УДК 515.146.3

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ПРО КОГОМОЛОГІЇ ЧЕТВЕРНОЇ ГРУПИ КЛЕЙНА

A free resolution of the trivial G-module $\mathbb{Z}$, where G is the Klein four-group, is constructed. Its relation with the standard resolution is established. Also $H^2(G, M)$ for some modules M is calculated.

Ми будуємо вільну резольвенту тривіального G-модуля $\mathbb{Z}$, де G — четверна група Клейна, встановлюємо зв'язок із стандартною резольвентою та обчислюємо $H^2(G, M)$ для деяких модулів M.

Theory of group cohomology is widely used in the theory of representations and the theory of groups, in particular, for the description of special classes of groups. Thus group cohomology plays an important role in the study of group extensions, for instance, in the study of Chernikov groups [1]. In the last case the corresponding G-modules are just dual to integral representations. The usual way to calculate cohomologies is by the standard resolution [2, 3]. Nevertheless, sometimes it is convenient to simplify this resolution. We propose a simplified resolution for the Klein four-group and use it to calculate cohomologies for duals of indecomposable integral representations with at most 3 irreducible components.

Let $G = \langle a, b \mid a^2 = b^2 = (ab)^2 = 1\rangle$ be the Klein four-group. We construct a free resolution of the trivial $\mathbb{Z}G$-module $\mathbb{Z}$, which can be used to calculate cohomologies of this group.

A resolution of $\mathbb{Z}$ for the cyclic group $C_2 = \langle a\rangle$ is well-known:

$$P_A : \ \dots \xrightarrow{a-1} \mathbb{Z}C_2 \xrightarrow{a+1} \mathbb{Z}C_2 \xrightarrow{a-1} \mathbb{Z}C_2 \longrightarrow \dots$$

From the Künneth formulas [3] it follows that a resolution for $G \cong \langle a\rangle \times \langle b\rangle$ can be constructed as $P = P_A \otimes_{\mathbb{Z}} P_B$, where P_A is a resolution for the first factor and P_B is a resolution for the second factor. We write the resolution P_A for the first factor C_2 as

$$\dots \longrightarrow Rx^3 \longrightarrow Rx^2 \longrightarrow Rx \longrightarrow R$$

with the differential $dx^k = (a + (-1)^k)x^{k-1}$, and the resolution P_B for the second factor as

$$\dots \longrightarrow Ry^3 \longrightarrow Ry^2 \longrightarrow Ry \longrightarrow R$$

with the differential $dy^k = (b + (-1)^k)y^{k-1}$. Then the n-th component

$$P_n = \bigoplus_{i+j=n} P_{A,i} \otimes P_{B,j}$$

can be considered as the module of homogeneous polynomials of degree n from $R[x, y]$, where $R = \mathbb{Z}G$ and

$$d(x^i y^j) = (a + (-1)^i)x^{i-1}y^j + (-1)^i(b + (-1)^j)x^i y^{j-1}.$$

So we can write the matrix defining this differential as

$$\begin{pmatrix} a+1 & 1-b & 0 & 0 & \dots \\ 0 & a-1 & b+1 & 0 & \dots \\ 0 & 0 & a+1 & 1-b & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

if n is even and as

$$\begin{pmatrix} a-1 & -(b+1) & 0 & 0 & \dots \\ 0 & a+1 & b-1 & 0 & \dots \\ 0 & 0 & a-1 & -(b+1) & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

if n is odd. Note that for $n = 2$ this results was obtained in [4].

Recall that in the standard resolution

$$F: \ \dots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow \mathbb{Z} \longrightarrow 0,$$

the $\mathbb{Z}G$-module F_n has a basis $[g_1|g_2|\dots|g_n]$, where $g_i \in G\backslash\{1\}$ (we also set $[g_1|g_2|\dots|g_n] = 0$ if some $g_i = 1$) and

$$d[g_1|g_2|\dots|g_n] = g_1[g_2|\dots|g_n] + \sum_{i=1}^{n-1}(-1)^i[g_1|g_2|\dots|g_i g_{i+1}|\dots|g_n] + \\ +(-1)^n[g_1|g_2|\dots|g_n].$$

There is a map $\sigma : F \longrightarrow P$, which defines a homotopy equivalence of these resolutions such that

$$\begin{array}{lll} \sigma_1[a] = x, & \sigma_1[b] = y, & \sigma_2[a|a] = x^2, \\ \sigma_2[b|b] = y^2, & \sigma_2[a|b] = 0, & \sigma_2[b|a] = -xy, \\ \sigma_2[ab|ab] = bx^2 - xy + y^2, & \sigma_2[ab|b] = ay^2, & \sigma_2[ab|a] = bx^2 + xy, \\ \sigma_2[a|ab] = x^2, & \sigma_2[b|ab] = -xy + ay^2. & \end{array} \tag{1}$$

We calculate $H^2(G, M)$, for G-modules M such that M as an abelian group is mQ, where Q is the quasicyclic p-group (or the group of type p^∞). Then the action of G on M is given by an integral p-adic representation of G [1]. We consider the cases when M is indecomposable and not faithful as $\mathbb{Z}_p G$-module.

If $m = 1$, there are 4 such representations $M_{\alpha,\beta}$ (α, $\beta \in \{1, -1\}$) which map $a \mapsto \alpha$, $b \mapsto \beta$. Evidently M_{+-} (M_{-+}) can be obtained from M_{--} if we replace a by ab (resp. b by ab). So we only have to calculate cohomology for M_{++} and M_{--}.

For M_{++} that's why $a = b = 1$, $a + 1 = b + 1 = 2$, $a - 1 = b - 1 = 0$ we have

$$\partial\gamma(x^3) = (a-1)\gamma(x^2) = 0,$$

$$\partial\gamma(y^3) = (b-1)\gamma(y^2) = 0,$$

$$\partial\gamma(x^2y) = (a+1)\gamma(xy) + (b-1)\gamma(x^2) = 2\gamma(xy),$$

as well as $\partial\gamma(xy^2)=2\gamma(xy)$.

We can replace γ by $\partial\xi$ for some $\xi : P_1 \to M_{++}$. Note that

$$\partial\xi(x^2)=(a+1)\xi(x)=2\xi(x),$$
$$\partial\xi(y^2)=(b+1)\xi(y)=2\xi(y),$$
$$\partial\xi(xy)=(a-1)\xi(y)-(b-1)\xi(x)=0.$$

As M is a divisible group, choosing appropriate $\xi(x)$ and $\xi(y)$, we can make $\gamma(x^2)=\gamma(y^2)=0$. Therefore, $H^2(G,M_{++})\simeq \mathbb{Z}/2\mathbb{Z}$ and the non-zero element γ of this group can be chosen as $\gamma(x^2)=\gamma(y^2)=0$, $\gamma(xy)=\varepsilon$, where ε is the unique element of Q of order 2.

Just in the same way we obtain that $H^2(G,M_{--})\simeq(\mathbb{Z}/2\mathbb{Z})^2$ and its elements are the classes of cocycles γ such that $\gamma(x^2)$ and $\gamma(y^2)$ are from $\{0,\varepsilon\}$, while $\gamma(xy)=0$.

If $m=2$, there is an exact sequence

$$0\longrightarrow M_{\alpha,\beta}\longrightarrow M\longrightarrow M_{\alpha',\beta'}\longrightarrow 0. \tag{2}$$

Moreover, if M is indecomposable, $(\alpha,\beta)\neq(\alpha',\beta')$ and M is defined by (α,β) and (α',β'). Note that if there is a sequence (2), there is also an exact sequence

$$0\longrightarrow M_{\alpha',\beta'}\longrightarrow M\longrightarrow M_{\alpha,\beta}\longrightarrow 0.$$

As before, applying an automorphism of G, we can suppose that $(\alpha,\beta)=(1,1)$ and $(\alpha',\beta')=(-1,-1)$ or $(\alpha,\beta)=(-1,1)$ and $(\alpha',\beta')=(1,-1)$.

Let $0\longrightarrow M_{-+}\longrightarrow M\longrightarrow M_{+-}\longrightarrow 0$ be exact. Then M corresponds to the representation of G such that

$$a\to\begin{pmatrix}-1&1\\0&1\end{pmatrix},\qquad b\to\begin{pmatrix}1&-1\\0&-1\end{pmatrix}.$$

Thus

$$a-1=\begin{pmatrix}-2&1\\0&0\end{pmatrix},\qquad b-1=\begin{pmatrix}0&-1\\0&-2\end{pmatrix},$$
$$a+1=\begin{pmatrix}0&1\\0&2\end{pmatrix},\qquad b+1=\begin{pmatrix}2&-1\\0&0\end{pmatrix}.$$

Let

$$\gamma(x^2)=\begin{pmatrix}u_1\\v_1\end{pmatrix},\qquad \gamma(y^2)=\begin{pmatrix}u_2\\v_2\end{pmatrix},\qquad \gamma(xy)=\begin{pmatrix}u_3\\v_3\end{pmatrix}.$$

Then

$$\partial\gamma(x^3)=(a-1)\begin{pmatrix}u_1\\v_1\end{pmatrix}=\begin{pmatrix}-2u_1+v_1\\0\end{pmatrix}=0,$$
$$\partial\gamma(y^3)=(b-1)\begin{pmatrix}u_2\\v_2\end{pmatrix}=\begin{pmatrix}-v_2\\-2v_2\end{pmatrix}=0,$$
$$\partial\gamma(x^2y)=(a+1)\begin{pmatrix}u_3\\v_3\end{pmatrix}+(b-1)\begin{pmatrix}u_1\\v_1\end{pmatrix}=\begin{pmatrix}v_3-v_1\\2v_3-2v_1\end{pmatrix}=0,$$

$$\partial\gamma(xy^2) = (a-1)\begin{pmatrix} u_2 \\ v_2 \end{pmatrix} - (b+1)\begin{pmatrix} u_3 \\ v_3 \end{pmatrix} = \begin{pmatrix} -2u_2 + v_2 - 2u_3 + v_3 \\ 0 \end{pmatrix} = 0.$$

So, we have $v_1 = v_3 = 2u_1 = 2u_2 + 2u_3$, $v_2 = 0$.

Let

$$\xi(x) = \begin{pmatrix} c_1 \\ d_1 \end{pmatrix}, \qquad \xi(y) = \begin{pmatrix} c_2 \\ d_2 \end{pmatrix}.$$

Then

$$\partial\xi(x^2) = (a+1)\begin{pmatrix} c_1 \\ d_1 \end{pmatrix} = \begin{pmatrix} d_1 \\ 2d_1 \end{pmatrix},$$

$$\partial\xi(y^2) = (b+1)\begin{pmatrix} c_2 \\ d_2 \end{pmatrix} = \begin{pmatrix} 2c_2 - d_2 \\ 0 \end{pmatrix},$$

$$\partial\xi(xy) = (a-1)\begin{pmatrix} c_2 \\ d_2 \end{pmatrix} - (b-1)\begin{pmatrix} c_1 \\ d_1 \end{pmatrix} = \begin{pmatrix} -2c_2 + d_2 + d_1 \\ 2d_1 \end{pmatrix},$$

Therefore, changing γ by $\gamma + \partial\xi$, we can make $u_1 = u_2 = 0$, whence also $v_1 = v_3 = 0$, $2u_3 = 0$. Thus $H^2(G, M) \simeq \mathbb{Z}/2\mathbb{Z}$ and the non-zero elements γ of this group is the class of the cycle γ such that

$$\gamma(x^2) = \gamma(y^2) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad \gamma(xy) = \begin{pmatrix} \varepsilon \\ 0 \end{pmatrix}.$$

Let now $0 \longrightarrow M_{--} \longrightarrow M \longrightarrow M_{++} \longrightarrow 0$ is exact, i.e.

$$a \to \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad b \to \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix},$$

$$a - 1 = b - 1 = \begin{pmatrix} -2 & 1 \\ 0 & 0 \end{pmatrix}, \qquad a + 1 = b + 1 = \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}.$$

Let

$$\gamma(x^2) = \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \qquad \gamma(y^2) = \begin{pmatrix} u_2 \\ v_2 \end{pmatrix}, \qquad \gamma(xy) = \begin{pmatrix} u_3 \\ v_3 \end{pmatrix}.$$

Then

$$\partial\gamma(x^3) = (a-1)\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} -2u_1 + v_1 \\ 0 \end{pmatrix} = 0,$$

$$\partial\gamma(y^3) = (b-1)\begin{pmatrix} u_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} -2u_2 + v_2 \\ 0 \end{pmatrix} = 0,$$

$$\partial\gamma(x^2y) = (a+1)\begin{pmatrix} u_3 \\ v_3 \end{pmatrix} + (b-1)\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} v_3 - 2u_1 + v_1 \\ 2v_3 \end{pmatrix} = 0,$$

$$\partial\gamma(xy^2) = (a-1)\begin{pmatrix} u_2 \\ v_2 \end{pmatrix} - (b+1)\begin{pmatrix} u_3 \\ v_3 \end{pmatrix} = \begin{pmatrix} -2u_2 + v_2 - v_3 \\ -2v_3 \end{pmatrix} = 0.$$

So, we have $v_3 = 0$, $2u_1 = v_1$, $2u_2 = v_2$.

Let

$$\xi(x) = \begin{pmatrix} c_1 \\ d_1 \end{pmatrix}, \qquad \xi(y) = \begin{pmatrix} c_2 \\ d_2 \end{pmatrix}.$$

Then

$$\partial\xi(x^2) = (a+1)\begin{pmatrix} c_1 \\ d_1 \end{pmatrix} = \begin{pmatrix} d_1 \\ 2d_1 \end{pmatrix},$$

$$\partial\xi(y^2) = (b+1)\begin{pmatrix} c_2 \\ d_2 \end{pmatrix} = \begin{pmatrix} d_2 \\ 2d_2 \end{pmatrix},$$

$$\partial\xi(xy) = (a-1)\begin{pmatrix} c_2 \\ d_2 \end{pmatrix} - (b-1)\begin{pmatrix} c_1 \\ d_1 \end{pmatrix} = \begin{pmatrix} -2c_2 + d_2 - 2c_1 + d_1 \\ 0 \end{pmatrix}.$$

Hence, changing γ by $\gamma + \partial\xi$, we can make $u_1 = u_2 = 0$ as well as $u_3 = 0$ (as M is divisible). Therefore, $H^2(G, M) = 0$.

Let $m = 3$. If M is indecomposable, there is a chain of submodules

$$M = M_0 \supset M_1 \supset M_2 \supset M_3 = 0$$

such that all quotients $L_i = M_{i-1}/M_i$ are of the form M_{α_i,β_i} and all M_{α_i,β_i} are different. Moreover, we can change the ordering of L_i arbitrarily. Up to an automorphism of G, there are four cases:

1) M_1 is cyclic $(\alpha_1, \beta_1) = (1, 1)$, $(\alpha_2, \beta_2) = (1, -1)$, $(\alpha_3, \beta_3) = (-1, 1)$;
2) M_2 is cyclic $(\alpha_1, \beta_1) = (-1, -1)$, $(\alpha_2, \beta_2) = (-1, 1)$, $(\alpha_3, \beta_3) = (1, -1)$;
3) M_3 is not cyclic $(\alpha_1, \beta_1) = (-1, 1)$, $(\alpha_2, \beta_2) = (1, -1)$, $(\alpha_3, \beta_3) = (1, 1)$;
4) M_4 is not cyclic $(\alpha_1, \beta_1) = (-1, -1)$, $(\alpha_2, \beta_2) = (-1, 1)$, $(\alpha_3, \beta_3) = (1, -1)$.

Case 1. Here

$$a \to \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad b \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

$$a - 1 = \begin{pmatrix} -2 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad b - 1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$a + 1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \qquad b + 1 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

Let

$$\gamma(x^2) = \begin{pmatrix} u_1 \\ v_1 \\ w_1 \end{pmatrix}, \qquad \gamma(y^2) = \begin{pmatrix} u_2 \\ v_2 \\ w_2 \end{pmatrix}, \qquad \gamma(xy) = \begin{pmatrix} u_3 \\ v_3 \\ w_3 \end{pmatrix}.$$

Then

$$\partial\gamma(x^3) = (a-1)\begin{pmatrix} u_1 \\ v_1 \\ w_1 \end{pmatrix} = \begin{pmatrix} -2u_1 + w_1 \\ 0 \\ 0 \end{pmatrix} = 0,$$

$$\partial\gamma(y^3) = (b-1)\begin{pmatrix} u_2 \\ v_2 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 \\ -2v_2 + w_2 \\ 0 \end{pmatrix} = 0,$$

$$\partial\gamma(x^2y) = (a+1)\begin{pmatrix} u_3 \\ v_3 \\ w_3 \end{pmatrix} + (b-1)\begin{pmatrix} u_1 \\ v_1 \\ w_1 \end{pmatrix} = \begin{pmatrix} w_3 \\ 2v_3 - 2v_1 + w_1 \\ 2w_3 \end{pmatrix} = 0,$$

$$\partial\gamma(xy^2) = (a-1)\begin{pmatrix} u_2 \\ v_2 \\ w_2 \end{pmatrix} - (b+1)\begin{pmatrix} u_3 \\ v_3 \\ w_3 \end{pmatrix} = \begin{pmatrix} -2u_2 + w_2 - 2u_3 \\ -w_3 \\ -2w_3 \end{pmatrix} = 0.$$

So, we have $w_3 = 0$, $w_1 = 2u_1$, $w_2 = 2v_2$.

Let

$$\xi(x) = \begin{pmatrix} c_1 \\ d_1 \\ f_1 \end{pmatrix}, \qquad \xi(y) = \begin{pmatrix} c_2 \\ d_2 \\ f_2 \end{pmatrix}.$$

Then

$$\partial\xi(x^2) = (a+1)\begin{pmatrix} c_1 \\ d_1 \\ f_1 \end{pmatrix} = \begin{pmatrix} f_1 \\ 2d_1 \\ 2f_1 \end{pmatrix},$$

$$\partial\xi(y^2) = (b+1)\begin{pmatrix} c_2 \\ d_2 \\ f_2 \end{pmatrix} = \begin{pmatrix} 2c_2 \\ f_2 \\ 2f_2 \end{pmatrix},$$

$$\partial\xi(xy) = (a-1)\begin{pmatrix} c_2 \\ d_2 \\ f_2 \end{pmatrix} - (b-1)\begin{pmatrix} c_1 \\ d_1 \\ f_1 \end{pmatrix} = \begin{pmatrix} -2c_2 + f_2 \\ 2d_1 - f_2 \\ 0 \end{pmatrix}.$$

So we can make $u_1 = u_2 = v_1 = v_2 = 0$, which gives $H^2(M) = (\mathbb{Z}/2\mathbb{Z})^2$, consisting of the classes of cocycles γ such that

$$\gamma(x^2) = \gamma(y^2) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \gamma(xy) = \begin{pmatrix} u \\ v \\ 0 \end{pmatrix},$$

where $u, v \in \{0, \varepsilon\}$.

The calculations in other cases are quite similar, so we only present the results, with some comments in Case 3.

Case 2.

$$a \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad b \to \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}.$$

Here we have $H^2(G, M) = \mathbb{Z}/2\mathbb{Z}$ and the nonzero element of this group is the class of the cocycle γ with $\gamma(x^2) = \gamma(y^2) = 0$, while

$$\gamma(xy) = \begin{pmatrix} \varepsilon \\ 0 \\ 0 \end{pmatrix}.$$

Case 3.

$$a \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad b \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

For $\gamma : G \to M$ such that

$$\gamma(x) = \begin{pmatrix} c_1 \\ d_1 \\ f_1 \end{pmatrix}, \qquad \gamma(y) = \begin{pmatrix} c_2 \\ d_2 \\ f_2 \end{pmatrix},$$

we obtain

$$\partial\gamma(x^2) = \begin{pmatrix} 2c_1 + f_1 \\ 2d_1 \\ 0 \end{pmatrix}, \qquad \partial\gamma(y^2) = \begin{pmatrix} 2c_2 + d_2 \\ 0 \\ 2f_2 \end{pmatrix}, \qquad \partial\gamma(xy) = \begin{pmatrix} f_2 - d_1 \\ 2d_1 \\ -2f_2 \end{pmatrix}.$$

Therefore, changing ξ by $\xi + \partial\gamma$, we can make

$$\xi(x^2) = \begin{pmatrix} 0 \\ 0 \\ w \end{pmatrix}, \qquad \xi(y^2) = \begin{pmatrix} 0 \\ v \\ 0 \end{pmatrix}, \qquad \xi(xy) = \begin{pmatrix} u_3 \\ v_3 \\ w_3 \end{pmatrix}.$$

The condition $\partial\xi = 0$ implies that $w = v = 0$, $v_3 = w_3 = 2u_3$ and $2v_3 = 0$. Hence $v_3 \in \{0, \varepsilon\}$, whence $H^2(G, M) = \mathbb{Z}/2\mathbb{Z}$ with the non-zero element being the class of the cocycle ξ such that $\xi(x^2) = \xi(y^2) = 0$,

$$\xi(xy) = \begin{pmatrix} \varepsilon' \\ \varepsilon \\ \varepsilon \end{pmatrix},$$

where ε' is an element of order 4 (any of two such elements can be chosen).

Case 4.

$$a \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad b \to \begin{pmatrix} -1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Here $H^2(G, M) \simeq (\mathbb{Z}/2\mathbb{Z})^2$ consists of the classes of cocycles ξ such that

$$\xi(x^2) = \begin{pmatrix} 0 \\ v \\ v \end{pmatrix}, \qquad \xi(y^2) = \begin{pmatrix} 0 \\ 0 \\ w \end{pmatrix}, \qquad \xi(xy) = \begin{pmatrix} 0 \\ v \\ v + w \end{pmatrix},$$

where $v, w \in \{0, \varepsilon\}$.

Note that, using formula (1), we can find the cocycles in the "standard" form. For instance, in Case 4 above, we obtain:

$$\gamma(a, a) = \gamma(a, ab) = \begin{pmatrix} 0 \\ v \\ v \end{pmatrix}, \quad \gamma(a, b) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \gamma(b, b) = \gamma(ab, b) = \begin{pmatrix} 0 \\ 0 \\ w \end{pmatrix},$$

$$\gamma(b,a) = \begin{pmatrix} 0 \\ v \\ v+w \end{pmatrix}, \quad \gamma(b,ab) = \begin{pmatrix} w \\ v \\ v+w \end{pmatrix},$$

$$\gamma(ab,a) = \begin{pmatrix} v \\ 0 \\ v+w \end{pmatrix}, \quad \gamma(ab,ab) = \begin{pmatrix} v \\ 0 \\ w \end{pmatrix}.$$

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