INSTITUTE OF EXPERIMENTAL PHYSICS
SLOVAK ACADEMY OF SCIENCES

# THE 15th SMALL TRIANGLE MEETING on theoretical physics

October 27–30, 2013 | Stará Lesná

# Boundary-Layer Method in the Theory of Tunnel Ionization of an Atom by Constant Uniform Electric Field

**O.K. Reity, V.K. Reity, V.Yu. Lazur**

*Department of Theoretical Physics, Uzhhorod National University, 54 Voloshyna Street, Uzhhorod 88000, Ukraine*

**Abstract**

Within the paraxial Fock-Leontovich approximation the three-dimensional version of the WKB method is developed for the Schrödinger equation with an arbitrary axially symmetrical potential of barrier type which does not permit the complete separation of variables. By means of the elaborated recurrent scheme of WKB expansions the wave functions in the problem of an atom in a constant uniform electric field are constructed in the classically forbidden and allowed regions. This has allowed for the first time to calculate the first two terms of the asymptotic (at small intensity of electric field) behavior of probability of tunneling ionization of an arbitrary atom (not $H$-like one) in a constant uniform electric field, taking into account the centrifugal energy.

## Introduction

In this paper we consider the decay of atoms and ions in a constant electric field. The common feature of these problem is the fact that the effects under consideration are determined by the behavior of the valence electron in an atom at large distances from the nucleus. The removal of an electron from an atom or an ion placed in a constant electric field occurs as a result of the tunnel effect. The difficulty in this problem consists of the fact that the barrier through which the electron penetrates is a three-dimensional one. In order to overcome this difficulty the problem is artificially reduced to a one-dimensional problem [1, 2], but the method of introducing the effective potential barrier is not justified and the obtained result is incorrect. In two cases the problem can be solved exactly. In the case of a hydrogen atom in the ground state [3] placed in an electric field the variables in the Schrödinger equation for the wave function of the electron can be separated in parabolic coordinates so that the problem is reduced to a one-dimensional problem. The Schrödinger equation can also be solved for the wave function of an electron situated in a spherically symmetrical field of force of zero range and in a constant electric field [4].

Of practical interest is the case when the intensity of the external electric field is much smaller than the intensity of the characteristic atomic fields. If this condition is satisfied the breakup of the atomic particle occurs slowly compared to the characteristic atomic times and the leaking out of the electron takes place primarily in directions close to the direction of the electric field. Therefore, in order to determine the frequency of the passage of the electron through the barrier it is convenient to use the idea of the boundary-layer method, i. e. to solve the Schrödinger equation near an axis directed along the electric field and passing through the atomic nucleus. This idea was used for calculating the leading term of the tunnel ionization rate of an atom in a constant uniform electric field in non-relativistic [5] and relativistic [6, 7] cases. Also on the basis of the boundary-layer method the relativistic two-center problem was solved at large intercenter distances [8]. In the present paper we use this method for finding two terms of asymptotic behavior (at small intensity of the external electric field) of the probability rate of tunnel ionization of an atom in a constant uniform electric field taking into account the centrifugal energy.

## The boundary-layer method for the Schrödinger equation with axially symmetrical potentials

Consider an axially symmetrical problem, when two classically allowed ranges are separated by a potential barrier. Then the direction of the most probable tunneling is the potential symmetry axis $z$, the axis $\rho$ is perpendicular to $z$, $\phi$ is the azimuthal angle.

The stationary Schrödinger equation is ($m_e = |e| = \hbar = 1$)

$$\Delta\Psi + 2(E - V)\Psi = 0. \tag{1}$$

where $V = V(z, \rho)$ is the effective potential energy of the interaction of the electron with the external field not allowing a complete separation of variables in the equation (1).

Since the potential $V$ is axially symmetrical, the Hamiltonian commutes with the operator of projection of total angular momentum of the electron onto a potential symmetry axis $z$, and equation (1) permits separation of a variable $\phi$. For this purpose we represent the solution of (1) in

$$\Psi = \psi(z, \rho)e^{im\phi}, \tag{2}$$

where $\psi(z, \rho)$ is a new unknown function, $m = 0, \pm1, \pm2, ...$ is the projection of the total angular momentum of the electron onto a potential symmetry axis $z$. By substituting (2) into (1), we obtain the differential equation

$$\Delta\psi + \left[\frac{2}{\hbar^2}(E - V) - \frac{m^2}{\rho^2}\right]\psi = 0, \tag{3}$$

where the Planck constant $\hbar$ is renewed.

We seek a solution of equation (3) in the form of a WKB expansion:

$$\psi = \varphi\, e^{\frac{S}{\hbar}}, \quad \varphi = \sum_{n=0}^{\infty} \hbar^n \varphi^{(n)}. \tag{4}$$

Having substituted $\psi$, determined by (4), into (3) and equated to zero the coefficients of each power of $\hbar$, we arrive at the hierarchy of equations

$$(\vec{\nabla} S)^2 = q^2, \qquad q^2 = 2(V - E); \tag{5}$$

$$2\vec{\nabla} S \vec{\nabla} \varphi^{(0)} + \Delta S\, \varphi^{(0)} = 0; \tag{6}$$

$$2\vec{\nabla} S \vec{\nabla} \varphi^{(n+1)} + \Delta S\, \varphi^{(n+1)} = (m^2/\rho^2)\varphi^{(n)} - \Delta\varphi^{(n)}, \tag{7}$$

where $n = 0, 1, 2, \ldots$. Unfortunately, equations (5)–(7), similarly to the initial equation (1), do not permit exact separation of variables. In order to solve this problem, we use the idea of the boundary-layer method.

We seek the solutions of equations (5)–(7) in the below-barrier range, where, unlike for the classically allowed range, the wave function is often localized in the vicinity of the most probable tunnelling direction, that substantially simplifies the whole problem: it is natural to expand all the quantities in equations (5)–(7), including the solutions, in the vicinity of the $z$-axis.

Consider equation (5) and assume that

$$V\,(z, \rho) = \sum_{k=0}^{\infty} V_k(z) \rho^{2k}, \quad V_k = \frac{1}{k!} \frac{\partial^k V(z, 0)}{\partial \rho^{2k}}. \tag{8}$$

According to the above speculations, the solution of equation (8) can also be represented in the form of an expansion in powers of coordinate the $\rho$:

$$S\,(z, \rho) = \sum_{k=0}^{\infty} s_k(z) \rho^{2k}. \tag{9}$$

By inserting (9) into (5) and equating to zero the coefficients of each power of $\rho$, we obtain

$$(s_0')^2 = q_0^2, \quad q_0 = \sqrt{2(V_0 - E)}; \tag{10}$$

$$s_0' s_1' + 2s_1^2 = V_1; \tag{11}$$

$$s_0' s_2' + 8 s_1 s_2 = V_2 - \left(s_1'\right)^2/2; \tag{12}$$

$$s_0' s_k' + 4k s_1 s_k = V_k - \frac{1}{2}\sum_{j=1}^{k-1} s_j' s_{k-j}' - 2\sum_{j=1}^{k-2} (j+1)(k-j) s_{j+1} s_{k-j}, \quad k = 3, 4, 5, \ldots \tag{13}$$

from which the values $s_n$ $(n = 0, 1, 2, \ldots)$ are successively determined. Here the prime means the derivative with respect to $z$. Note that if in the expansion (9) the coefficients of negative and odd powers of $\rho$ are taken into account, after substitution of (9) into (5) they will be equal to zero. We shall consider the first three equations of the given system.

It is easy to show that the solution of equation (10) is

$$s_0 = \pm \int q_0 dz + \text{const}. \tag{14}$$

Since in the below-barrier range the wave function should decrease exponentially with increasing $z$, in (16) we select the negative sign.

Equation (11) is the nonlinear Riccati differential equation and are not solvable analytically in a general case. However, by making the substitution

$$s_1 = \frac{q_0(z)}{2}\left(\frac{1}{2}\frac{q_0'(z)}{q_0(z)} - \frac{\sigma'(z)}{\sigma(z)}\right), \tag{15}$$

one can proceed from (11) to the linear second-order equation

$$\sigma'' + \left[\frac{1}{4}\left(\frac{q_0'}{q_0}\right)^2 - \frac{1}{2}\frac{q_0''}{q_0} - \frac{2V_1}{q_0^2}\right]\sigma = 0, \tag{16}$$

which after substitution $q_0 \to \pm i p_0$ coincides with the equation obtained by Sumetsky within the parabolic equation method [9].

The equations for $s_2$, $s_3$,... are linear and integrated in quadratures:

$$s_2 = \frac{q_0^2}{\sigma^4}\left\{\int \frac{\sigma^4}{q_0^3}\left[\frac{(s_1')^2}{2} - V_2\right] dz + \text{const}\right\}, \tag{17}$$

$$s_k = \left(\frac{q_0}{\sigma^2}\right)^k \left\{\int \frac{\sigma^{2k}}{q_0^{k+1}}\left[\frac{1}{2}\sum_{j=1}^{k-1} s_j' s_{k-j}' + 2\sum_{j=1}^{k-2}(j+1)(k-j)s_{j+1}s_{k-j} - \right.\right.$$
$$\left.\left. - V_k\right] dz + \text{const}\right\}, \tag{18}$$

The solutions of the equations (6), (7) are sought in the form

$$\varphi^{(n)} = \rho^{|m|}\sum_{k=0}^{\infty}\varphi_k^{(n)}(z)\rho^{2k}. \tag{19}$$

By substituting (19) into the corresponding equations and equating to zero the coefficients of each power of $\rho$, we obtain the system of ordinary first-order differential equations for $\varphi_k^{(n)}(z)$ which are solvable. So for the equation (6) these functions to within a constant common multiplier are of the form:

$$\varphi_0^{(0)} = \frac{1}{\sqrt{q_0}}\left(\frac{\sqrt{q_0}}{\sigma}\right)^{|m|+1}, \tag{20}$$

$$\varphi_k^{(0)} = \frac{1}{\sqrt{q_0}}\left(\frac{\sqrt{q_0}}{\sigma}\right)^{|m|+2k+1}\left\{\int \frac{1}{\sqrt{q_0}}\left(\frac{\sigma}{\sqrt{q_0}}\right)^{|m|+2k+1}\sum_{j=1}^{k}\left(s_k'\varphi_{k-j}^{(0)\,\prime} + [2(j+1)\right.\right.$$
$$\left.\left.\times\ (|m| + 2k - j + 1)s_{k+1} + s_k''/2]\,\varphi_{k-j}^{(0)}\right) dz + \text{const}\right\},\ k = 1, 2, 3, \ldots, \tag{21}$$

and for the equations (7) –

$$\varphi_0^{(n)} = \frac{1}{\sqrt{q_0}}\left(\frac{\sqrt{q_0}}{\sigma}\right)^{|m|+1}\left\{\int \frac{1}{\sqrt{q_0}}\left(\frac{\sigma}{\sqrt{q_0}}\right)^{|m|+1}\left[2(|m|+1)\varphi_1^{(n-1)} + \frac{\varphi_0^{(n-1)''}}{2}\right] dz + \text{const}\right\}, \tag{22}$$

$$\varphi_k^{(n)} = \frac{1}{\sqrt{q_0}}\left(\frac{\sqrt{q_0}}{\sigma}\right)^{|m|+2k+1}\left\{\int \frac{1}{\sqrt{q_0}}\left(\frac{\sigma}{\sqrt{q_0}}\right)^{|m|+2k+1}\left[\sum_{j=1}^{k}\left(s_k'\varphi_{k-j}^{(n)\,\prime} + [2(j+1)(|m|+2k-j+1)s_{k+1} + s_k''/2]\,\varphi_{k-j}^{(n-1)}\right) + 2(k+1)(|m|+k+1)\varphi_{k+1}^{(n-1)} + \varphi_k^{(n-1)''}/2\right] dz + \text{const}\right\}, \tag{23}$$

where $k = 1, 2, 3, \ldots$. The latter equation is really correct for all $n, k \geqslant 0$, because $\varphi_k^{(n)} \equiv 0$ when $n, k < 0$.

Note that if it is necessary to find the wave function to within terms $O(\hbar^{N+1})$, then in the expansion (19) for $\varphi^{(0)}$ one has to take into account the first $N+1$ terms, for $\varphi^{(1)}$ – $N$ terms, ..., for $\varphi^{(N)}$ – one leading term, and in the expansion (9) for the function $S$ giving stronger exponential dependence the corrections $s_0$, $s_1$, ..., $s_{N+1}$ should be taken into account. For example, if the problem consists in finding wave function to within terms $O(\hbar^2)$ as well as in the problem of the breakup of an atom in a constant uniform electric field considered here then we can make replacement $\varphi^{(1)} \to \varphi^{(0)} S^{(1)}$ i. e. represent the solution of (3) in the form

$$\psi = \varphi^{(0)}\, e^{\frac{S}{\hbar}}[1 + \hbar S^{(1)}] \simeq \varphi^{(0)}\, \exp\left\{\frac{S}{\hbar} + \hbar S^{(1)}\right\}, \tag{24}$$

where $S$ and $\varphi^{(0)}$ are determined by

$$S = -\int q_0(z)dz + s_1(z)\rho^2 + s_2(z)\rho^4, \quad \varphi^{(0)} = \varphi_0^{(0)}(z) + \varphi_1^{(0)}(z)\rho^2, \quad \varphi^{(1)} = \varphi_0^{(1)}(z), \tag{25}$$

and $S^{(1)}$ satisfies the equation

$$2\vec{\nabla}S\vec{\nabla}S^{(1)} = m^2/\rho^2 - \Delta\varphi^{(0)}/\varphi^{(0)}. \tag{26}$$

Saving in the expansion of $S^{(1)}$ in powers of $\rho^2$ only the leading term we obtain the following solution of (26):

$$S^{(1)}(z) = \int \frac{\varphi_0^{(0)''} + 4(|m|+1)\varphi_1^{(0)}}{2q_0\varphi_0^{(0)}} dz + \text{const}. \tag{27}$$

## Quasiclassical solutions of the problem of an atom in a constant uniform electric field

If the arbitrary (not H-like) atom (ion) is placed in the constant uniform electric field being antiparallel to the axis $z$, then an interaction potential at distances much more than atomic size ($r \gg 2Z/\gamma^2$, $\gamma = \sqrt{-2E}$) has the following asymptotic behavior:

$$V \sim -\frac{Z}{r} - \mathcal{E}z. \tag{28}$$

where $Z$ is the charge of atomic core, $\mathcal{E} = |\vec{\mathcal{E}}| = \text{const}$ is the intensity of electric field.

The leading term $V_0$ of the expansion (8) which we shall call "potential" has a form of the potential with a barrier (see figure 1).

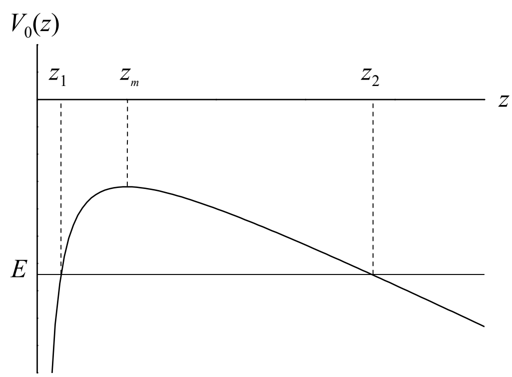


Figure 1: The "potential" $V_0(z)$; $z_1$, $z_2$ are roots of equation $q_0(z) = 0$, $z_m = \sqrt{Z/\mathcal{E}}$ is the maximum point.

When $\mathcal{E} \ll \gamma^4/4Z$ then the tuning points of the "potential" equal

$$\begin{aligned} z_1 &= \frac{\gamma^2 - \sqrt{\gamma^4 - 16\mathcal{E}Z}}{4\mathcal{E}} \simeq \frac{2Z}{\gamma^2} + \frac{8Z^2\mathcal{E}}{\gamma^6}, \\ z_2 &= \frac{\gamma^2 + \sqrt{\gamma^4 - 16\mathcal{E}Z}}{4\mathcal{E}} \simeq \frac{\gamma^2}{2\mathcal{E}} - \frac{2Z}{\gamma^2} - \frac{8Z^2\mathcal{E}}{\gamma^6}, \end{aligned} \tag{29}$$

the below-barrier region is quite wide ($z_1 \ll z_2$) and there is the region $z_1 \ll z \ll z_b$ ($z_b < z_m$) where an electric field can be considered as a perturbation. In this region the wave function should be close to the asymptotic behavior (at $\rho/z \ll 1$, $r \sim z \gg z_1$) of the Coulomb (atomic) wave function $\Psi_0$ [3]:

$$\Psi \xrightarrow[z_1 \ll z \ll z_b]{} \Psi_0^{(as)}, \tag{30}$$

$$\Psi_0^{(as)}(\vec{r}) = R_{nl}^{(as)}(r) Y_{lm}^{(as)}(\vec{n}), \quad \vec{n} = \vec{r}/r, \tag{31}$$

where

$$R_{nl}^{(as)} \simeq a z^{Z/\gamma-1}\left[1-\frac{(Z/\gamma-l-1)(Z/\gamma+l)}{2\gamma z}+\left(\frac{Z}{\gamma}-1\right)\frac{\rho^2}{2z^2}\right] \times \exp\left\{-\gamma z-\frac{\gamma\rho^2}{2z}+\frac{\gamma\rho^4}{8z^3}\right\}, \tag{32}$$

$$Y_{lm}^{(as)} \simeq A_{lm}\left(\frac{\rho}{z}\right)^{|m|}\left\{1-\left[\frac{l(l+1)}{|m|+1}+|m|\right]\frac{\rho^2}{2z^2}\right\}e^{im\phi}, \tag{33}$$

$$A_{lm}=\frac{(-1)^{(m+|m|)/2}}{2^{|m|}|m|!}\sqrt{\frac{2l+1}{4\pi}\frac{(l+|m|)!}{(l-|m|)!}}. \tag{34}$$

Here $l$ is the orbital angular moment of an atom, $a$ is the asymptotic coefficient.

Let us find the quasiclassical localized wave function $\Psi$ in the range $z_b < z < z_2$.

$$s_0(z)=-\int_{z_1}^{z} q_0(x)dx+\ln C_0, \quad C_0=\text{const}. \tag{35}$$

From boundary condition (30) it follows that

$$s_0 \xrightarrow[z_1\ll z\ll z_b]{} -\gamma z+\frac{Z}{\gamma}\ln z-\frac{Z^2}{2\gamma^3 z}+\ln(\sqrt{\gamma}\,aA_{lm})+O(z^{-2}). \tag{36}$$

In order to calculate the integral (35) at $z_1 \ll z \ll z_b$ we represent $q_0$ in the form

$$q_0(z)=\sqrt{\frac{2\mathcal{E}(z-z_1)(z_2-z)}{z}}\simeq\sqrt{2\mathcal{E}z_2}\sqrt{\frac{z-z_1}{z}}\left(1-\frac{z}{2z_2}\right), \tag{37}$$

Then

$$\int_{z_1}^{z} q_0(x)dx=\sqrt{2\mathcal{E}z_2}\left\{\sqrt{z(z-z_1)}-\frac{z_1}{2}\ln\frac{2\sqrt{z(z-z_1)}+2z-z_1}{z_1}- -\frac{1}{8z_2}\left[(2z-z_1)\sqrt{z(z-z_1)}-z_1^2\ln(1+\sqrt{1-z_1/z})\right]\right\}. \tag{38}$$

Due to $z \gg z_1$ one can expand (38) in powers of the parameter $z_1/z \ll 1$ in the following way

$$\int_{z_1}^{z} q_0(x)dx=\sqrt{2\mathcal{E}z_2}\left(z-\frac{z_1}{2}-\frac{z_1}{2}\ln\frac{4z}{z_1}+\frac{z_1^2}{8z}-\frac{z^2}{2z_2}\right). \tag{39}$$

Taking into account (29) we have

$$s_0=-\gamma z+\frac{Z}{\gamma}+\frac{Z}{\gamma}\ln\frac{2\gamma^2 z}{Z}-\frac{Z^2}{2\gamma^3 z}+\frac{\mathcal{E}z^2}{2\gamma}+\ln C_0. \tag{40}$$

Equating the expressions (36) and (40), we obtain the constant

$$C_0 = \sqrt{\gamma}\left(\frac{Z}{2e\gamma^2}\right)^{Z/\gamma} aA_{lm}, \quad e = 2.718... \tag{41}$$

The non-linear Riccati equation (11) for potential (28) is of the form

$$-q_0 s_1' + 2s_1^2 = \frac{Z}{2z^3} \tag{42}$$

and satisfied limiting condition

$$s_1 \xrightarrow[z_1 \ll z \ll z_b]{} -\frac{\gamma}{2z} + O(z^{-3}). \tag{43}$$

The equation (42) can be reduced to the Bernoulli equation for $f(z)$ being solvable exactly. But in the given problem it is necessary to obtain asymptotic solution at large $z$ only. Let us represent the solution of (42) in the form

$$s_1(z) = \sum_{n=1}^{\infty} s_{1n}(z), \tag{44}$$

where $s_{10}(z) \sim 1/z$ is the solution of the equation (42) without the right-hand side, $s_{1n}(z)$ $(n = 1, 2, \ldots)$, ... are the corrections $\sim 1/z^{n+1}$ and satisfy the 1-st order linear differential equations. Taking into account the first two terms of (42) the condition (43) we have obtained that

$$s_1 = -\frac{\gamma + q_0}{4z}\left[1 + \frac{Z(\gamma + q_0)}{4\gamma^3 z}\right]. \tag{45}$$

and

$$\sigma(z) = \frac{2\gamma z\sqrt{q_0}}{\gamma + \sqrt{\gamma^2 - 2\mathcal{E}z}}\left[1 + \frac{Z(14\mathcal{E}z - 3\gamma^2 - \gamma q_0)}{4\gamma^3 z q_0} + \frac{3Z\mathcal{E}}{\gamma^4}\ln\frac{\sqrt{2\mathcal{E}z}}{\gamma + q_0}\right], \tag{46}$$

The substitution of $s_1$ and $\sigma$ into (17) gives

$$s_2 = \frac{(\gamma + q_0)^3}{64\gamma z^3 q_0}, \tag{47}$$

Similarly, from formulae (21) and (27) one can find the following expressions for $\varphi_1^{(0)}$ and $S_1$. Therefore, the asymptotic behavior of quasiclassical localized wave function in the below-barrier region is of the form

$$\Psi = C_0\rho^{|m|}\left(\varphi_0^{(0)} + \varphi_1^{(0)}\rho^2\right)\exp\left[-\int_{z_1}^{z} q_0(x)dx + s_1\rho^2 + s_2\rho^4 + S_1 + im\phi\right]. \tag{48}$$

## Wave function in the classically allowed domain. Width of below-barrier resonance

Transition through the turning point $z = z_2$ into classically allowed domain $z > z_2$ is performed within the Zwaan [3]). Then

$$\widetilde{\Psi} = \widetilde{C}_0 \left( \tilde{\varphi}_0^{(0)} + \tilde{\varphi}_1^{(0)} \rho^2 \right) \exp \left[ i \int_{z_2}^{z} p_0 dx + \tilde{s}_1 \rho^2 + \tilde{s}_2 \rho^4 + \widetilde{S}_1 + im\phi \right], \tag{49}$$

where

$$\widetilde{C}_0 = C_0 \exp \left( - \int_{z_1}^{z_2} q_0(z) dz + \frac{i\pi}{4} \right), \tag{50}$$

$p_0(z) = \sqrt{2(E - V_0)}$ is the quasiclassical momentum in one-dimensional case, and $\tilde{s}_1$, $\tilde{\sigma}$, $\tilde{s}_2$, $\tilde{\varphi}_0^{(0)}$, $\tilde{\varphi}_1^{(0)}$, $\widetilde{S}_1$ are obtained from corresponding quantities $s_1$, $\sigma$, $s_2$, $\varphi_0^{(0)}$, $\varphi_1^{(0)}$, $S_1$ by means of the formal replacement $q_0 \to -ip_0$.

The ionization rate is equal to the total probability flux through the plane which is perpendicular to $z$-axis and located in the domain $z > z_2$:

$$w = \frac{i}{2} \int\limits_S \left( \widetilde{\Psi} \vec{\nabla} \widetilde{\Psi}^* - \widetilde{\Psi}^* \vec{\nabla} \widetilde{\Psi} \right) d\vec{S} = \frac{i}{2} \int_0^{2\pi} \int_0^{\infty} \left( \widetilde{\Psi} \frac{\partial \widetilde{\Psi}^*}{\partial z} - \widetilde{\Psi}^* \frac{\partial \widetilde{\Psi}}{\partial z} \right) \rho d\rho d\phi, \tag{51}$$

Having substituted (49) into (51) and calculating the integral, we obtain:

$$w = \pi C_0^2 |m|! \left( \frac{\mathcal{E}}{\gamma^3} \right)^{|m|+1} e^{-2J} \left[ 1 + \frac{l(l+1) - (|m|+1)(Z/\gamma + 3|m|/2 + 2)}{2\gamma^3} \mathcal{E} \right], \tag{52}$$

where

$$J = \int\limits_{z_1}^{z_2} q_0(z) dz. \tag{53}$$

The barrier integral (53) can be expressed through elliptic integrals of 1-st $K(k)$ and 2-nd $E(k)$ kind. For the asymptotic expression of (53) one can use the technique elaborated by Chibisov [10]:

$$J = \frac{\gamma^3}{3\mathcal{E}} - \frac{Z}{\gamma} + \frac{17 Z^2 \mathcal{E}}{4\gamma^5} + \frac{Z}{\gamma} \left( 1 + \frac{3Z\mathcal{E}}{2\gamma^4} \right) \ln \frac{Z\mathcal{E}}{4\gamma^4}. \tag{54}$$

Taking into account (54) the ionization rate can be written in the form

$$w = \frac{a^2(2l+1)}{2^{|m|+1}|m|!\gamma^{|m|}} \frac{(l+|m|)!}{(l-|m|)!} \left( \frac{2\gamma^2}{\mathcal{E}} \right)^{2Z/\gamma - |m| - 1} e^{-\frac{2\gamma^3}{3\mathcal{E}}} \left\{ 1 + \frac{\mathcal{E}}{4\gamma^3} \times \right.$$
$$\left. \left[ \frac{2Z^2}{\gamma^2} \left( 6 \ln \frac{4\gamma^4}{Z\mathcal{E}} - 17 \right) + 2l(l+1) - (|m|+1)(2Z/\gamma + 3|m| + 4) \right] \right\}. \tag{55}$$

The leading term of (55) coincides with the result of the paper [5] (see corrections for this paper in [11])

## References

[1] S. N. Kaplan, G. A. Paulikas, and R. V. Pule, *Phys. Rev.* **131**, 2574 (1963).

[2] G. Darewich and S. Neamtan, *Nucl. Instr. Meth.* **21**, 241 (1963).

[3] L. D. Landau and E. M. Lifshitz, Quantum Mechanics: Nonrelativistic Theory [in Russian] (Course of Theoretical Physics, Vol. 3), Nauka, Moscow (1989); English transl. prev. ed., Oxford Univ. Press, Oxford (1975).

[4] Yu. N. Demkov, G. F. Drukarev, *JETP* **47**, 918 (1964).

[5] B. M. Smirnov and M. I. Chibisov, *Zh. Eksp. Teor. Fiz.* **49**, 841 (1965).

[6] O. K. Reity, V. K. Reity, V. Yu. Lazur, Proc. 13-th Small Triangle Meeting, Inst. Exp. Phys. SAS, Košice, 94 (2011).

[7] O. K. Reity, V. K. Reity, V. Yu. Lazur, *Uzhhorod University Scientific Herald, Series Physics*, No 27, 97 (2010).

[8] O. K. Reity, V. Yu. Lazur, A. V. Katernoha, *J. Phys. B* **35**, 1 (2002).

[9] M. Yu. Sumetsky, *Teor. Mat. Fiz.* **45**, 64 (1980).

[10] M. I. Chibisov, *Zhurn. Tekhn. Fiz.* **51**, 470 (1981).

[11] A. M. Perelomov, V. S. Popov, M. V. Terent'ev, *JETP* **23**, 924 (1966).