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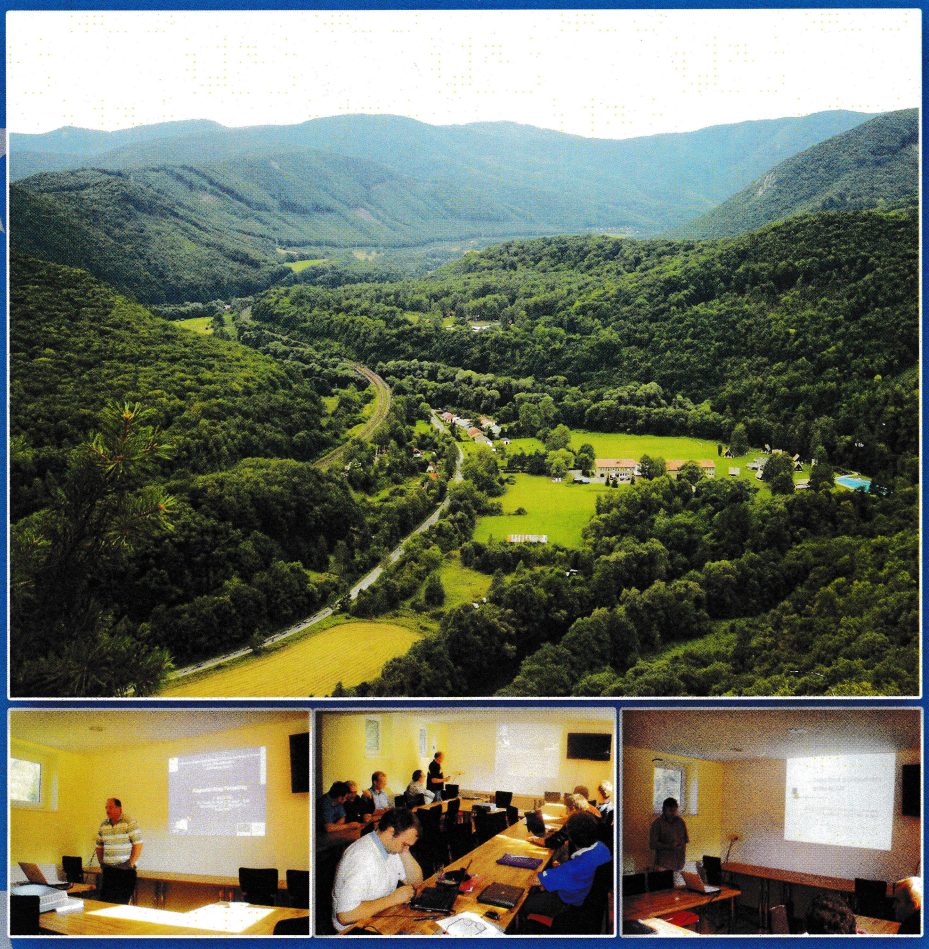

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Relativistic Theory of Tunnel Ionization of Heavy Atoms and Ions in the Constant Homogenous Electric Field

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Abstract

Within the paraxial Fock-Leontovich approximation, the new method of calculation of the Dirac wave functions is developed for three-dimensional axially symmetrical problems which do not permit the complete separation of variables. By means of the elaborated quasi-classical method the relativistic wave functions for the H-like atom in the weak constant homogeneous electric field are constructed in the below-barrier and classically allowed ranges. General analytical expression for probability of ionization of the atom in an external electric field is obtained. The comparison of the found formulas with results known previously is carried out.

Introduction

The problem of hydrogen atom in an electric field has fundamental meaning for a quantum mechanics and the atomic physics and has many applications (see, for example, [1, 2, 3] and the references therein). Since the twenties [4], properties of an energy spectrum of hydrogen atom and other atoms in external fields were rather intensively studied in the framework of the Schrödinger equation.

As is known, in the nonrelativistic case the homogeneous electric field removes degeneration of energy levels of hydrogen atom with respect to orbital and magnetic quantum numbers, and states, which differ by a sign of a magnetic quantum number, remain degenerated. The energy spectrum of these states is not discrete any more; it consists of the series of the fuzzy levels whose width increases when field intensity increases. This phenomenon is called the Stark effect [2]. In order to estimate the influence of the electric field of the given intensity F on atom which is in some quantum state, it is necessary to determine the corresponding position E_r and width Γ of this level. The level width defines the probability of ionization of the quasi-stationary state of an atom in the electric field; it is inversely proportional to an atom lifetime in this state: $\Gamma \sim \hbar/\tau_{ion}$ [1].

The mechanism of decay of atom in an electric field via below-barrier transition of electrons from the field of atomic core into the continuum has been clarified at the first stage of studying this process [4]. In the field, which is small in comparison with intraatomic one ($\sim 5 \times 10^9$ V/cm), time of electronic escape is large when compared with atomic time. Therefore, the electronic state can be considered as quasi-stationary one with energy $E = E_r - i\Gamma/2$.

The quasi-classical theory of decay of atomic particles in an electric field, that was created in 60s (see, for example, [5]), has allowed to obtain the useful analytical formulas for ionization probability, which are asymptotically exact in the limiting case of weak fields. Both neutral atoms [1, 5, 6, 7], and negative ions of type H^-, J^- etc. [8] were considered.

Rather recently (see [9, 10] and references therein) within the imaginary time method the quasi-classical theory of ionization of atoms and ions under the influence of constant and homogeneous electric and magnetic fields has been elaborated taking into account the Coulomb interaction between an electron and atomic core during tunnelling process. The $1/n$-expansion method (n is the principal quantum number) occupies the important place among new quantum mechanical methods of investigation of processes of interaction of atomic particles with electric and magnetic fields. This approach is

rather effective for highly exited (Rydberg) states of atoms and molecules, and when effects in strong external fields are consided (see, for example, [11]),

Note that in all specified papers, the emphasis was on the nonrelativistic aspects of the theory of ionization of atoms and ions by an electric field. At the same time the interior logic of development of study of atomic systems with a high degree of ionization (the multiply charged ions) dictates, obviously, formulation of various qualitatively new problems, similar to those which were previously solved only for neutral or weakly ionized atoms. Essentially relativistic character of motion of electrons in the fields created by multiply charged ions (the characteristic velocity of the electron in H-like ions with nuclear charge Z is $\sim \alpha Z c$; α is the fine structure constant, c is the velocity of light) is the main feature of such ions that distinguishes them from neutral atoms. Thus, the consistent theory of Stark ionization of such systems should be relativistic because relativistic effects are not small corrections, and fundamentally determine the orders of spectral characteristics.

In order to construct such a theory one should have the solution of the relativistic problem of motion of an electron in the field of nucleus and in the constant external electric field. Since the Dirac equation with such superpositional potential does not permit complete separation of variables in any orthogonal system of coordinates, the given problem has no exact analytical solution, and numerical methods are rather onerous.

The relativistic calculations of the linear Stark effect are carried out by means of perturbation theory [12, 13], and quadratic Stark effect was treated by means of RCGF method in the form of the expansion in powers of $Z\alpha$ [14]. However, the most of papers was basically devoted to position of quasi-stationary level, and there are only rare cases of calculation of width Γ in the relativistic case. Rather recently the probability of ionization of s-level, whose binding energy can be of order of the rest energy, in electric and magnetic fields has been calculated by means of generalization of the imaginary time method. However, in the general case, widths of quasi-stationary states are not found until now.

Due to such situation in the theory and due to intensive experimental researches during last years, asymptotic methods of calculation of ionization probability, which are based on clear physical ideas about below-barrier electron transition, are gaining in importance. From this point of view it is worthwhile to use the WKB method (or quasi-classical approximation) which enables to find the approximative analytical solutions of the relativistic problem and to express required ionization probability in terms of quantum penetrability of the potential barrier which separates domains of discrete and continuous spectra. As is known, this method has rather high accuracy even for small quantum numbers.

Quasi-classical approximation for the Dirac equation with an axially symmetrical potential

Consider an axially symmetrical problem, when two classically allowed ranges are separated by a potential barrier. Then the direction of the most probable tunnelling is the potential symmetry axis z, the axis ρ is perpendicular to z, φ is a azimuth angle.

For the bispinor Ψ the stationary Dirac equation is of the form ($m_e = e = \hbar = 1$)

$$\begin{array}{l} c\vec{\sigma}\vec{p}\xi = (E - V + c^2)\eta \\ c\vec{\sigma}\vec{p}\eta = (E - V - c^2)\xi \end{array}, \qquad \Psi = \begin{pmatrix} \xi \\ \eta \end{pmatrix} \tag{1}$$

where $\vec{p} = -i\vec{\nabla}$ is the momentum operator, c is the velocity of light, $\vec{\sigma}$ are the Pauli matrices, E is the electron energy including c^2, $V = V(z, \rho)$ is the effective potential energy of the interaction of the electron with the external field not allowing complete separation of variables in the Dirac equation.

By inserting the first equation of (1) into the second one and using the substitution

$$\xi = (W^+)^{1/2}\Phi, \qquad W^{\pm} = E - V \pm c^2, \tag{2}$$

we arrive at the matrix equation

$$\Delta\Phi + k^2\Phi = 0, \qquad k^2 = \frac{1}{\hbar^2 c^2}\left[(E-V)^2 - c^4\right] - \frac{\Delta V}{2W^+} - \frac{3}{4}\left(\frac{\vec{\nabla}V}{W^+}\right) + \frac{i}{W^+}\vec{\sigma}\left[\vec{\nabla}V, \vec{\nabla}\right]. \tag{3}$$

Here we have restored in an obvious way the reduced Planck constant $\hbar$. Since the potential V is axially symmetrical, the Hamiltonian commutes with the operator of projection of total angular momentum of the electron onto a potential symmetry axis z, and equation (3) permits separation of a variable φ. For this purpose we represent the solution of (3) in the form

$$\Phi = \begin{pmatrix} F_1(z,\rho)\exp\left[i\left(m-1/2\right)\varphi\right] \\ F_2(z,\rho)\exp\left[i\left(m+1/2\right)\varphi\right] \end{pmatrix}, \tag{4}$$

where $F_{1,2}$ are new unknown functions, m is the projection of the total angular momentum of the electron onto a potential symmetry axis z. By substituting (4) into (3), we obtain the matrix differential equation

$$(\Delta + \hat{\partial})F = (\hbar^{-2}q^2 + \gamma)F, \quad F = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}, \quad q = \frac{1}{c}\left[c^4 - (E-V)^2\right]^{1/2}, \tag{5}$$

$$\hat{\partial} = \frac{1}{W^+}\left(\frac{\partial V}{\partial\rho}\frac{\partial}{\partial z} - \frac{\partial V}{\partial z}\frac{\partial}{\partial\rho}\right)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \gamma = \begin{pmatrix} a_{m-1/2} & b_{m+1/2} \\ b_{m-1/2} & a_{-m-1/2} \end{pmatrix}, \tag{6}$$

$$a_\mu(z,\rho) = \frac{\mu^2}{\rho^2} + \frac{1}{W^+}\left[\frac{\mu}{\rho}\frac{\partial V}{\partial\rho} + \frac{\Delta V}{2} + \frac{3}{4}\frac{\left(\vec{\nabla}V\right)^2}{W^+}\right], \quad b_\mu(z,\rho) = -\frac{\mu}{\rho W^+}\frac{\partial V}{\partial z}. \tag{7}$$

We seek a solution of equation (5) in the form of a WKB expansion:

$$F = \varphi\exp(\hbar^{-1}S), \quad \varphi = \sum_{n=0}^{\infty}\hbar^n\varphi^{(n)}. \tag{8}$$

Here $\varphi^{(n)}$ is a bispinor (the upper component corresponds to the function F_1, the lower to F_2). Having substituted F, determined by (8), into (5) and equated to zero the coefficients of each power of $\hbar$, we arrive at the hierarchy of equations

$$\left(\vec{\nabla}S\right)^2 - q^2 = 0, \tag{9}$$

$$2\vec{\nabla}S\cdot\vec{\nabla}\varphi^{(0)} + \Delta S\varphi^{(0)} + \hat{\partial}S\varphi^{(0)} = 0, \tag{10}$$

$$2\vec{\nabla}\cdot S\vec{\nabla}\varphi^{(n+1)} + \Delta S\varphi^{(n+1)} + \hat{\partial}S\varphi^{(n+1)} + \Delta\varphi^{(n)} + \hat{\partial}\varphi^{(n)} - \gamma\varphi^{(n)} = 0, \tag{11}$$

where $n = 0, 1, 2, \ldots$. Unfortunately, equations (9)–(11), similarly to the initial equation (1), do not permit exact separation of variables. In order to solve this problem, we use the idea of the boundary-layer method.

We seek the solutions of equations (9)–(11) in the below-barrier range, where, unlike for the classically allowed range, the wave function is often localized in the vicinity of the most probable tunnelling direction, that substantially simplifies the whole problem: it is natural to expand all the quantities in equations (9)–(11), including the solutions, in the vicinity of the z-axis.

Consider equation (9) and assume that

$$q^2(z,\rho) = q_0^2(z) + \sum_{k=1}^{\infty}Q_k(z)\rho^{2k}, \quad q_0^2(z) = q^2(z,0), \quad Q_k = \frac{1}{(2k)!}\frac{\partial^{2k}q^2(z,0)}{\partial\rho^{2k}}. \tag{12}$$

According to the above speculations, the solution of equation (9) can also be represented in the form of an expansion in powers of coordinate the ρ:

$$S(z,\rho) = \sum_{n=0}^{\infty} S_n(z)\rho^{2n}. \tag{13}$$

By inserting (13) into (9) and equating to zero the coefficients of each power of ρ, we obtain the recurrent system of first-order differential equations

$$(S_0')^2 - q_0^2 = 0, \tag{14}$$
$$2S_0'S_1' + 4S_1^2 - Q_1 = 0, \tag{15}$$

and so on, from which the values S_n (n =0,1,2,...) are successively determined. Here the prime means the derivative with respect to z. Note that if in the expansion (13) the coefficients of negative and odd powers of ρ are taken into account, after substitution of (13) into (9) they will be equal to zero. The similar situation will arise later for the functions $\varphi^{(n)}$. We shall consider the first three equations of the given system. It is easy to show that the solution of equation (14) is

$$S_0 = \pm \int q_0 dz + C_0, \quad C_0 = const. \tag{16}$$

Since in the below-barrier range the wave function should decrease exponentially with increasing z, in (16) we select the negative sign.

Equation (15) is the nonlinear Riccati differential equation and are not solvable analytically in a general case. However, by making the substitution

$$S_1 = \frac{q_0(z)}{2}\left(\frac{1}{2}\frac{q_0'(z)}{q_0(z)} - \frac{\sigma'(z)}{\sigma(z)}\right), \tag{17}$$

one can proceed from (15) to the linear second-order equation

$$\sigma'' + \left[\frac{1}{4}\left(\frac{q_0'}{q_0}\right)^2 - \frac{1}{2}\frac{q_0''}{q_0} - \frac{Q_1}{q_0^2}\right]\sigma = 0. \tag{18}$$

Note that in the nonrelativistic limit $c \to \infty$ equation (18) is transformed into a similar equation, obtained by Sumetsky [15] by solving the Schrödinger equation with an axially symmetrical potential by the parabolic equation method.

The solutions of the equations (10), (11) are sought in the form

$$\varphi^{(n)}(z,\rho) = \begin{pmatrix} \rho^{|m-1/2|}\sum\limits_{k=0}^{\infty}\varphi_{1k}^{(n)}(z)\rho^{2k} \\ \rho^{|m+1/2|}\sum\limits_{k=0}^{\infty}\varphi_{2k}^{(n)}(z)\rho^{2k} \end{pmatrix}. \tag{19}$$

By substituting (19) into the corresponding equations and equating to zero the coefficients of each power of ρ in the each of the two components, we obtain the system of ordinary first-order differential equations, which is solvable. For $m > 0$ the solutions are expressed as integrals:

$$\varphi_{10}^{(0)} = \frac{C_2^+}{\sigma}\left(\frac{\sqrt{q_0}}{\sigma}\right)^{p-2}, \quad \varphi_{20}^{(0)} = \frac{C_2^{(+)}}{\sigma}\left(\frac{\sqrt{q_0}}{\sigma}\right)^{p-1}\left[\int \frac{\sigma A_1(z)}{q_0\sqrt{q_0}}dz + C_3^+\right], \tag{20}$$

where

$$A_1(z) = \left(S_1V_0' - q_0V_1\right)/W_0^+, \quad W_0^{\pm} = W^{\pm}(z,0), \quad p = |m| + 3/2, \tag{21}$$

$$V_0(z) = V(z,0), \quad V_k(z) = \frac{1}{(2k)!}\frac{\partial^{2k}V(z,0)}{\partial\rho^{2k}}. \tag{22}$$

For $m < 0$ these solutions are obtained from (20) by making the replacements $\varphi_{10}^{(0)} \to \varphi_{20}^{(0)}$, $\varphi_{20}^{(0)} \to -\varphi_{10}^{(0)}$, $C_k^+ \to C_k^-$, $(k = 2,3)$.

Note that if it is necessary to find the first l terms of the expansion (13), then in each function $\varphi^{(n)}$ of (19) one has to take into account the first $l-n-1$ $(n = 0,1,...,l-2)$ terms of the expansion in ρ. In the given paper we take into account the first two terms of (13), and the leading term of $\varphi^{(0)}$.

The lower component η of Ψ is obtained from the upper one ξ by the operation

$$\xi \xrightarrow[W^+ \to W^-]{} \eta. \tag{23}$$

Thus we have obtained the solution Ψ of equation (1) within constants C_0, $C_k^{\pm}$ $(k = 2,3)$. To determine them, one should take a certain potential and normalize the wave function. In the next section, we shall consider the potential of a Coulomb center and constant homogenous electric field.

Quasi-classical solutions of the relativistic problem of atom in the constant homogeneous electric field

If the H-like atom is placed in the constant homogeneous electric field, then an interaction potential in the cylindrical coordinates $\{z,\rho,\varphi\}$ is of the form:

$$V(z,\rho) = -\frac{Z}{\sqrt{z^2+\rho^2}} - Fz, \tag{24}$$

where Z is the nuclear charge, $F = const$ is the intensity of electric field.

If we write the quantity $q_0(z)$ as $q_0 = \sqrt{2(U_{eff} - E_{eff})}$, then the effective energy $E_{eff} = -\lambda^2/2$ $(\lambda = c\sqrt{1-\varepsilon^2}$, $\varepsilon = E/c^2)$ and effective potential

$$U_{eff}(z,\varepsilon) = \varepsilon V_0 - V_0^2/2c^2 \tag{25}$$

correspond to the expressions (5), (12). The form of the effective potential $U_{eff}(z,\varepsilon)$ is shown in figure 1.

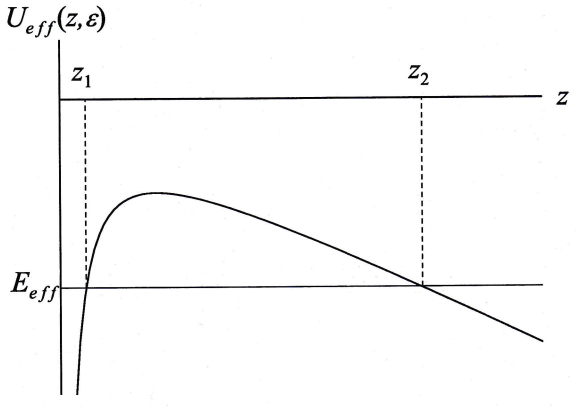

Figure 1: The effective potential $U_{eff}(z,\varepsilon)$

Let z_0 denotes the value of z satisfying the condition $z_1 \ll z_0 \ll z_2$ where $z_{1,2}$ are the turning points. The wave function is quasi-classical when $z \geqslant z_0$. At $z \sim z_0$ the solution of the Dirac equation with the potential (24) we shall match with asymptotics of the unperturbed atomic wave function Ψ_0:

$$\Psi \xrightarrow[z_1 \ll z \ll z_2]{} \Psi_0, \tag{26}$$

$$\Psi_0(\vec{r}) = \begin{pmatrix} f(r)\,\Omega_{jlm}(\vec{n}) \\ ig(r)\,\Omega_{jl'm}(\vec{n}) \end{pmatrix}, \quad l = j \pm 1/2, \quad l' = 2j - l, \quad \vec{n} = \vec{r}/r, \tag{27}$$

$$\left.\begin{matrix} f(r) \\ g(r) \end{matrix}\right\} = \pm\sqrt{1 \pm \varepsilon}\, A r^{\frac{\varepsilon Z}{\lambda} - 1} e^{-\lambda r}[1 + O(r^{-1})], \quad r \gg \frac{Z}{\lambda^2}, \tag{28}$$

which should be expanded in powers of ρ. For hydrogen-like ions the asymptotic coefficient A is of the form:

$$A = \lambda\,(2\lambda)^{\varepsilon Z/\lambda} \left(\frac{Z/\lambda - k}{2Z\Gamma\,(\varepsilon Z/\lambda - \gamma + 1)\,\Gamma\,(\varepsilon Z/\lambda + \gamma + 1)} \right)^{1/2}, \quad \gamma = \sqrt{k^2 - (\alpha Z)^2}. \tag{29}$$

Let us find the wave function Ψ in the range $z_0 < z < z_2$ within terms $O(F)$. For this purpose it is necessary to find the functions $S_0(z)$, $S_1(z)$, $\varphi_{10}^{(0)}(z)$, $\varphi_{20}^{(0)}(z)$, $\tilde{\varphi}_{10}^{(0)}(z)$, and $\tilde{\varphi}_{20}^{(0)}(z)$. In the range $z_0 < z < z_2$, an influence of the Coulomb potential is weak and, therefore, the quantity $q_0 = c^{-1}\sqrt{c^4 - (E + Z/z + Fz)^2}$ we can expand as follows

$$q_0 \approx q^{(0)} - \frac{Z(E + Fz)}{c^2 z q^{(0)}}, \quad q^{(0)}(z) = \frac{1}{c}\sqrt{c^4 - (E + Fz)^2}. \tag{30}$$

Then

$$S_0(z) = \int q_0 dz + C_0 \approx \frac{E + Fz}{2F} q^{(0)} + \left(\frac{c^3}{2F} - \frac{Z}{c} \right) \arcsin \frac{E + Fz}{c^2} - \frac{\varepsilon Z}{\lambda} \log \left| \frac{z}{\lambda^2 + \lambda q^{(0)} - \varepsilon F z} \right| + C_0. \tag{31}$$

From the boundary condition (26) it follows that

$$S_0 \xrightarrow[z_1 \ll z \ll z_2]{} \lambda z - \frac{\varepsilon Z}{\lambda} \log |z|, \tag{32}$$

Having imposed (32), we obtain

$$S_0(z) \approx \frac{E + Fz}{2F} q^{(0)} + \left(\frac{c^3}{2F} - \frac{Z}{c} \right) \left(\arcsin \frac{E + Fz}{c^2} - \arcsin \varepsilon \right) - \frac{\varepsilon Z}{\lambda} \log \left| \frac{2\lambda^2 z}{\lambda^2 + \lambda q^{(0)} - \varepsilon F z} \right| - \frac{E\lambda}{2F}. \tag{33}$$

In order to find S_1 one has to solve the Riccati equation (15) which in the case of potential (24) becomes

$$2q_0 S_1' + 4S_1^2 = \frac{Z}{z^3} \frac{E - V_0}{c^2}. \tag{34}$$

According to the boundary condition (26)

$$S_1 \xrightarrow[z_1 \ll z \ll z_2]{} \frac{\lambda}{2z}. \tag{35}$$

In the zeroth approximation, we neglect the right-hand side of the equation (34) and obtain its approximative solution

$$S_1 = \frac{1}{2\displaystyle\int \frac{dz}{q_0} + C_1}. \tag{36}$$

Having replaced q_0 by the approximative expression (30) and taking into account the boundary condition (35) we finally arrive at

$$S_1 = \frac{F}{2c \left(\arcsin \dfrac{E + Fz}{c^2} - \arcsin \varepsilon \right)}. \tag{37}$$

In a similar way, using the formulas (20) and boundary condition (26) we obtain the functions $\varphi_{10}^{(0)}(z)$, $\varphi_{20}^{(0)}(z)$, $\tilde{\varphi}_{10}^{(0)}(z)$, $\tilde{\varphi}_{20}^{(0)}(z)$. For example, for $m > 0$ the quasi-classically localized solution of the Dirac equation with the potential (24), which is constructed in the below-barrier range, is

$$\Psi = AC_2^{(+)} \begin{pmatrix} \sqrt{\frac{\lambda(c^2+E+Fz)}{c^2 q^{(0)}}} \left(\frac{2S_1}{\lambda}\right)^{m+1/2} \rho^{m-1/2} e^{i(m-1/2)\varphi} \\ O(\rho^{m+1/2}) \\ i\sqrt{\frac{\lambda(c^2-E-Fz)}{c^2 q^{(0)}}} \left(\frac{2S_1}{\lambda}\right)^{m+1/2} \rho^{m-1/2} e^{i(m-1/2)\varphi} \\ O(\rho^{m+1/2}) \end{pmatrix} \left(\frac{2\lambda^2 z}{\lambda^2 + \lambda q^{(0)} - \varepsilon F z}\right)^{\frac{\varepsilon Z}{\lambda}}$$

$$\times \exp\left\{-\frac{(E+Fz)q^{(0)} - E\lambda}{2F} - \left(\frac{c^3}{2F} - \frac{Z}{c}\right)\left(\arcsin\frac{E+Fz}{c^2} - \arcsin\varepsilon\right) - S_1\rho^2\right\}, \quad (38)$$

$$C_2^{(+)} = \frac{(-1)^{m+1/2}\mathrm{sgn}k}{2^{m-1/2}(m-1/2)!}\sqrt{\frac{(j+m)!}{4\pi(j-m)!}}. \quad (39)$$

Wave function in the classically allowed range. Width of below-barrier resonance

Let us continue the obtained solutions into the classically allowed range $z > z_2$. For this purpose we will use the method of bypass of the turning point $z = z_2$ in the complex plane (the Zwaan method). Then transition through the turning point $z = z_2$ is reduced to the replacement $q^{(0)} \to -ip^{(0)}$ in the wave function where $p^{(0)} = c^{-1}\sqrt{(E+Fz)^2 - c^4}$. In doing so we obtain

$$\Psi = AC_2^{(+)} \begin{pmatrix} \sqrt{\frac{\lambda(E+Fz+c^2)}{c^2 p^{(0)}}} \left(\frac{2S_1}{\lambda}\right)^{m+1/2} \rho^{m-1/2} e^{i(m-1/2)\varphi} \\ O(\rho^{m+1/2}) \\ \sqrt{\frac{\lambda(E+Fz-c^2)}{c^2 p^{(0)}}} \left(\frac{2S_1}{\lambda}\right)^{m+1/2} \rho^{m-1/2} e^{i(m-1/2)\varphi} \\ O(\rho^{m+1/2}) \end{pmatrix} \left(\frac{2\lambda^2 z}{\lambda^2 - \varepsilon F z - i\lambda p^{(0)}}\right)^{\frac{\varepsilon Z}{\lambda}}$$

$$\times \exp\left\{\frac{(E+Fz)ip^{(0)} + E\lambda}{2F} - \left(\frac{c^3}{2F} - \frac{Z}{c}\right)\left(\arccos\varepsilon + i\log\frac{E+Fz+cp^{(0)}}{c^2}\right) + \frac{i\pi}{4} - S_1\rho^2\right\}, \quad (40)$$

where

$$S_1 = \frac{F}{2c\left(\arccos\varepsilon + i\log\dfrac{E+Fz+cp^{(0)}}{c^2}\right)}. \quad (41)$$

The energy of the quasi-stationary state is $E = E_r - i\Gamma/2$, where E_r and Γ are the position and width of resonance, respectively. The quantity Γ is positive; it determinate the probability of system decay per time unit: $W = \Gamma/\hbar$. The ionization probability is equal to the total probability flux through the plane which is perpendicular to z-axis:

$$W = c\int\limits_S \Psi^+\vec{\alpha}\Psi d\vec{S} = c\int\limits_0^\infty\int\limits_0^{2\pi} (\Psi^+\alpha_z\Psi)\rho d\rho d\varphi. \quad (42)$$

Having substituted (40) into (42) and calculated the double integral, we arrive at the following expression for W:

$$W(Z,F) = \frac{2\lambda A^2}{(|m|-1/2)!}\frac{(j+|m|)!}{(j-|m|)!}\frac{(4\lambda^4)^{\frac{\varepsilon Z}{\lambda}}}{(4\lambda^2 c\arccos\varepsilon)^{|m|+1/2}}F^{|m|+1/2-\frac{2\varepsilon Z}{\lambda}} \times \exp\left\{\frac{E\lambda - c^3\arccos\varepsilon}{F} + \frac{2Z}{c}\arccos\varepsilon\right\}. \quad (43)$$

For s-states ($j = |m| = 1/2$) this formula is simplified as follows

$$W(Z,F) = \frac{A^2(2\lambda^2)^{\frac{2\varepsilon Z}{\lambda}}}{2\lambda c\arccos\varepsilon}F^{1-\frac{2\varepsilon Z}{\lambda}}\exp\left\{\frac{E\lambda - c^3\arccos\varepsilon}{F} + \frac{2Z}{c}\arccos\varepsilon\right\}, \quad (44)$$

and coincides with the result of paper [16] that was obtained in the framework of the method of imaginary time. Note, however, that there is not the asymptotic coefficient given clearly.

In the nonrelativistic limit $c \to \infty$ and at $Z = 1$ the expression (43) becomes the well-known formula for the probability of ionization of the hydrogen atom [1]:

$$W_H = \frac{4}{F}\exp\left(-\frac{2}{3F}\right). \quad (45)$$

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