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WKB Method for the Dirac Equation with a Strong Coulomb Field and Its Application to the Theory of Two-Dimensional Supercritical Atoms

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Abstract
Solutions of the Dirac equation in a strong external field are obtained in the WKB approximation. A field is considered strong if the electron binding energy exceeds $2mc^2$ and the descrete spectrum levels may be lowered into the lower continuum. The wave functions in the classically allowed and forbidden regions are found and the conditions for matching them on transition through the turning point are obtained. The WKB method is applied to the following problems: 1) generalization of the Bohr-Sommerfeld quantization conditions with allowance for relativistic effects and the spin in 2+1 dimensions; 2) energy and width of the quasistationary level in the lower continuum.

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I. Introduction

It is known [1,2] that in three spatial dimensions the expression for the electron ground state energy in the Coulomb field of a point-charge $Z|e|$ becomes purely imaginary when $Z>137$, and that its interpretation as electron energy no longer has a physical meaning. To determine the electron energy spectrum in the Coulomb field with such a charge we need to eliminate the singularity of the Coulomb potential of a point-charge at $r=0$ by cutting off the Coulomb potential at small distances. This is equivalent to taking into account of the nucleus size. In three space dimensions the electron energy spectrum in the Coulomb field regulated at small distances was first considered in [3]. With increasing Z in the region $Z>137$, the electron energy levels in such a field were found to decrease, become negative, and may cross the boundary of the lower energy continuum, $E=-mc^2$. The value of $Z|e|=Z_{cr}|e|$ at which the lowest electron energy level cross the boundary of the lower energy continuum is called the critical charge for the electron ground state [2,4-5]. If Z continues to grow and enters the transcritical region with $Z>Z_{cr}$, the lowest electron energy level "sinks" into the lower energy continuum, which result in a rearrangement of the vacuum of the QED. This rearrangement is constrained by Pauli's exclusion principle. If the electron ground state at $Z<Z_{cr}$ is vacant, two electron-positron pairs are created; if it is half-occupied, one pair is created; and if it is occupied, no pairs are created. The Coulomb potential is repulsive for the created positrons, so they go to infinity. Hence at $Z>Z_{cr}$ a quasistationary state appears in the lower energy continuum and the new vacuum of QED, which corresponds to the filling of all the electron states with $E<-mc^2$, has the total electric charge $2e$ [2, 4-5]. Indeed, all the electron states with $E<-mc^2$ (the Dirac sea) were filled at $Z<Z_{cr}$, so electrons created by the strong Coulomb field with $Z>Z_{cr}$ cannot be described by means of a convenient wave function, and the notion of charged vacuum was introduced to describe these states [4-8]. In terms of the new vacuum, the density of electric charge $\rho(r)$ is classical. It is a function characterising the spatial distribution of the real electric charge appearing in the new (charged) vacuum, while in terms of the old (uncharged) vacuum this function should be interpreted as the probability of two electrons (with charge $2e$) being present at a given point in space.

We would like to see how the same system behaves in two dimensions. With this aim we shall apply the WKB method to the Dirac equation in a strong Coulomb field. Such approach works rather well for states with energy both $0<\varepsilon<1$ and $\varepsilon<-1$ (in mc^2 units). The obtained by this way quasiclassical formulae for the energy of quasistationary levels of the Dirac equation solutions in the lower continuum in (2+1) dimensions allow to consider a wide range of problems in the theory of supercritical atoms.

II. The Dirac equation in an external Coulomb field in 2+1 dimensions

Since [9] in 2+1 dimensions the Dirac algebra may be represented in terms of the Pauli matrices as $\gamma^0=\sigma^3$, $\gamma^k=i\sigma^k$, the Dirac equation for an electron minimally coupled to an external electromagnetic field has the form ($\hbar=c=m=1$)

$$(i\partial/\partial t - H_D)\Psi = 0, \tag{1}$$

where

$$H_D = \vec{\alpha}\vec{p} + \beta - eA^0 I = \sigma^1 p_2 - \sigma^2 p_1 + \sigma^3 - eA^0 I \tag{2}$$

is the Dirac Hamiltonian, $p_\mu = i\partial_\mu + eA_\mu$ is the operator of generalized momentum of the electron, $-e$ is the electric charge of electron (e>0), A_μ – the vector potential of the external electromagnetic field, and the conserved total angular momentum has only a single component, namely, $J_z=L_z+S_z$, where $L_z = -i\partial/\partial\varphi$ and $S_z = \sigma^3/2$.

Let us apply the Dirac equation (1), (2) to study two-dimensional hydrogen-like ion with nuclear charge $eZ >> 1$. Consider the problem neglecting the nucleus size and assuming the vector potential to be Coulomb

$$A^0(r) = -Ze/r, \quad A^x = A^y = 0 \tag{3}$$

for $0 \le r < \infty$.

The solutions of the Dirac equation (1) in the field (3) we seek in the polar coordinates in the form

$$\Psi(t,\vec{x}) = \frac{1}{\sqrt{2\pi}} exp(-i\varepsilon t + il\varphi)\psi(r,\varphi), \tag{4}$$

where ε – the energy, l – an integer number and

$$\psi(r,\varphi) = \frac{1}{\sqrt{r}}\begin{pmatrix} F(r) \\ G(r)e^{i\varphi}\end{pmatrix}. \tag{5}$$

Note that the function (5) is an eigenfunction of the the Dirac Hamiltonian H_D and the total angular momentum J_z with eigenvalues ε and l+1/2, respectively.

Substituting (4) and (5) into (1), we obtain

$$\begin{aligned} &\frac{dF}{dr} - \frac{l+1/2}{r}F + (\varepsilon + 1 - V(r))G = 0, \\ &\frac{dG}{dr} + \frac{l+1/2}{r}G - (\varepsilon - 1 - V(r))F = 0, \end{aligned} \tag{6}$$

where $V(r) = -Z\alpha/r$, $\alpha=e^2\approx 1/137$ is the fine structure constant.

The exact solutions and the energy eigenvalues with ε<1 corresponding to stationary states of the Dirac equation may be found in full analogy with the case of three space dimensions [1]:

$$F = \sqrt{1+\varepsilon}\ e^{-\rho/2}\rho^\gamma(Q_1+Q_2), \quad G = \sqrt{1-\varepsilon}\ e^{-\rho/2}\rho^\gamma(Q_1-Q_2), \tag{7}$$

where

$$\rho = 2\lambda r, \qquad \lambda = \sqrt{1-\varepsilon^2}, \quad \gamma = \sqrt{(l+1/2)^2 - (Z\alpha)^2}.$$

The value of γ is to be found by studying the behavior of the wave function at small r. The functions Q_1 and Q_2 which rendered the solutions of (6) finite at ρ=0 are given in terms of the conuent hypergeometric function $F(a;\ b;\ z)$ as:

$$Q_1 = AF(\gamma - \varepsilon Z\alpha/\lambda, 2\tilde{a}+1; \rho), \qquad Q_2 = BF(\gamma - \varepsilon Z\alpha/\lambda + 1, 2\gamma+1; \rho).$$

The relation between constants A and B and the energy eigenvalues are defined by

$$B = \frac{\gamma - \varepsilon Z\alpha/\lambda}{l + 1/2 + Z\alpha/\lambda}A, \qquad \varepsilon = \left[1 + \frac{(Z\alpha)^2}{\left(n_r + \sqrt{(l+1/2)^2 - (Z\alpha)^2}\right)^2}\right]^{-1/2}, \tag{8}$$

where n_r=0,1,2,..., if $l \ge 0$, and n_r=1,2,3,... if $l < 0$.

It is seen that $\varepsilon_0 = \sqrt{1-(2Z\alpha)^2}$ for l=n_r=0 and becomes zero at $Z\alpha = 1/2$, whereas in three spatial dimensions ε_0 equals zero at $Z\alpha = 1$. Thus, in two space dimensions the expression for the electron ground state energy in the Coulomb field of a point-charge $Z|e|$ no longer has a physical meaning at a much lower value of $Z\alpha = 1/2$, and the corresponding solution of the Dirac equation oscillates near the point $r \to 0$.

III. Discrete spectrum at 2Z>137. Critical charge.

To determine the electron energy spectrum in the Coulomb field with the charge 2Z>137 we need to eliminate the singularity of the Coulomb potential of a point-charge at r=0 by cutting off the Coulomb potential at small distance r_N. This is equivalent to taking into account of the nucleus size. Consider the Coulomb potential in the form:

$$V(r) = \begin{cases} -\dfrac{Z\alpha}{r}, & r > r_N, \\ -\dfrac{Z\alpha}{r} f\left(\dfrac{r}{r_N}\right), & r \le r_N. \end{cases} \tag{9}$$

Here $f(x)$ – cutoff function, $0 \le x = r/r_N \le 1$. Most often the following models are used: $f(x) = 1$ and $f(x) = (3-x^2)/2$. In given paper the first model $f(x)=1$ is considered.

Eliminating $G(r)$ from (6), we arrive at the equation for the function $F(r)$:

$$\frac{d^2F(r)}{dr^2} - \frac{l^2 - 1/4}{r^2}F(r) + \left[\left(\varepsilon + \frac{Z\alpha}{r_N}\right)^2 - 1\right]F(r) = 0, \quad r < r_N .$$

The general solution of this equation is

$$F(r) = \sqrt{r}\left[A_1 J_{|l|}(kr) + B_1 N_{|l|}(kr)\right] \quad r < r_N, \tag{10}$$

where $J_n(x)$ and $N_n(x)$ – the Bessel and the Neumann functions of integer order n, and $k = \sqrt{(\varepsilon + Z\alpha/r_N)^2 - 1}$. From (6) function $G(r)$ can be obtained as

$$G(r) = sgn\left(l + \frac{1}{2}\right)\frac{k\sqrt{r}}{\varepsilon + Z\alpha/r + 1}\left[A_1 J_{l_1}(kr) + B_1 N_{l_1}(kr)\right] \quad ,r < r_N, \tag{11}$$

where $l_1 = |l| + sgn(l + 1/2)$. In order for the function $F(r)$ to be finite at the point r=0 we need to set B_1=0.

In external region $r > r_N$ the potential $V(r)$ is Coulomb but we take into account both the signs of the quantity γ. Thus the finite at $r \to \infty$ solutions of the Dirac system (6) are determined by (7), and the functions Q_1 and Q_2 are

$$Q_j = C_j \Psi(1/2 - \chi_j + \gamma, 2\gamma + 1; \rho), \quad j = 1,2,$$
$$\chi_1 = Z\alpha\varepsilon/\lambda + 1/2, \quad \chi_2 = Z\alpha\varepsilon/\lambda - 1/2, \quad \rho = 2\lambda r, \quad C_{1,2} = const, \tag{12}$$

where $\Psi(a; b; z)$ – the irregular solution of the conuent hypergeometric equation; the regular solution $F(a; b; z)$ is inapplicable because it is infinite at $r \to \infty$. Substituting (12), (7) into (6) and using the recurrent relations between conuent hypergeometric functions, we find the relation between the constant C_1 and C_2:

$$C_2/C_1 = Z\alpha/\lambda - l - 1/2$$

The matching the internal and external solutions at the point r=r_N gives the equation

$$\left.\frac{\sqrt{1-\varepsilon}}{\sqrt{1+\varepsilon}}\frac{Q_1 - Q_2}{Q_1 + Q_2}\right|_{r=r_N} = A_l, \quad A_l = \left(\frac{G(r)}{F(r)}\right)_{r=r_N-0} = sgn(l + 1/2)\frac{k}{\varepsilon + Z\alpha/r_N + 1}\frac{J_{l_1}(kr)}{J_{|l|}(kr)}. \tag{13}$$

determining the Dirac equation spectrum. Rewrite (13) in explicit form:

$$x\frac{d}{dx}ln\Psi(\gamma-\varepsilon Z\alpha/\lambda,2\gamma+1;x)=\frac{1-\beta A_l}{1+\beta A_l}\left(\frac{Z\alpha}{\lambda}+l+\frac{1}{2}\right)+\frac{\varepsilon Z\alpha}{\lambda}-\gamma, \tag{14}$$

where $x=2\lambda r_N$, $\beta=\sqrt{(1+\varepsilon)/(1-\varepsilon)}$.

Representing function $\Psi(a;\ b;\ z)$ by $F(a;\ b;\ z)$ and restricting ourselves to the leading term of the asymptotic expansion of $F(a;\ b;\ z)$ (in the small parameter z) we obtain the transcendental equation

$$\gamma\frac{D^+ + D^-}{D^+ - D^-}=B_l,\ D^{\pm}=x^{\pm\gamma}\frac{\Gamma(-\varepsilon Z\alpha/\lambda\pm\gamma)}{\Gamma(1\pm 2\gamma)},\ B_l=\frac{l+1/2-Z\alpha A_l+\beta[Z\alpha-(l+1/2)A_l]}{1+\beta A_l} \tag{15}$$

Consider two ranges of the discrete spectrum:

1) $Z\alpha<|l+1/2|$. Here γ is real, and the transcendental equation for ε has the form:

$$\frac{\varepsilon Z\alpha}{\lambda}=\frac{(-1)^{n_r}}{\pi}arcsin\left\{\frac{\pi(2\lambda r_N)^{2\gamma}}{2\gamma\Gamma^2(2\gamma)}\frac{sin[\pi(\varepsilon Z\alpha/\lambda+\gamma)]}{sin(2\pi\gamma)}\frac{\Gamma(\varepsilon Z\alpha/\lambda+\gamma)}{\Gamma(\varepsilon Z\alpha/\lambda-\gamma)}\frac{B_l-\gamma}{B_l+\gamma}\right\}+n_r+\gamma. \tag{16}$$

It is easy to see that the expression (16) at $r_N\to 0$ transits into the analogue of the Bohr-Sommerfeld formula (8) in the two-dimensional case.

2) $Z\alpha\ge|l+1/2|$. Here $\gamma=i\theta$, $\theta=\sqrt{(Z\alpha)^2-(l+1/2)^2}$ is real. The levels of the energy ε are the solutions of the transcendental equation:

$$\Phi-arctg\frac{B_l}{\theta}=\pi\left(n_r+\frac{sgn(l+1/2)}{2}\right),\qquad \Phi=arg\frac{\Gamma(1+2i\theta)}{\Gamma(-\varepsilon Z\alpha/\lambda+i\theta)}-\theta\, ln(2\lambda r_N). \tag{17}$$

Formulae (16), (17) describe the discrete energy spectrum in range $-1\le\varepsilon\le 1$. Note, however, that for the numerical calculations of spectrum the expression (15) is more applicable.

At $\varepsilon\to -1$ in the formula (17)

$$\Phi=arg\,\Gamma(1+2i\theta)-\theta\, ln(2Z\alpha r_N),\ B_l=l+1/2-Z\alpha A_l,\ A_l=sgn(l+1/2)\frac{J_{l_1}(kr)}{J_{|l|}(kr)},$$

and from equation (17) we obtain the known [10] transcendental for Z_{cr}.

IV. WKB method for the Dirac equation in the strong external field

For finding the quasiclassical solutions of the system (6) it is convenient to write equations (6) in the matrix form:

$$\psi'=\frac{1}{\hbar}D\psi,\quad \psi=\begin{pmatrix}F\\G\end{pmatrix},\qquad D=\begin{pmatrix}\hbar\aleph/r & -(\varepsilon+1-V(r))\\ \varepsilon-1-V(r) & -\hbar\aleph/r\end{pmatrix}. \tag{18}$$

Here we have restored in an obvious view a Planck constant $\hbar$, the prime denotes the derivative with respect to r, $\aleph=(l+1/2)$, and the external electrostatic potential is $V(r)=-eA^0(r)$. The solution of the matrix equation (18) we shall look as the formal expansion in powers of $\hbar$:

$$\psi=\varphi\, exp\left(\int^r ydr\right),\qquad y(r)=\frac{1}{\hbar}y_{-1}(r)+y_0(r)+\hbar y_1(r)+\dots,\quad \varphi(r)=\sum_{n=0}^{\infty}\hbar^n\varphi^{(n)}(r), \tag{19}$$

where the upper (lower) component $\varphi_F^{(n)}$ ($\varphi_G^{(n)}$) of the vector $\varphi^{(n)}$ corresponds to the radial wave function F (G). By substituting (19) into (18) and equating to zero the coefficient of each power of $\hbar$, we obtain the recurrence system

$$\begin{aligned}&(D-y_{-1})\varphi^{(0)}=0,\\ &(D-y_{-1})\varphi^{(n+1)}=\varphi^{(n)\prime}+\sum_{k=0}^{n}y_{n-k}\varphi^{(k)},\qquad n=0,1,\dots\end{aligned} \tag{20}$$

Using the first two equation of system (20) by the left and right vectors technique we find the terms y_{-1}, y_0 and $\varphi^{(0)}$. Solving the next equations of the system (20) by the similar procedure one can sequentially find the terms y_2, y_3,..., $\varphi^{(2)}$, $\varphi^{(3)}$,... in the expansions (19). But formulae for them are rather cumberous, therefore in applications ones usually restrict themselves to only first terms.

Actually the reason of this is the fact that the expansions in powers of $\hbar$ (19) in the general case don`t convergent and are asymptotic series, the finite number of terms of which gives the good approximation for the wave function, if a parameter of an expansion (the Dirac constant $\hbar$) is rather small. So we obtained (to within a normalization constant)

$$\psi = \frac{1}{\sqrt{qQ_{\mp}}} exp\left[\int^{r}\left(\pm q + \frac{V'(r)}{2qQ_{\mp}}\right)dr\right]\begin{pmatrix}1+\varepsilon - V(r)\\ \mp Q_{\mp}\end{pmatrix}. \qquad (21)$$

The wave function of quasistationary state has the various look in the various regions.

I. The region $r_0 < r < r_-$ is classically allowed; there the wave functions (21) oscillate

$$G = C_1^{\mp}\left(\frac{\mathring{a} - V + 1}{p}\right)^{\frac{1}{2}} cos\,\grave{E}_1, \quad F = C_1^{\pm}\, sgn\,\aleph\left(\frac{\mathring{a} - V - 1}{p}\right)^{\frac{1}{2}} cos\,\grave{E}_2. \qquad (23)$$

Here r_0, r_- and r_+ – turning points,

$$p(r) = \sqrt{(\mathring{a} - V)^2 - 1 - \frac{\aleph^2}{r^2}}$$

– quasiclassical moment for the radial motion of a particle, $C_1^{\mp}$ – normalization constants.

$$\grave{E}_1 = \int_{r_-}^{r}\left(p - \frac{\aleph w}{pr}\right)dr + \frac{\eth}{4}, \qquad \grave{E}_2 = \int_{r_-}^{r_1}\left(p - \frac{\aleph \widetilde{w}}{pr}\right)dr + \frac{\eth}{4},$$

$$w = \frac{1}{2}\left(\frac{V'}{c^2 + E_I - V} - \frac{1}{r}\right), \qquad \widetilde{w} = \frac{1}{2}\left(\frac{V'}{c^2 - E_I + V} - \frac{1}{r}\right).$$

Signs $\pm$ correspond to values $\aleph > 0$ i $\aleph < 0$. If a width γ of a level is small (it will be shown later) the wave function of quasistationary state can be normalized on a single particle localized in the region I, neglecting a its penetrability into the classically forbidden regions $r < r_0$ and $r > r_-$ [11]. Here $cos^2\,\Theta_i(r)$ can be replaced with average value 1/2:

$$\left|C_1^{\mp}\right| = \left\{\int_{r_0}^{r_-}\frac{\mathring{a} - V(r)}{p(r)}dr\right\}^{-1/2} = \left(\frac{2}{T}\right)^{1/2},$$

where T – the frequency period of a relativistic particle inside a potential well.

II. The below-barrier region $r_- < r < r_+$ - classically forbidden. Here $p = iq$, and quantities q, y_{-1} i y_0 are real. As known [11] the wave function should exponentially damp inside of this region. So the solutions of the Dirac system (6) in the below-barrier region for $\aleph < 0$ are

$$\psi = \frac{C_2^-}{\sqrt{qQ_-}} exp\left[\int_{r_+}^{r}\left(-q - \frac{V'(r)}{2qQ_-}\right)dr\right]\begin{pmatrix}-Q_-\\ \varepsilon - 1 - V(r)\end{pmatrix}, \qquad (24)$$

III. In the region $r > r_+$ a divergent wave corresponds to the quasistationary state (taking off positron) for $\aleph < 0$:

$$\varnothing = \frac{C_3^-}{\sqrt{pP_-}} exp\left[\int_{r_+}^{r}\left(ip + \frac{V'(r)}{2pP_-}\right)dr\right]\begin{pmatrix}iP_-\\ \mathring{a} - 1 - V(r)\end{pmatrix}, \qquad (25)$$

where $P_{\pm} = p \pm i\aleph/r$. The formulae (23)-(25) include the whole range of values of r (except for range $r < r_0$ for which the view of a wave function here is not written out), except for neighbourhoods of turning points r_- i r_+. For bypass of these points and sewing the solutions we shall use the usual method [11]. Closely to the r_- i r_+ the system (6) reduces to the Schrödinger equation with the effective potential linearly depending on $r - r_{\pm}$, the solution of which expressed through the Airy function; one can sew by the more elegant Zwaan method. So the relation between the constants in various regions is of the form

$$C_2^\pm = iC_3^\pm = \sigma C_1^\pm \left[\frac{|\aleph|}{\left(r_-^2+\aleph^2\right)^{1/2}+r_-}\right]^\sigma \exp\left[-\int_{r_-}^{r_+}\left(q+\sigma\frac{V'(r)}{qQ_\pm}\right)dr\right], \qquad (26)$$

where $\sigma = sgn\,\aleph/2$.

Though the formulae (23)-(26) essentially differ from the formulae of nonrelativistic quasiclassics and are more complicated from them, their application to concrete problems does not meet difficulties, because all quantities in functions F and G express in quadratures.

V. Position and width of quasistationary levels in the lower continuum

Let us find the energy of quasistationary states that are the prolongation of the discrete spectrum levels into supercritical region $Z>Z_{cr}$, $\varepsilon<-1$. Neglecting the penetrability of a barrier in the region $r_- < r < r_+$ we obtain from (23) the quantization condition:

$$\int_{r_-}^{r_+}\left(p-\frac{\aleph w}{pr}\right)dr = \pi\left(n+\frac{1}{2}\right), \qquad n=0,\ 1,\ 2,\ldots \qquad (27)$$

The equation (27) determines the real part of the level energy $\varepsilon_{nl} = \varepsilon - i\gamma/2$. It is easy to show that the condition (27) reproduces the exact expression of the energy spectrum in the case $0<\varepsilon<1$.

Calculating the integral in (27) for the potential (9) and taking into account that $|\varepsilon| << Z\alpha/r_N$, we arrive at the transcendental equation ε:

$$\frac{\varepsilon Z\alpha}{2k}\ln\frac{|\varepsilon|Z\alpha+kg}{|\varepsilon|Z\alpha-kg} - g\ln\frac{r_N e\mu}{2g^2} + \sigma\arccos\frac{g^2-\varepsilon\aleph^2}{Z\alpha\mu} + I = \pi\left(n+\frac{1}{2}\right), \qquad (28)$$

where

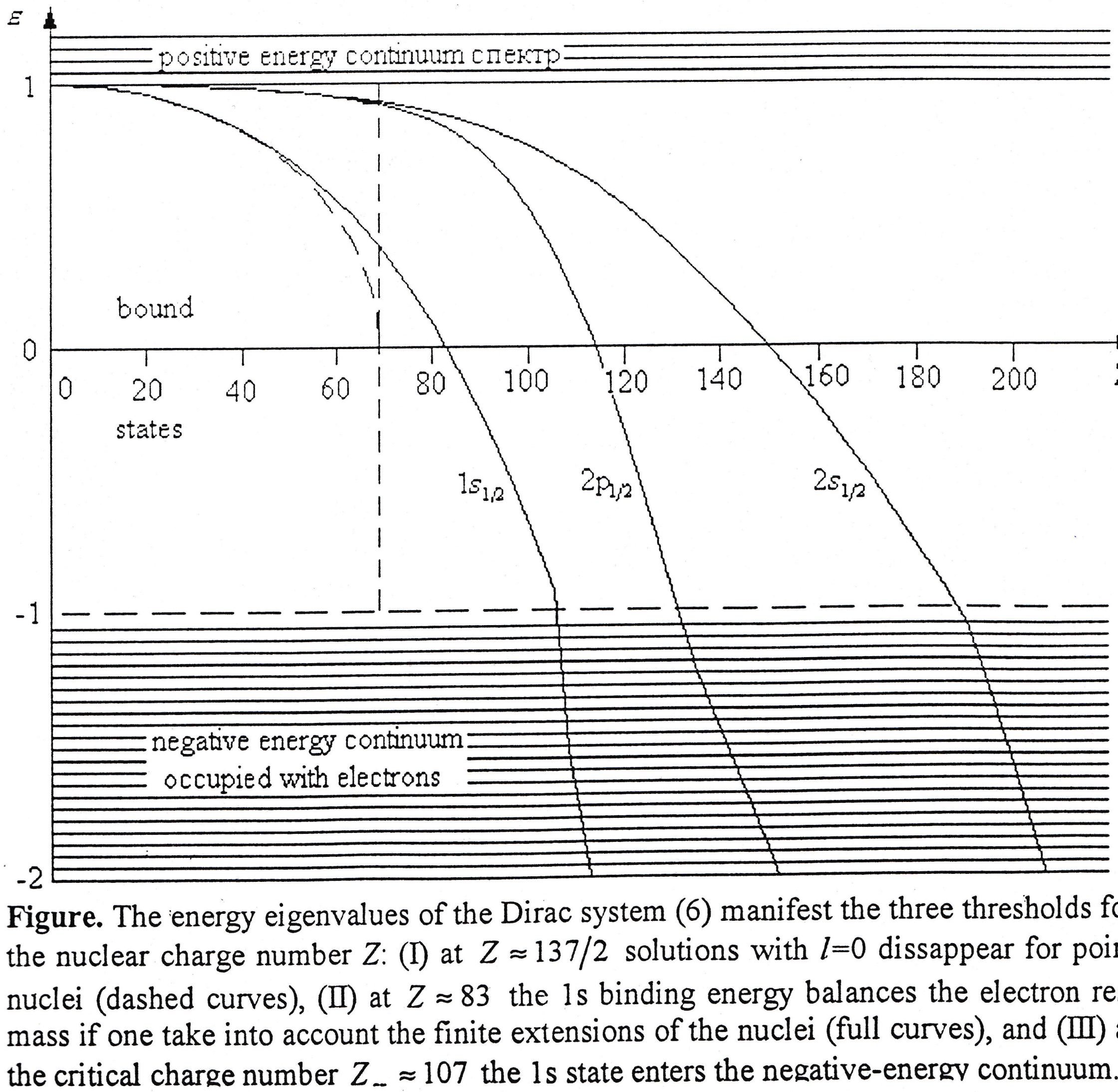

Figure. The energy eigenvalues of the Dirac system (6) manifest the three thresholds for the nuclear charge number Z: (I) at $Z \approx 137/2$ solutions with l=0 dissappear for point nuclei (dashed curves), (II) at $Z \approx 83$ the 1s binding energy balances the electron rest mass if one take into account the finite extensions of the nuclei (full curves), and (III) at the critical charge number $Z_{-} \approx 107$ the 1s state enters the negative-energy continuum.

$$I = Z\alpha\int_{x_0}^{1}\left[\sqrt{f^2(x)-\frac{\rho^2}{x^2}}+\frac{\aleph}{2(Z\alpha)^2}\left(\frac{f'(x)}{f(x)}+\frac{1}{x}\right)\frac{1}{\sqrt{x^2 f^2(x)-\rho^2}}\right]dx,\ e=2.718...$$

The numerical solutions of equations (16), (17) and (28) (for model I) for three lowest states are shown in figure.

Let now us pass to determination of the level width $\gamma = -2 Im\,\varepsilon_{nl}$ that coincides with the probability of the spontaneous creation of positrons. From the equations (6) we obtain the expression for γ

$$\gamma = 2 Im\left[G^*(r)F(r)\right]_{r\to\infty}.$$

By the obtained formulae for G and F γ takes the form

$$\gamma = \gamma_0\, exp\left\{-2\pi Z\alpha\left[\sqrt{1+1/k^2}-\sqrt{1-\rho^2}\right]\right\}, \qquad T=\frac{1}{\gamma_0}=-\frac{2}{k^2}\left[\varepsilon g+\frac{Z\alpha}{2k}ln\left(\frac{|\varepsilon|Z\alpha+kg}{|\varepsilon|Z\alpha-kg}\right)\right].$$

VI. Conclusions

In this paper by the exact solutions of the Dirac equation with a strong Coulomb field in 2+1 dimensions we construct the discrete energy spectrum in the range $-1\le\varepsilon\le 1$. The WKB-solutions of the two-dimensional Dirac equation are obtained. By the obtained quasiclassical formulae we find the spectrum of quasistationary levels (its position and width) in the lower energy continuum $\varepsilon < -1$ for a spherical superheavy nuclear with a charge $Z > Z_{cr}$ (see figure 2). The comparision of values of critical charge Z_{cr} obtained from exact solutions of the Dirac equation with Z_{cr} obtained from the quasiclassical formula (28) shows a good coinciding (see figure 2). Note that in the ground state for the model I at r_N=0.03 $Z_{cr}\approx 107$ and 170 in (2+1)- and (3+1)-dimensional QED, respectively. Thus, the Dirac vacuum in two space dimensions in the presence of a strong Coulomb field is unstable against electron-positron production at significantly smaller values of the critical charge than in the case of three spatial dimensions. Another difference between these two cases results from the fact that electrons confined to a plane behave like a spinless fermion. So if the ground electron state at $Z < Z_{cr}$ is vacant, one pair is created; if it is occupied, no pairs are created.

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