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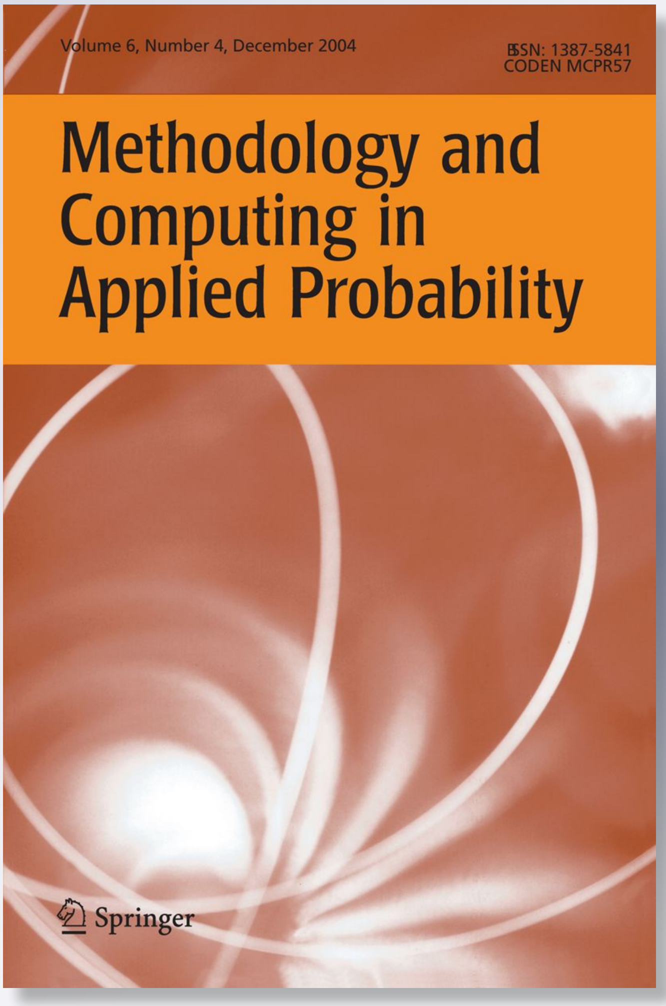

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Simulation of Cox Processes Driven by Random Gaussian Field

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Abstract In this paper we consider the Cox processes directed by random log Gaussian homogeneous field. We construct models for such processes with some accuracy and reliability.

Keywords Cox processes · Simulation · Accuracy · Reliability

AMS 2000 Subject Classification 60G55

1 Introduction

We deal with the Cox processes directed by random log Gaussian fields. Using different simulation methods of Gaussian fields and processes (Kozachenko and Rozora 2004; Kozachenko et al. 2005) we propose a simulation method of Cox processes directed by any Gaussian processes or fields with accuracy and reliability determined beforehand. Also we can weaken the Gaussianity assumption and consider Cox processes directed by Sub-Gaussian processes or processes from Orlicz spaces of random variables (see Buldygin and Kozachenko 2000). Gaussian processes properties which are used during the simulation can be found in Adler (1990), Cramér and Leadbetter (1967), Dudley (1973), Fernique (1975), Piterbarg (2000), Talagrand (1987).

Results may be used in simulation of insurance company activities. The risk process of an insurance company (according to the Encyclopedia of actuarial science (V.3)) may be represented in the form $R_t = u + c(t) - \sum_{i=1}^{N_t} \xi_i$ where $u > 0$ is initial capital of the company, $c(t)$ is the premium function, ξ_i denotes the independent

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claims, N_t is the claim arrival point process, and N_t may be a Cox process. That is we can simulate the risk process with determined accuracy and reliability. Moreover, the paper results may be used for the simulation of Cox processes in statistical analysis of spatial point patterns (see Brix and Møller 2001; Møller et al. 1998; Møller and Waagepetersen 2004).

Let $\{\Omega, \mathfrak{F}, \mathbf{P}\}$ be a standard probability space, $\mathbf{T}$ be a hyperparallelepiped in $\mathbf{R}^n$, $\mathfrak{B}$ be a σ-algebra of Borel sets on $\mathbf{T}$, $\left\{Y\left(\vec{t}\right),\ \vec{t} \in \mathbf{T}\right\}$ be a centered homogeneous Gaussian random field. Let the random field $\left\{Y\left(\vec{t}\right),\ \vec{t} \in \mathbf{T}\right\}$ be mean square continuous, $C\left(\vec{t}-\vec{s}\right) = \mathbf{E}Y\left(\vec{t}\right)Y\left(\vec{s}\right)$.

Definition 1.1 The random process $\{\nu(B),\ B \in \mathfrak{B}\}$ is said to be a log Gaussian Cox process driven by the random field $\exp\left\{Y\left(\vec{t}\right)\right\}$ if the following assertions hold:

1) if $B_1 \cap B_2 = \emptyset$, $B_1, B_2 \in \mathfrak{B}$, then the random variables $\nu(B_1)$ and $\nu(B_2)$ are independent;
2) $\mathbf{P}\left\{\nu(B) = k \,/\, Y\left(\vec{t}\right),\ \vec{t} \in \mathbf{T}\right\} = \frac{1}{k!}\exp\{-\mu(B)\}\,(\mu(B))^k$, $k = 0, 1, 2, \ldots$, where $\mu(B) = \int\limits_B \exp\left\{Y\left(\vec{t}, \cdot\right)\right\} d\vec{t}$, and $Y\left(\vec{t}, \cdot\right)$, $\vec{t} \in \mathbf{T}$, is the sample function of the process $\left\{Y\left(\vec{t}\right),\ \vec{t} \in \mathbf{T}\right\}$.

Since $\{\nu(B),\ B \in \mathfrak{B}\}$ is a double stochastic random process, the model of this process is constructed in two stages. At first we simulate the Gaussian field $\left\{Y\left(\vec{t}\right),\ \vec{t} \in \mathbf{T}\right\}$, then we consider some partitioning $D_{\mathbf{T}}$ of the domain $\mathbf{T}$ and on every element of the partitioning $D_{\mathbf{T}}$ we construct the model of Poisson random variable with corresponding mean.

Let $\mathbf{T} = [0, T] \times \ldots \times [0, T]$, $T \in \mathbf{R}_+$, and choose the partitioning of $D_{\mathbf{T}}$ in the following way:

$$B_{i_1,\ldots,i_n} = \left\{ \left[t_1^{i_1}, t_1^{i_1+1}\right) \times \ldots \times \left[t_n^{i_n}, t_n^{i_n+1}\right) \,\middle|\, t_m^{i_m} < t_m^{i_m+1},\right.$$
$$\left. t_m^{i_m+1} - t_m^{i_m} = d = \frac{T}{k},\ k \in \mathbf{N},\ m = \overline{1, n},\ i_m = \overline{0, k-1} \right\}.$$

Denote by $\widetilde{Y}\left(\vec{t}\right)$ – the model of the field $Y\left(\vec{t}\right)$, $\widetilde{\mu}(B_{i_1,\ldots,i_n}) = \int_{B_{i_1,\ldots,i_n}} \exp\left\{\widetilde{Y}\left(\vec{t}\right)\right\} d\vec{t}$, $\widetilde{\nu}(B_{i_1,\ldots,i_n})$ – the model of $\nu(B_{i_1,\ldots,i_n})$, that is the model of a Poisson random variable with mean $\widetilde{\mu}(B_{i_1,\ldots,i_n})$.

Since $\widetilde{\nu}(B_{i_1,\ldots,i_n})$ is the number of points which belong to domain $B_{i_1,\ldots,i_n}$, we allocate these points in $B_{i_1,\ldots,i_n}$ by all means. If $\widetilde{\nu}(B_{i_1,\ldots,i_n}) = 1$, we place this point in the center of the domain.

It is evident that the model $\widetilde{\nu}(B_{i_1,\ldots,i_n})$ is admissible if the conditional probabilities $p_{kY}(B_{i_1,\ldots,i_n}) = \mathbf{P}\left\{\nu(B_{i_1,\ldots,i_n}) = k \,/\, Y\left(\vec{t}\right),\ \vec{t} \in \mathbf{T}\right\}$ and $\widetilde{p}_{kY}(B_{i_1,\ldots,i_n}) = \mathbf{P}\left\{\widetilde{\nu}(B_{i_1,\ldots,i_n}) = k \,/\, \widetilde{Y}\left(\vec{t}\right),\ \vec{t} \in \mathbf{T}\right\}$ differ little and the probability of the event that the number of points $\nu(B_{i_1,\ldots,i_n})$ (respectively $\widetilde{\nu}(B_{i_1,\ldots,i_n})$) is more than one is little as well. Therefore the problem of simulation of the log Gaussian Cox process consists of two problems. The first is the problem of the choice of domain $\mathbf{T}$ partitioning and the second is the construction of the model of the field $Y\left(\vec{t}\right)$.

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2 Partitioning of the Domain T

Partitioning of the domain **T** is chosen in such a way that the following inequality holds true

$$\mathbf{P}\left\{\nu\left(B_{i_1,\ldots,i_n}\right) > 1\right\} < \delta, \tag{1}$$

where δ is given and small.

Theorem 2.1 *The inequality (1) holds true if we set*

$$d = \frac{T}{k} \le \left[2\delta \exp\left\{-2C(\vec{0})\right\}\right]^{\frac{1}{2n}}, \tag{2}$$

where $C(\cdot)$ is the covariation function of the field $Y\left(\vec{t}\right)$.

Proof Since

$$\mathbf{P}\left\{\nu\left(B_{i_1,\ldots,i_n}\right)>1\right\}=\mathbf{E}\left\{1-\exp\{-\mu\left(B_{i_1,\ldots,i_n}\right)\}-\mu\left(B_{i_1,\ldots,i_n}\right)\exp\{-\mu\left(B_{i_1,\ldots,i_n}\right)\}\right\},$$

then it is sufficient to choose such partitioning, that the following inequality holds true

$$\mathbf{E}\left\{1-\exp\left\{-\mu\left(B_{i_1,\ldots,i_n}\right)\right\}-\mu\left(B_{i_1,\ldots,i_n}\right)\exp\left\{-\mu\left(B_{i_1,\ldots,i_n}\right)\right\}\right\} < \delta.$$

By virtue of the inequality $1-\exp\{-x\}(1+x) \le \frac{x^2}{2}$ as $x > 0$ the preceding inequality holds true if

$$\mathbf{E}\left\{\left[\mu\left(B_{i_1,\ldots,i_n}\right)\right]^2\right\} < 2\delta.$$

For $\xi = N\left(0,\sigma^2\right)$, we have $\mathbf{E}\exp\{\lambda\xi\} = \exp\left\{\frac{\lambda^2\sigma^2}{2}\right\}$ and

$$\begin{aligned}
\mathbf{E}\left\{\left[\mu\left(B_{i_1,\ldots,i_n}\right)\right]^2\right\} &= \mathbf{E}\left\{\int\limits_{B_{i_1,\ldots,i_n}} \exp\left\{Y\left(\vec{t}\right)\right\}d\vec{t} \int\limits_{B_{i_1,\ldots,i_n}} \exp\left\{Y\left(\vec{s}\right)\right\}d\vec{s}\right\} \\
&= \iint\limits_{B_{i_1,\ldots,i_n}\times B_{i_1,\ldots,i_n}} \mathbf{E}\left\{\exp\left\{Y\left(\vec{t}\right)+Y\left(\vec{s}\right)\right\}\right\}d\vec{t}d\vec{s} \\
&= \iint\limits_{B_{i_1,\ldots,i_n}\times B_{i_1,\ldots,i_n}} \exp\left\{\frac{\mathbf{E}\left\{\left(Y\left(\vec{t}\right)+Y\left(\vec{s}\right)\right)^2\right\}}{2}\right\}d\vec{t}d\vec{s} \\
&= \iint\limits_{B_{i_1,\ldots,i_n}\times B_{i_1,\ldots,i_n}} \exp\left\{\frac{\mathbf{E}\left\{Y^2\left(\vec{t}\right)\right\}}{2} + \mathbf{E}\left\{Y\left(\vec{t}\right)Y\left(\vec{s}\right)\right\} + \frac{\mathbf{E}\left\{Y^2\left(\vec{s}\right)\right\}}{2}\right\}d\vec{t}d\vec{s} \\
&= \iint\limits_{B_{i_1,\ldots,i_n}\times B_{i_1,\ldots,i_n}} \exp\left\{C(\vec{0}) + C\left(\vec{t}-\vec{s}\right)\right\}d\vec{t}d\vec{s} \le d^{2n}\exp\left\{2C(\vec{0})\right\}.
\end{aligned}$$

The assertion of this theorem follows from the last inequality. □

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3 Construction of the Model of the Field $Y(\vec{t})$

Let $\{\mathbf{R}_+^n, \mathfrak{U}, \nu\}$–be measurable space, where $\mathfrak{U}$–is a σ-algebra of Borel sets and ν–is a finite measure. We consider centered, homogeneous in wide sense, continuous in mean square random fields. Therefore the covariance function $C(\vec{\tau})$ of these fields can be presented in the form $C(\vec{\tau}) = \int\limits_{\mathbf{R}_+^n} \cos(\vec{\lambda}, \vec{\tau}) d\nu(\vec{\lambda})$, where $\nu(\vec{\lambda})$, $\vec{\lambda} \in \mathbf{R}_+^n$-is a finite measure such that $\nu(\mathbf{R}_+^n) = C(\vec{0})$, see for example Gikhman and Skorokhod (1971). Then it follows from Karhunen theorem that the field $Y(\vec{t})$ can be presented in the form

$$Y(\vec{t}) = \int\limits_{\mathbf{R}_+^n} \cos(\vec{\lambda}, \vec{t}) dZ_1(\vec{\lambda}) + \int\limits_{\mathbf{R}_+^n} \sin(\vec{\lambda}, \vec{t}) dZ_2(\vec{\lambda}),$$

where $Z_1(S)$ and $Z_2(S)$, $S \in \mathfrak{U}$, are uncorrelated random measures subordinated to the measure ν, that is $\mathbf{E}\{Z_i(S_1) Z_i(S_2)\} = \nu(S_1 \cap S_2)$, $S_1, S_2 \in \mathfrak{U}$, $i = 1, 2$. The following sum $\widetilde{Y}(\vec{t})$ will be considered as a model of the field $Y(\vec{t})$:

$$\begin{aligned}\widetilde{Y}(\vec{t}) = & \sum_{i_1,\ldots,i_n=0}^{N-1} \cos\left(\vec{t}, \vec{\lambda}(\lambda_1^{i_1}, \ldots, \lambda_n^{i_n})\right) Z_1(\Delta(i_1, \ldots, i_n)) \\ & + \sum_{i_1,\ldots,i_n=0}^{N-1} \sin\left(\vec{t}, \vec{\lambda}(\lambda_1^{i_1}, \ldots, \lambda_n^{i_n})\right) Z_2(\Delta(i_1, \ldots, i_n)),\end{aligned} \tag{3}$$

where $\vec{\lambda}\left(\lambda_1^{i_1}, \ldots, \lambda_n^{i_n}\right)$ are the points of partitioning D_{Λ^n} :

$$\begin{aligned}\Delta(i_1, \ldots, i_n) = \Big\{ & [\lambda_1^{i_1}, \lambda_1^{i_1+1}) \times \ldots \times [\lambda_n^{i_n}, \lambda_n^{i_n+1}) \mid \lambda_m^{i_m} < \lambda_m^{i_m+1}, \\ & \lambda_m^{i_m+1} - \lambda_m^{i_m} = \frac{\Lambda}{N}, \ \Lambda \in \mathbf{R}_+, \ N \in \mathbf{N}, \ m = \overline{1, n}, \ i_m = \overline{1, N-1} \Big\}.\end{aligned}$$

Since the field $Y(\vec{t})$ is Gaussian, it follows from Karhunen theorem that $Z_1(S)$ and $Z_2(S)$ are Gaussian as well.

4 Approximation of Log Gaussian Cox Process with Given Accuracy and Reliability

We want to construct such a model of log Gaussian Cox process $\{\nu(B), B \in \mathfrak{B}\}$, that the conditional probabilities $p_{kY}(B_{i_1,\ldots,i_n})$ and $\widetilde{p}_{kY}(B_{i_1,\ldots,i_n})$ differ little with probability close to one. Therefore we present the following definition.

Definition 4.1 The model of log Gaussian Cox process approximates that process with accuracy α, $0 < \alpha < 1$, and reliability $1 - \gamma$, $0 < \gamma < 1$, if the following inequality holds true

$$\mathbf{P}\left\{ \max_{B_{i_1,\ldots,i_n} \in \mathfrak{B}} \left| p_{kY}\left(B_{i_1,\ldots,i_n}\right) - \widetilde{p}_{kY}\left(B_{i_1,\ldots,i_n}\right) \right| > \alpha \right\} < \gamma.$$

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Let us estimate the difference $|\, p_{kY}(B) - \widetilde{p}_{kY}(B)\,|$, $B \in \mathfrak{B}$. Let $k \neq 0$, it follows from Lagrange formula that

$$
\begin{aligned}
|\, p_{kY}(B) - \widetilde{p}_{kY}(B)\,| &= \left| \frac{\exp\{-\mu(B)\}\,(\mu(B))^k}{k!} - \frac{\exp\{-\widetilde{\mu}(B)\}\,(\widetilde{\mu}(B))^k}{k!} \right| \\
&= |\mu(B) - \widetilde{\mu}(B)|\,\frac{1}{k!}\exp\left\{-\hat{\mu}(B)\right\}\left(\hat{\mu}(B)\right)^{k-1}\left|k - \hat{\mu}(B)\right| \\
&= \begin{cases} |\mu(B)-\widetilde{\mu}(B)|\dfrac{1}{(k-1)!}\exp\left\{-\hat{\mu}(B)\right\}\left(\hat{\mu}(B)\right)^{k-1} \\ \qquad \leq |\mu(B)-\widetilde{\mu}(B)|\,,\; k \geq \hat{\mu}(B)\,; \\ \mu(B)-\widetilde{\mu}(B)|\,\dfrac{1}{k!}\exp\left\{-\hat{\mu}(B)\right\}\left(\hat{\mu}(B)\right)^{k} \\ \qquad \leq |\mu(B)-\widetilde{\mu}(B)|\,,\quad k < \hat{\mu}(B)\,. \end{cases}
\end{aligned}
$$

If $k = 0$, then

$$
\begin{aligned}
|p_{0Y}(B) - \widetilde{p}_{0Y}(B)| &= |\exp\{-\mu(B)\} - \exp\{-\widetilde{\mu}(B)\}| \\
&\leq |\mu(B) - \widetilde{\mu}(B)|\exp\left\{-\hat{\mu}(B)\right\} \leq |\mu(B) - \widetilde{\mu}(B)|\,.
\end{aligned}
$$

That is, we have

$$
\begin{aligned}
&\mathbf{P}\left\{\max_{B_{i_1,\ldots,i_n}\in\mathfrak{B}} \left|p_{kY}\left(B_{i_1,\ldots,i_n}\right)-\widetilde{p}_{kY}\left(B_{i_1,\ldots,i_n}\right)\right| > \alpha\right\} \\
&\quad\leq \mathbf{P}\left\{\max_{B_{i_1,\ldots,i_n}\in\mathfrak{B}} \left|\mu\left(B_{i_1,\ldots,i_n}\right)-\widetilde{\mu}\left(B_{i_1,\ldots,i_n}\right)\right| > \alpha\right\}.
\end{aligned}
$$

Lemma 4.1 *Let $Y(\vec{t})$ be a homogeneous, centered, continuous in mean square random field, then $\forall p > 1$ we have*

$$
\begin{aligned}
&\mathbf{P}\left\{\max_{B_{i_1,\ldots,i_n}\in\mathfrak{B}} \left|\mu\left(B_{i_1,\ldots,i_n}\right)-\widetilde{\mu}\left(B_{i_1,\ldots,i_n}\right)\right| > \alpha\right\} \\
&\quad\leq \frac{2\,k^n\,M_N^p\,p^{\frac{p}{2}}\exp\left\{-\frac{p}{2}+\frac{p^2 v_2}{2}C(\vec{0})\right\}}{\alpha^p},
\end{aligned} \tag{4}
$$

where

$$
\begin{aligned}
M_N &= d^n v_1^{\frac{1}{2}} A_N^{\frac{1}{2}}, \\
A_N &= 2^{2-2a} n^{2a}\frac{d^{2a}\Lambda^{2a}}{N^{2a}}\nu\left(\Lambda^n\right) + C(\vec{0}) - \nu\left(\Lambda^n\right),
\end{aligned} \tag{5}
$$

$\mu(B) = \int_B \exp\left\{Y\left(\vec{t},\cdot\right)\right\}d\vec{t}$, $\quad Y\left(\vec{t},\cdot\right)$, $\vec{t} \in \mathbf{T}$ *– is a sample function of the field* $Y\left(\vec{t}\right)$, $C(\vec{\tau})$ *– is the covariance function of the field* $Y\left(\vec{t}\right)$, $v_2 = \frac{v_1}{v_1-1}$, v_1 *– any positive number,* $a \in [0, 1]$.

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Proof

$$\begin{aligned}
&\mathbf{P}\left\{\max_{B_{i_1,\ldots,i_n}\in\mathfrak{B}}\left|\mu\left(B_{i_1,\ldots,i_n}\right)-\widetilde{\mu}\left(B_{i_1,\ldots,i_n}\right)\right|>\alpha\right\}\\
&\quad\le\sum_{i_1,\ldots,i_n=0}^{k}\mathbf{P}\left\{\left|\mu\left(B_{i_1,\ldots,i_n}\right)-\widetilde{\mu}\left(B_{i_1,\ldots,i_n}\right)\right|>\alpha\right\}\\
&\quad\le k^n\max_{B_{i_1,\ldots,i_n}\in\mathfrak{B}}\mathbf{P}\left\{\left|\mu\left(B_{i_1,\ldots,i_n}\right)-\widetilde{\mu}\left(B_{i_1,\ldots,i_n}\right)\right|>\alpha\right\}. \qquad (6)
\end{aligned}$$

It follows from Tchebychev inequality that

$$\mathbf{P}\left\{\left|\mu\left(B_{i_1,\ldots,i_n}\right)-\widetilde{\mu}\left(B_{i_1,\ldots,i_n}\right)\right|>\alpha\right\}\le\frac{\mathbf{E}\left\{\left|\mu\left(B_{i_1,\ldots,i_n}\right)-\widetilde{\mu}\left(B_{i_1,\ldots,i_n}\right)\right|^p\right\}}{\alpha^p}. \qquad (7)$$

Let $\frac{1}{p}+\frac{1}{q}=1$, $p,q>0$. By virtue of Holder inequality we have:

$$\begin{aligned}
\left|\mu\left(B_{i_1,\ldots,i_n}\right)-\widetilde{\mu}\left(B_{i_1,\ldots,i_n}\right)\right|&\le\int\limits_{B_{i_1,\ldots,i_n}}\left|\exp\left\{Y\left(\vec{t}\right)\right\}-\exp\left\{\widetilde{Y}\left(\vec{t}\right)\right\}\right|d\vec{t}\\
&\le\left(\int\limits_{B_{i_1,\ldots,i_n}}\left|\exp\left\{Y\left(\vec{t}\right)\right\}-\exp\left\{\widetilde{Y}\left(\vec{t}\right)\right\}\right|^p d\vec{t}\right)^{\frac{1}{p}}\left(\int\limits_{B_{i_1,\ldots,i_n}}1^q d\vec{t}\right)^{\frac{1}{q}}\\
&=d^{n\left(1-\frac{1}{p}\right)}\left(\int\limits_{B_{i_1,\ldots,i_n}}\left|\exp\left\{Y\left(\vec{t}\right)\right\}-\exp\left\{\widetilde{Y}\left(\vec{t}\right)\right\}\right|^p d\vec{t}\right)^{\frac{1}{p}} \qquad (8)
\end{aligned}$$

It follows from (8) and (7) that

$$\begin{aligned}
&\mathbf{P}\left\{\left|\mu\left(B_{i_1,\ldots,i_n}\right)-\widetilde{\mu}\left(B_{i_1,\ldots,i_n}\right)\right|>\alpha\right\}\\
&\quad\le\frac{\mathbf{E}\left\{d^{np-n}\int_{B_{i_1,\ldots,i_n}}\left|\exp\left\{Y\left(\vec{t}\right)\right\}-\exp\left\{\widetilde{Y}\left(\vec{t}\right)\right\}\right|^p d\vec{t}\right\}}{\alpha^p}\\
&\quad=\frac{d^{np-n}\int_{B_{i_1,\ldots,i_n}}\mathbf{E}\left\{\left|\exp\left\{Y\left(\vec{t}\right)\right\}-\exp\left\{\widetilde{Y}\left(\vec{t}\right)\right\}\right|^p\right\}d\vec{t}}{\alpha^p}. \qquad (9)
\end{aligned}$$

Let us estimate $\mathbf{E}\{\left|\exp\left\{Y\left(\vec{t}\right)\right\}-\exp\left\{\widetilde{Y}\left(\vec{t}\right)\right\}\right|^p\}$. Let $\frac{1}{v_1}+\frac{1}{v_2}=1$. It follows from the inequality $|\exp\{x\}-\exp\{y\}|\le|x-y|\exp\{\max(x,\ y)\}$ and Holder inequality that

$$\begin{aligned}
&\mathbf{E}\left\{\left|\exp\left\{Y\left(\vec{t}\right)\right\}-\exp\left\{\widetilde{Y}\left(\vec{t}\right)\right\}\right|^p\right\}\\
&\quad\le\mathbf{E}\left\{\left|Y\left(\vec{t}\right)-\widetilde{Y}\left(\vec{t}\right)\right|^p\exp\left\{p\max\left(Y\left(\vec{t}\right),\ \widetilde{Y}\left(\vec{t}\right)\right)\right\}\right\}\\
&\quad\le\left(\mathbf{E}\left\{\left|Y\left(\vec{t}\right)-\widetilde{Y}\left(\vec{t}\right)\right|^{pv_1}\right\}\right)^{\frac{1}{v_1}}\left(\mathbf{E}\left\{\exp\left\{p\,v_2\max\left(Y\left(\vec{t}\right),\ \widetilde{Y}\left(\vec{t}\right)\right)\right\}\right\}\right)^{\frac{1}{v_2}}. \qquad (10)
\end{aligned}$$

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If ξ is Gaussian random variable with $\mathbf{E}\{\xi\} = 0$, $\mathbf{E}\left\{\xi^2\right\} = \sigma^2$, then

$$\mathbf{E}\left\{|\xi|^p\right\} = c_p\,\left(\sigma^2\right)^{\frac{p}{2}}, \tag{11}$$

where $c_p = \frac{2^{\frac{p}{2}}}{\sqrt{\pi}}\Gamma\left(\frac{p+1}{2}\right)$. It follows from Stirling formula that

$$c_p \le \sqrt{2}p^{\frac{p}{2}}\exp\left\{-\frac{p}{2}\right\}. \tag{12}$$

It follows from (11) that

$$\mathbf{E}\left\{\left|\,Y\left(\vec{t}\right) - \widetilde{Y}\left(\vec{t}\right)\right|^{pv_1}\right\} = c_{pv_1}\left(\mathbf{E}\left\{\left|\,Y\left(\vec{t}\right) - \widetilde{Y}\left(\vec{t}\right)\right|^2\right\}\right)^{\frac{pv_1}{2}}.$$

Since $\mathbf{E}\left\{Y^2\left(\vec{t}\right)\right\} = C(\vec{0})$, $\mathbf{E}\left\{\widetilde{Y}^2\left(\vec{t}\right)\right\} = \nu\left(\Lambda^n\right)$, then

$$\mathbf{E}\left\{\left|\,Y\left(\vec{t}\right) - \widetilde{Y}\left(\vec{t}\right)\right|^2\right\} = C\left(\vec{0}\right) + \nu\left(\Lambda^n\right) - 2\,\mathbf{E}\left\{Y\left(\vec{t}\right)\widetilde{Y}\left(\vec{t}\right)\right\}.$$

$$\begin{aligned}
\mathbf{E}\left\{Y(\vec{t})\,\widetilde{Y}(\vec{t})\right\} = \mathbf{E}\Bigg\{\Bigg(&\sum_{i_1,\ldots,i_n=0}^{N-1}\int\limits_{\Delta(i_1,\ldots,i_n)}\cos(\vec{t},\vec{\lambda})dZ_1(\vec{\lambda}) + \sum_{i_1,\ldots,i_n=0}^{N-1}\int\limits_{\Delta(i_1,\ldots,i_n)}\sin(\vec{t},\vec{\lambda})dZ_2(\vec{\lambda})\\
&+\int\limits_{\mathbf{R}^n\setminus\Lambda^n}\cos\left(\vec{t},\vec{\lambda}\right)dZ_1(\vec{\lambda}) + \int\limits_{\mathbf{R}^n\setminus\Lambda^n}\sin\left(\vec{t},\vec{\lambda}\right)dZ_2(\vec{\lambda})\Bigg)\\
&\times\Bigg(\sum_{i_1,\ldots,i_n=0}^{N-1}\int\limits_{\Delta(i_1,\ldots,i_n)}\cos\left(\vec{t},\vec{\lambda}\left(\lambda_1^{i_1},\ldots,\lambda_n^{i_n}\right)\right)dZ_1(\vec{\lambda})\\
&+\sum_{i_1,\ldots,i_n=0}^{N-1}\int\limits_{\Delta(i_1,\ldots,i_n)}\sin\left(\vec{t},\vec{\lambda}\left(\lambda_1^{i_1},\ldots,\lambda_n^{i_n}\right)\right)dZ_2(\vec{\lambda})\Bigg)\Bigg\}\\
=&\sum_{i_1,\ldots,i_n=0}^{N-1}\int\limits_{\Delta(i_1,\ldots,i_n)}\cos\left(\vec{t},\vec{\lambda}\right)\cos\left(\vec{t},\vec{\lambda}\left(\lambda_1^{i_1},\ldots,\lambda_n^{i_n}\right)\right)d\nu(\vec{\lambda})\\
&+\sum_{i_1,\ldots,i_n=0}^{N-1}\int\limits_{\Delta(i_1,\ldots,i_n)}\sin\left(\vec{t},\vec{\lambda}\right)\sin\left(\vec{t},\vec{\lambda}\left(\lambda_1^{i_1},\ldots,\lambda_n^{i_n}\right)\right)d\nu(\vec{\lambda})\\
=&\sum_{i_1,\ldots,i_n=0}^{N-1}\int\limits_{\Delta(i_1,\ldots,i_n)}\cos\left(\vec{t},\vec{\lambda}-\vec{\lambda}\left(\lambda_1^{i_1},\ldots,\lambda_n^{i_n}\right)\right)d\nu(\vec{\lambda}).
\end{aligned}$$

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$$\begin{aligned}
&\mathbf{E}\left\{\left| Y\left(\vec{t}\right) - \widetilde{Y}\left(\vec{t}\right)\right|^2\right\}\\
&= 2\,\nu\left(\Lambda^n\right) - 2\,\mathbf{E}\left\{Y\left(\vec{t}\right)\widetilde{Y}\left(\vec{t}\right)\right\} + C(\vec{0}) - \nu\left(\Lambda^n\right)\\
&= 2\sum_{i_1,\dots,i_n=0}^{N-1}\ \int\limits_{\Delta(i_1,\dots,i_n)} \left(1-\cos\left(\vec{t}, \vec{\lambda}-\vec{\lambda}\left(\lambda_1^{i_1},\dots,\lambda_n^{i_n}\right)\right)\right)d\nu\left(\vec{\lambda}\right) + C(\vec{0}) - \nu\left(\Lambda^n\right)\\
&= 4\sum_{i_1,\dots,i_n=0}^{N-1}\ \int\limits_{\Delta(i_1,\dots,i_n)} \sin^2\frac{\left(\vec{t}, \vec{\lambda}-\vec{\lambda}\left(\lambda_1^{i_1},\dots,\lambda_n^{i_n}\right)\right)}{2}d\nu\left(\vec{\lambda}\right) + C(\vec{0}) - \nu\left(\Lambda^n\right)\\
&\le 4\sum_{i_1,\dots,i_n=0}^{N-1}\ \int\limits_{\Delta(i_1,\dots,i_n)} \left(\frac{\vec{t}, \vec{\lambda}-\vec{\lambda}\left(\lambda_1^{i_1},\dots,\lambda_n^{i_n}\right)}{2}\right)^{2a}d\nu\left(\vec{\lambda}\right) + C(\vec{0}) - \nu\left(\Lambda^n\right),
\end{aligned}$$

$a \in [0,\ 1]$. Since $\left(\vec{e},\ \vec{f}\right) \le \left(\sum\limits_{i=1}^{n} e_i^2\right)^{\frac{1}{2}}\left(\sum\limits_{i=1}^{n} f_i^2\right)^{\frac{1}{2}}$ and $\lambda_m - \lambda_m^{i_m} \le \lambda_m^{i_m+1} - \lambda_m^{i_m} = \frac{\Lambda}{N}$, then

$$\begin{aligned}
&\mathbf{E}\left\{\left| Y\left(\vec{t}\right) - \widetilde{Y}\left(\vec{t}\right)\right|^2\right\}\\
&\le 4\sum_{i_1,\dots,i_n=0}^{N-1}\ \int\limits_{\Delta(i_1,\dots,i_n)} \frac{\left(\sum\limits_{m=1}^{n} t_m^2\right)^a\left(\sum\limits_{m=1}^{n}\left(\lambda_m-\lambda_m^{i_m}\right)^2\right)^a}{2^{2a}}d\nu\left(\vec{\lambda}\right) + C(\vec{0}) - \nu\left(\Lambda^n\right)\\
&= 2^{2-2a}\sum_{i_1,\dots,i_n=0}^{N-1}\ \int\limits_{\Delta(i_1,\dots,i_n)} \left(nd^2\right)^a\left(n\frac{\Lambda^2}{N^2}\right)^a d\nu\left(\vec{\lambda}\right) + C(\vec{0}) - \nu\left(\Lambda^n\right)\\
&= 2^{2-2a}n^{2a}\frac{d^{2a}\Lambda^{2a}}{N^{2a}}\nu\left(\Lambda^n\right) + C(\vec{0}) - \nu\left(\Lambda^n\right).
\end{aligned}$$

Therefore

$$\begin{aligned}
&\mathbf{E}\left\{\left| Y\left(\vec{t}\right) - \widetilde{Y}\left(\vec{t}\right)\right|^{pv_1}\right\} \le c_{pv_1}A_N^{\frac{pv_1}{2}},\\
&A_N = 2^{2-2a}n^{2a}\frac{d^{2a}\Lambda^{2a}}{N^{2a}}\nu\left(\Lambda^n\right) + C(\vec{0}) - \nu\left(\Lambda^n\right). \qquad (13)
\end{aligned}$$

Let us estimate $\mathbf{E}\left\{\exp\left\{p\,v_2\max\left(Y\left(\vec{t}\right),\ \widetilde{Y}\left(\vec{t}\right)\right)\right\}\right\}$. If $\xi = N\left(0,\ \sigma^2\right)$ then for all $\lambda \in \mathbf{R}$ $\mathbf{E}\left\{\exp\left\{\lambda\xi\right\}\right\} = \exp\left\{\frac{\lambda^2\sigma^2}{2}\right\}$. Therefore

$$\begin{aligned}
&\mathbf{E}\left\{\exp\left\{p\,v_2\max\left(Y\left(\vec{t}\right),\widetilde{Y}\left(\vec{t}\right)\right)\right\}\right\} \le \mathbf{E}\left\{\exp\left\{p\,v_2 Y\left(\vec{t}\right)\right\}\right\} + \mathbf{E}\left\{\exp\left\{p\,v_2\widetilde{Y}\left(\vec{t}\right)\right\}\right\}\\
&\quad= \exp\left\{\frac{(p\,v_2)^2}{2}B(\vec{0})\right\} + \exp\left\{\frac{(p\,v_2)^2}{2}\nu\left(\Lambda^n\right)\right\} \le 2\exp\left\{\frac{(p\,v_2)^2}{2}C(\vec{0})\right\}. \qquad (14)
\end{aligned}$$

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It follows from (13), (14) and (10) that:

$$\mathbf{E}\left\{\left|\exp\left\{Y\left(\vec{t}\right)\right\}-\exp\left\{\widetilde{Y}\left(\vec{t}\right)\right\}\right|^{p}\right\} \le c_{pv_1}^{\frac{1}{v_1}} A_N^{\frac{p}{2}} 2^{\frac{1}{v_2}} \exp\left\{\frac{p^2 v_2}{2} C(\vec{0})\right\}.$$

Now the assertion of the lemma follows from (6). □

Lemma 4.2 *Let $Y\left(\vec{t}\right)$ be a homogeneous, centered, continuous in mean square random field. If $M_N < \alpha \exp\left\{\frac{1}{2} - v_2 C(\vec{0})\right\}$ then we have*

$$\mathbf{P}\left\{\max_{B_{i_1,\dots,i_n}\in\mathfrak{B}}\left|\mu\left(B_{i_1,\dots,i_n}\right)-\widetilde{\mu}\left(B_{i_1,\dots,i_n}\right)\right|>\alpha\right\}$$

$$\le 2\,k^n\left(\frac{1-2\ln\frac{M_N}{\alpha}}{2v_2 C(\vec{0})}\right)^{\frac{1-2\ln\frac{M_N}{\alpha}}{4v_2 C(\vec{0})}}\exp\left\{-\frac{\left(1-2\ln\frac{M_N}{\alpha}\right)^2}{8v_2 C(\vec{0})}\right\},$$

where M_N and A_N are defined in (5).

Proof The assertion of this lemma follows from Lemma 4.1 if we set $p_0 = \frac{1-2\ln\frac{M_N}{\alpha}}{2v_2 C\left(\vec{0}\right)}$ in the function $f(p) = \frac{2\,k^n M_N^p\, p^{\frac{p}{2}}\exp\left\{-\frac{p}{2}+\frac{p^2 v_2}{2}C\left(\vec{0}\right)\right\}}{\alpha^p}$. This point is near the point where the function $f\left(p\right)$ has its minimum. □

Theorem 4.1 *Let $Y\left(\vec{t}\right)$ be a centered, homogeneous, continuous in mean square, Gaussian random field, which drives the log Gaussian Cox process $\{\nu\left(B\right),\ B\in\mathfrak{B}\}$. Then the model of this field constructed in Section 3 generates the log Gaussian process $\left\{\widetilde{\nu}(B_{i_1,\dots,i_n}),\quad B_{i_1,\dots,i_n}\subset\mathfrak{B}\right\}$, which approximates the process ν with accuracy α and reliability $1-\gamma$ if the following conditions hold true:*

$$M_N < \alpha\exp\left\{\frac{1}{2}-v_2 C(\vec{0})\right\},$$

$$2\,k^n\left(\frac{1-2\ln\frac{M_N}{\alpha}}{2v_2 C(\vec{0})}\right)^{\frac{1-2\ln\frac{M_N}{\alpha}}{4v_2 C(\vec{0})}}\exp\left\{-\frac{\left(1-2\ln\frac{M_N}{\alpha}\right)^2}{8v_2 C\left(\vec{0}\right)}\right\}<\gamma,$$

Table 1 Values of N for given $\delta, \alpha, 1-\gamma$ and β

δ	α	$1-\gamma$	β	d	N
0.01	0.01	0.99	1	0.253928	6,885
0.01	0.01	0.97	1	0.253928	5,181
0.01	0.03	0.97	1	0.253928	1,655
0.01	0.05	0.95	1	0.253928	833
0.01	0.03	0.97	10	0.361579	45
0.01	0.05	0.95	10	0.361579	26
0.02	0.03	0.97	10	0.429992	73
0.02	0.05	0.95	10	0.429992	42

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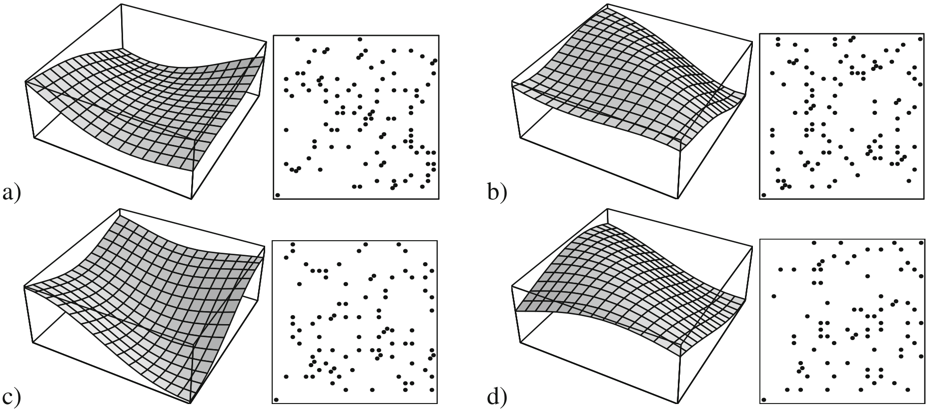

Fig. 1 Models of Gaussian random field $Y(\vec{t})$ and of the log Gaussian Cox process driven by $\exp\{Y(\vec{t})\}$: **a** $\delta = 0.01$, $\alpha = 0.03$, $1-\gamma = 0.97$; **b** $\delta = 0.01$, $\alpha = 0.05$, $1-\gamma = 0.95$; **c** $\delta = 0.02$, $\alpha = 0.03$, $1-\gamma = 0.97$; **d** $\delta = 0.02$, $\alpha = 0.05$, $1-\gamma = 0.95$

where

$$M_N = d^n v_1^{\frac{1}{2}} A_N^{\frac{1}{2}},$$
$$A_N = 2^{2-2a} n^{2a} \frac{d^{2a}\Lambda^{2a}}{N^{2a}} \nu\left(\Lambda^n\right) + C\left(\vec{0}\right) - \nu\left(\Lambda^n\right), \tag{15}$$

$C(\vec{\tau})$ *– is a covariance function of the field* $Y(\vec{t})$, $v_2 = \frac{v_1}{v_1-1}$, v_1 *– any positive number,* $a \in [0, 1]$.

Proof The assertion of this theorem follows from Definition 4.1 and Lemma 4.2. □

Example 1 Let $\{Y(\vec{t}), \vec{t} \in \mathbf{T}\}$, $\mathbf{T} = [0, T] \times [0, T]$, $T \in \mathbf{R}$ be a random field for which the assumption of Theorem 4.1 are hold. Let this field have the spectral density $f(\lambda_1, \lambda_2) = \exp\{-\beta(\lambda_1^2 + \lambda_2^2)\}$. In Table 1 we present values of N for simulation of the log Gaussian Cox process $\{\nu(B), B \in \mathfrak{B}\}$ driven by the field $\{Y(\vec{t}), \vec{t} \in \mathbf{T}\}$ with given accuracy α and reliability $1-\gamma$. All models are constructed in the domain $\mathbf{T} = [0, 10] \times [0, 10]$.

In Fig. 1 the models of Gaussian random field $Y(\vec{t})$ and the models of the log Gaussian Cox process driven by $\exp\{Y(\vec{t})\}$ are presented for the values of δ, α and $1-\gamma$ given in the last four rows of the Table 1.

5 Conclusions

All Gaussian homogeneous, continuous in mean square random fields or processes satisfy the conditions of Theorem 4.1. For example processes with the spectral density $f(\lambda) = \frac{g(\lambda)}{1+|\lambda|^\alpha}$ where $\alpha > 2$, $g(\lambda)$ is a restricted function or, for example, homogeneous fields with the spectral density $f(\lambda_1, \lambda_2) = \frac{g(\lambda_1,\lambda_2)}{1+|\lambda|^\alpha+|\lambda|^\beta}$ where $\alpha > 2$, $\beta > 2$, $g(\lambda_1, \lambda_2)$ is a restricted function.

In the simulation algorithm it does not point out how to choose Λ value in partitioning D_{Λ^n}. In general it cannot be done. The Λ choice depends on the spectral

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density of the field. Small enlargement of Λ value under the condition of other parameters fixation reduces to the strong increase of N value. In practice it is efficient to choose it as small as possible, but in such a way that $C(\vec{0}) - \nu(\Lambda^n)$ also remains as small as possible.

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