язки рівняння Дірака зі скалярно-векторним потенціалом кулонівського типу, а також трансцендентне рівняння, що визначає рівні основного та збуджених електронних станів у докритичній області $Z < Z_{\rm cr}$. Знайдено рівняння для величини критичного заряду ядра, за якого рівень опускається в нижній енергетичний континуум. Виявлено сильний вплив лоренцевої структури потенціалів взаємодії на критичний заряд та дискретний спектр ферміона в скалярному та векторному кулоноподібних полях.