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ESTIMATION OF COVARIANCE FUNCTIONS OF GAUSSIAN STOCHASTIC FIELDS AND THEIR SIMULATION

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Introduction

Nowadays, the theory of stochastic processes is widely used in various fields of science and not only. Using stochastic processes, we can describe a large number of production processes, as well as processes occurring in economics, finance, insurance, radiophysics, etc. Since the covariance function is one of the most important characteristics of stochastic processes, the tasks of evaluating this function and constructing the criteria for its identification are an actual direction in the theory of stochastic processes and are widely used in solving statistical problems of stochastic processes. Another actual direction in the theory of stochastic processes is computer simulation of stochastic processes and fields, which is an effective means of reproduction and prediction of various phenomena and processes of the environment. Due to the powerful possibilities of computer techniques the problems of numerical simulations become especially important and allow to predict the behavior of a random process. The given monograph is dedicated to these tasks, namely, the tasks of simulation of stochastic processes and fields and the problem of identifying the covariance function of stochastic processes and fields.

In the first chapter we consider the space of sub-Gaussian random variables $Sub(\Omega)$, the Orlicz spaces of random variables $L_U(\Omega)$ and the space of quadratically Gaussian random variables $SG_\Xi(\Omega)$. The concept of sub-Gaussian random value was introduced in 1960 by Kahan. Later, in 1985, Kozachenko and Ostrovsky introduced and investigated certain properties of φ-sub-Gaussian random variables. The partial case of φ-sub-Gaussian random variables are quadratically Gaussian random variables. Quadratically Gaussian stochastic processes appeared in the literature in the middle of the twentieth century and were intensively investigated by many scholars, in particular by Kozachenko and his students. Since estimates for quadratically Gaussian stochastic processes are used in the evaluation of the spectral and covariance functions of stochastic processes and fields and the construction of criteria for the identification of these characteristics, it is precisely to this class of random variables and stochastic processes we devotes considerable attention. In the first section we consider the problem of evaluation of the exponential moments of quadratic forms from random variables from the space $SG_\Xi(\Omega)$ and limits in square mean of such quadratic forms. The upper and lower estimates for distributions of quadratic forms of quadratically Gaussian random variables and limits in square mean of such quadratic forms are found. The necessary definitions and assertions about the random variables from the Orlicz spaces are given for further work.

The second chapter is devoted to the construction of models of Gaussian non-stationary stochastic processes with given accuracy and reliability. A well-known, suggested by Mikhailov, method for constructing models of Gaussian stationary processes, namely, the method of partition and randomization of the spectrum, was modified here. Using the modified method of partition and randomization of the spectrum we constructed the models of Gaussian non-stationary stochastic processes. In addition, were investigated the conditions for selecting a partition of a set are so that the constructed model approximates a Gaussian non-stationary stochastic process with given reliability and accuracy in the spaces $C(\mathbb{T})$ and $L_p(\mathbb{T})$.

The third chapter is devoted to the construction of models of Gaussian non-stationary stochastic fields with given accuracy and reliability. At the beginning of the section, we constructed a model of stochastic field and obtained the estimates of k-th moments of sub-Gaussian random variables. With the help of these estimates, we investigated the accuracy and reliability of the constructed models and established a sufficient condition that the model of a stochastic Gaussian non-stationary field approximates it with given reliability and accuracy in the space $L_p(\mathbb{T}), p \geqslant 1$. In addition, in Section 3.4, new estimates for Bessel functions of the first kind are found. Also, the differences between the Bessel functions with different arguments are considered. Estimates for differences between two and four functions are obtained.

In the fourth chapter, a separable, real, stationary Gaussian stochastic process $\xi(t)$ is considered. Using previously obtained inequalities, estimates for the deviation of normalized correlogram from the covariance function in the metric of the space $L_2(0, B)$, $0 < B < \infty$ are found for a stochastic process $\xi(t)$. Here we considered the case when the process $\xi(t)$ is a centered stochastic process and the case when the mean of the process is different from 0. The covariance function is evaluated using correlograms.

In the fifth chapter we proved the theorem on the deviation of the covariance function from its estimate, that is, the correlogram. The criteria for testing the hypotheses about the covariance function of the Gaussian stationary stochastic process and the Gaussian non-stationary stochastic process are formulated. A theorem on the deviation of a covariance function from its estimate is proved in the case when the value of the process is known only for a finite set of points. On the basis of this theorem, a criterion for testing the hypothesis about the covariance function of a Gaussian stationary stochastic process is formulated. We proposed a criterion for comparing two hypotheses about the covariance function and a criterion for testing the hypothesis about the covariance function of a Gaussian stochastic process in the case when the mean of this process is different from zero. All these results are based on the estimates of the norms of quadratically Gaussian stochastic processes in the space $L_p(T)$, $p \geqslant 1$, that were obtained in Section 1.6.

In the sixth chapter we find the estimates for the distribution of the supremum of quadratically Gaussian stochastic processes defined on $\mathbb{R}^+$. The obtained results are used, in particular, to stationary in the wide means quadratically Gaussian stochastic processes. For a real stationary Gaussian stochastic process, with the help of the obtained inequalities, we find estimates for the deviation of the correlogram from the covariance function in the uniform metric on $(0, \infty)$. We constructed a criterion for testing hypothesis about the covariance function of the process on the interval (a, b) by observing the trajectory of the process on a segment of arbitrary length.

In the seventh chapter homogeneous and isotropic mean-square continuous Gaussian random field $\xi(x)$ defined in $\mathbb{R}^n$ with $E\xi(x) = 0$ is considered. For this random field, we obtained estimates for the distribution of spherical mean deviations from the covariance function in L_2-metric and metric of the space $L_p(\Omega), p \geqslant 1$. In Section 7.3 we considered the case when the values of the field on a sphere are known. Using the obtained inequalities, we constructed the criteria for testing the hypotheses about the covariance function of a stochastic field. The evaluation is carried out by observing the stochastic field on the ball, and the spherical mean is used as the estimate of the covariance function.

Chapter 1
Orlicz, $SG_\Xi(\Omega)$ and $Sub(\Omega)$ spaces of random variables.

1.1. Orlicz spaces of random variables.

Definition 1.1. [88] A continuous even convex function $U = \{U(x), x \in \mathbb{R}\}$ is called Orlicz C-function, if it is monotone increasing, $U(0) = 0, U(x) > 0$, $x \neq 0$.

Example 1.1. The next functions are Orlicz C-functions:

1) $U(x) = a|x|^\alpha,\ x \in \mathbb{R}\ a > 0,\ \alpha \geqslant 1;$

2) $U(x) = c(\exp\{a|x|^\alpha\} - 1),\ x \in \mathbb{R},\ c > 0, a > 0, \alpha \geqslant 1;$

3) $U(x) = c(\exp\{\varphi(x)\} - 1),\ x \in \mathbb{R}, c > 0,\ \varphi = \{\varphi(x), x \in \mathbb{R}\}$–arbitrary Orlicz C-function.

The main properties of Orlicz C-function reviewed in the book [88].

Let $\{\Omega, \mathcal{B}, P\}$ is the probability space. Denote:

- $L_0(\Omega)$- space of all random variables defined on the probability space $\{\Omega, \mathcal{B}, P\}$;
- $L_p(\Omega)$-space of random variables with finite p-th absolute moment $(p \geqslant 1)$;
- $L_p^{(0)}(\Omega)$-space of zero-mean random variables with finite p-th absolute moment $(p \geqslant 1)$.

The space $L_p(\Omega)$ is Banach with respect to the norm

$$\|\xi\|_p = [E|\xi|^p]^{1/p}, \xi \in L_p(\Omega).$$

Example 1.2. Let $U(x) = |x|^p,\ x \in \mathbb{R},\ p \geqslant 1$. In this case $L_U(\Omega)$ is a $L_p(\Omega)$ space and Luxemburg norm $\|\xi\|_U$ and norm $\|\xi\|_p$ are equivalent.

Convergence in the space $L_2(\Omega)$ by the norm $\| \bullet \|_2$ called the convergence in the mean square and if $\xi_n \longrightarrow \xi$ in the space $L_2(\Omega)$, then we can write down $\xi = l.i.m._{n \longrightarrow \infty} \xi_n$.

Definition 1.2. [19] Let U-arbitrary Orlicz C-function. The Orlicz space, generated by the function $U(x)$, is defined as the family of random variables

$\xi \in L_U(\Omega)$ where for each function ξ there exists a constant $r_\xi > 0$ such that

$$EU\left(\frac{\xi}{r_\xi}\right) < \infty.$$

Theorem 1.1. *[19] The Orlicz space $L_U(\Omega)$ endowed with the Luxemburg norm*

$$\|\xi\|_U = inf\left\{r > 0\ :\ EU\left(\frac{\xi}{r}\right) \leqslant 1\right\} \tag{1.1}$$

is a Banach space and

$$L_U(\Omega) \subseteq L_1(\Omega). \tag{1.2}$$

The functional $\|\bullet\|_U$ can take value ∞ on the space $L_0(\Omega)$ and $\|\xi\|_U < \infty$ if and only if $\xi \in L_U(\Omega)$, namely

$$L_U(\Omega) = \{\xi \in L_0(\Omega) : \|\xi\|_U < \infty\}.$$

By $L_U(\Omega)$ we denote the Orlicz space generated by Orlicz C-function $U(x)$.

Since (1.2) is true, than the space of zero-mean random variables can be written as

$$L_U^{(0)}(\Omega) = \{\xi \in L_U(\Omega) : E\xi = 0\}.$$

Lemma 1.1. *[19] The space $L_U^{(0)}(\Omega)$ is Banach subspace in the $L_U(\Omega)$ space with respect to the norm $\| \bullet \|_U$.*

1.2. Orlicz space of exponential type.

Let's consider the spaces of Orlicz, for which there are corresponding ones exponential moments.

Definition 1.3. [19] Suppose that $\varphi = \{\varphi(x), x \in \mathbb{R}\}$ is an arbitrary C-function. The Orlicz space generated by the C-function

$$U(x) = \exp\{\varphi(x)\} - 1, x \in \mathbb{R},$$

is called an Orlicz space of exponential type.

We denote this space by $Exp_\varphi(\Omega)$ and the norm of the space $Exp_\varphi(\Omega)$ by $\| \bullet \|_{E_\varphi}$.

Random variables belonging to space of exponential type have power moments any order, yielding the inclusion

$$Exp_\varphi(\Omega) \subset L_p(\Omega)$$

for any $p \geqslant 1$. By Theorem 3.2 in the book [19], this is a topological embedding: that is, there exists a constant $c > 0$ such that

$$\|\xi\|_p \leqslant c\|\xi\|_{E_\varphi}.$$

for any $\xi \in Exp_\varphi(\Omega)$. In general, the calculation of c is cumbersome, but in the case of N-functions this constant can be represented in a form convenient for applications.

Remark 1.1. In what follows, we will write $Exp_{(\alpha)}(\Omega)$ instead of $Exp_\varphi(\Omega)$ when $\varphi = \{|x|^\alpha, x \in \mathbb{R}\}$, $\alpha \geqslant 1$, and the corresponding norm $\| \bullet \|_{E_\varphi}$ will be denoted $\| \bullet \|_{E(\alpha)}$.

Denote

$$Exp_\varphi^{(0)}(\Omega) = \{\xi \in Exp_\varphi(\Omega) : E\xi = 0\}.$$

From Lema 1.1 we obtainte that the space $Exp_\varphi^{(0)}(\Omega)$ is Banach subspace in the $Exp_\varphi(\Omega)$ space with respect to the norm $\| \bullet \|_U$. If we consider only centered random variables, then we can easily determine norms that will be equivalent to the Luxemburg norms.

Let γ is a Gaussian random variable with $(0, \sigma^2)$ parameters. This random variable belong to $L_U(\Omega)$ Orlicz space, where $U(x) = \exp\{x^2\} - 1$ and the norm of this random variable is equal to $C\| \bullet \|_{L_2}$.

1.3. Sub-Gaussian random variable

Definition 1.4. [19] A random variable χ is sub-Gaussian if there exists $a \geqslant 0$, such that the inequality

$$\mathsf{E}\exp\{\lambda\chi\} \leqslant \exp\left\{\frac{a^2\lambda^2}{2}\right\},$$

holds for all $\lambda \in \mathbb{R}$.

The space of all sub-Gaussian random variables defined on a common probability space $\{\Omega, \mathbf{B}, \mathcal{P}\}$ we denote $Sub(\Omega)$. The space $Sub(\Omega)$ is a Banach space with respect to the norm $\tau(\chi) = \sup\limits_{\lambda \neq 0}\left[\frac{2\ln \mathsf{E}\exp\{\lambda\chi\}}{\lambda^2}\right]^{\frac{1}{2}}$.

Lemma 1.2. *[19] Assume that $\xi_1, \xi_2, ..., \xi_n$ are independent sub-Gaussian random variables. Then*

$$\tau^2\left(\sum_{k=1}^n \xi_k\right) \leqslant \sum_{k=1}^n \tau^2(\xi_k)$$

Lemma 1.3. *[19] Let ξ be a zero-mean random variable such that $\mathsf{E}\xi^{2k+1} =$*

0 *i* $\theta(\xi) = \sup\limits_{k\geqslant 1}\left[\frac{2^k k!}{(2k)!}E\xi^{2k}\right]^{\frac{1}{2k}} < \infty$. *Then* $\xi \in Sub(\Omega)$ *and* $\tau(\xi) \leqslant \theta(\xi)$.

Definition 1.5. [19] Let T be a parametric set. A stochastic process $\xi = \{\xi(t), t \in T\}$ is called sub-Gaussian if for all $t \in T$, $\xi(t) \in Sub(\Omega)$ and $\sup_{t\in T}\tau(\xi(t)) < \infty$.

1.4. Space of square Gaussian random variables $SG_{\Xi}(\Omega)$

Let

- $\overline{\xi} = (\xi_1, \ldots, \xi_N)^T$ be N-dimensional Gaussian column vector ($N \geqslant 1$), $E\xi_k = 0$, $k = 0, \ldots, N$;
- $B = \text{cov}\overline{\xi} = \text{E}\overline{\xi}\overline{\xi}^{\text{T}}$ be the covariance matrix of the vector $\overline{\xi}$;
- $A = (a_{jk})_{j,k=1}^N$ be a symmetric matrix with real-valued entries ($A^T = A$).

Lemma 1.4. *[59] For* $|s| < 1$ *and* $D\left(\overline{\xi}^T A\overline{\xi}\right) = E\left(\overline{\xi}^T A\overline{\xi} - E\overline{\xi}^T A\overline{\xi}\right)^2 > 0$ *the next inequality holds*

$$E\exp\left\{\frac{s}{\sqrt{2}}\left(\frac{\overline{\xi}^T A\overline{\xi} - E\overline{\xi}^T A\overline{\xi}}{\left(D\left(\overline{\xi}^T A\overline{\xi}\right)\right)^{1/2}}\right)\right\} \leqslant (1-|s|)^{-1/2}\exp\left\{-\frac{|s|}{2}\right\}.$$

Remark 1.2. The conclusion of lemma also holds for an asymmetric matrix A. In this case one can use the immediate equality

$$\overline{\xi}^T A\overline{\xi} = \overline{\xi}^T\left(\frac{A+A^T}{2}\right)\overline{\xi},$$

and observe that the matrix $\frac{1}{2}(A + A^T)$ is symmetric.

Remark 1.3. Assume that $\overline{\xi} = (\xi_1, \ldots, \xi_{N_1})^T$, $N_1 \geqslant 1$ and $\overline{\eta} = (\eta_1, \ldots, \eta_{N_2})^T$, $N_2 \geqslant 1$ are zero-mean jointly Gaussian vectors and let $A = (a_{jk})$ be an $N_1 \times N_2$ matrix with real valued entries. Consider the quadratic form

$$\overline{\xi}^T A\overline{\eta} = \Sigma_{j=1}^{N_1}\Sigma_{k=1}^{N_2} a_{jk}\xi_j\eta_k.$$

Introducing a random $N_1 + N_2$-vector $(\overline{\xi};\overline{\eta}) = (\xi_1, \ldots, \xi_{N_1}, \eta_1, \ldots, \eta_{N_2})^T$

and a block $(N_1+N_2)\times(N_1+N_2)$-matrix

$$\widetilde{A}=\begin{pmatrix} 0 & \frac{A}{2} \\ \frac{A}{2} & 0 \end{pmatrix}$$

we have the equality

$$\overline{\xi}^T A\overline{\eta}=(\overline{\xi};\overline{\eta})^T\widetilde{A}(\overline{\xi};\overline{\eta}).$$

Corollary 1.1. *Let* $D\left(\overline{\xi}^T A\overline{\eta}\right)=E\left(\overline{\xi}^T A\overline{\eta}-E\overline{\xi}^T A\overline{\eta}\right)^2>0$. *Then forT* $|s|<1$ *the following inequality holds*

$$E\exp\left\{\frac{s}{\sqrt{2}}\left(\frac{\overline{\xi}^T A\overline{\eta}-E\overline{\xi}^T A\overline{\eta}}{\left(D\left(\overline{\xi}^T A\overline{\eta}\right)\right)^{1/2}}\right)\right\}\leqslant(1-|s|)^{-1/2}\exp\left\{-\frac{|s|}{2}\right\}.$$

Remark 1.4. It is an easy exercise to check that corollary 1.1 also holds for a linear combination of the form

$$\zeta=\Sigma_{l=1}^n\overline{\xi}_l^T A_l\overline{\eta}_l,$$

where $\overline{\xi}_1,\dots,\overline{\xi}_n,\overline{\eta}_1,\dots,\overline{\eta}_n$ are zero-mean jointly Gaussian random vectors whose dimensions can be arbitrary and where $A_1,\dots,A_n$ are symmetric matrices which fit these dimensions. In this case, the only restriction is the condition that the random variable ζ is nonsingular. However, this fact is obvious since the random variable ζ can always be represented as $\overline{\gamma}^T\widetilde{A}\overline{\gamma}$, where $\overline{\gamma}$ is a compound Gaussian vector formed by the vectors $\overline{\xi}_1,\dots,\overline{\xi}_n,\overline{\eta}_1,\dots,\overline{\eta}_n$, and the matrix $\widetilde{A}$ is built from the matrices $A_1,\dots,A_n$.

On a probability space $\{\Omega,\mathcal{B},P\}$, consider a family of random variables of the form $\overline{\xi}^T A\overline{\xi}-E\overline{\xi}^T A\overline{\xi}$, where $\overline{\xi}$ is a zero-mean Gaussian vector of an arbitrary dimension $N\geqslant 1$ defined on $\{\Omega,\mathcal{B},P\}$, and A is an arbitrary $N\times N$-symmetric matrix with real-valued entries.

Definition 1.6. [81] Let T be some parametric set, $\Xi=\{\xi_t,t\in T\}$ be the family of jointly Gaussian random variables, $E\xi_t=0$ (for example, ξ_t be Gaussian random process). The space $SG_\Xi(\Omega)$ is called the space of square Gaussian random variables, if random variables ζ from $SG_\Xi(\Omega)$ can be represented in the form

$$\zeta=\overline{\xi}^T A\overline{\xi}-E\overline{\xi}^T A\overline{\xi}, \tag{1.3}$$

where

• $\overline{\xi} = (\xi_1, \dots, \xi_N)^T$ is Gaussian random vector for $N \geqslant 1$, $E\overline{\xi} = 0$,
• random variables $\xi_i, i = 1, \dots, N$ belong to Ξ,
• A is an arbitrary symmetric matrix,
or random variables from $SG_\Xi(\Omega)$ are mean square limits of a sequence of random variables $\zeta_n = \overline{\xi}_n^T A_n \overline{\xi}_n - E\overline{\xi}_n^T A_n \overline{\xi}_n$, $n \geqslant 1$.

Remark 1.5. Assume that $\overline{\eta}$ and $\overline{\theta}$ are random vectors with components from Ξ, and C is a symmetric matrix. Then $\zeta = \overline{\eta}^T C\overline{\theta} - E\overline{\eta}^T C\overline{\theta}$ belongs to the space $SG_\Xi(\Omega)$.

Remark 1.6. Let $\overline{\eta}_i$, $i = 1, 2, \dots, n$ be the random vectors with components from Ξ, C_i be the symmetric matrices, $u_1, u_2, \dots, u_n$ be the arbitrary numbers. Then $\zeta = \sum_{i=1}^n u_i \left(\overline{\eta}_i^T C_i \overline{\eta}_i - E\overline{\eta}_i^T C_i \overline{\eta}_i\right)$ belongs to the space $SG_\Xi(\Omega)$.

In [19] was proved, that the space $SG_\Xi(\Omega)$ is a closed subspace of the space $Exp_\varphi^{(0)}(\Omega)$ with $\varphi(x) = |x|, x \in \mathbb{R}$ and $\| \bullet \|_{E_\varphi}$ and $L_2(\Omega)$-norm are equivalent. This means that space $SG_\Xi(\Omega)$ is Banach space relative to the norm $\|\zeta\| = \sqrt{E\zeta^2}$. For the random variables from $SG_\Xi(\Omega)$ the following lemma holds.

Lemma 1.5. *[81] Assume that $\zeta_i, i = 1, 2, \dots, n$ are random variables from $SG_\Xi(\Omega)$. Then for all $|s| < 1$ for all $\lambda_i \in R^1, i = 1, 2, \dots, n$ inequality*

$$E \exp\left\{\frac{s}{\sqrt{2}} \frac{\zeta}{(D\zeta)^{\frac{1}{2}}}\right\} \le R(|s|) \tag{1.4}$$

holds, where $\zeta = \sum_{i=1}^n \lambda_i \zeta_i$,

$$R(s) = \exp\left\{-\frac{s}{2}\right\} (1 - s)^{-\frac{1}{2}} \tag{1.5}$$

Definition 1.7. [81] A random vector $\overline{\zeta} \in \mathbb{R}^d$ is said to be square Gaussian, if all its components ζ_i, $i = 1, 2, \dots, d$, belong to the space $SG_\Xi(\Omega)$.

Definition 1.8. [81] A random process $\zeta = \{\zeta(t), t \in T\}$ is called the square Gaussian random process relative to family Ξ, if for all $t \in T$ random variables $\zeta(t)$ belong to the space $SG_\Xi(\Omega)$ and $\sup_{t \in T} E\zeta^2(t) < \infty$.

Let us consider the examples of square Gaussian random processes.

1. Let $\xi_1(t), \xi_2(t), \dots, \xi_n(t), t \in T$ be a family of zero-mean jointly Gaussian random processes and assume that for each $t \in T$ exist symmetric matrix $A(t)$. Then $\zeta(t) = \overline{\xi}^T(t) A(t) \overline{\xi}(t) - E\overline{\xi}^T(t) A(t) \overline{\xi}(t)$
be the square Gaussian random process with $\overline{\xi}^T(t) = (\xi_1(t), \xi_2(t), \dots, \xi_n(t))$.

2. The mean square limits of a sequence of random processes

$$\zeta_n(t) = \overline{\xi}_n^T(t) A_n(t) \overline{\xi}_n(t) - E\overline{\xi}_n^T(t) A_n(t) \overline{\xi}_n(t),$$

where $\overline{\xi}_n(t)$ are zero-mean Gaussian random vector-processes and $A_n(t)$ are symmetric matrices, be the square Gaussian random process.

3. Assume that $\xi = \{\xi(t), t \in T\}$ is zero-mean stationary Gaussian random process. Correlogram

$$\widehat{B}(\tau) = \frac{1}{V} \int_0^V \xi(t+\tau)\xi(t)dt - E\xi(t+\tau)\xi(t), \quad V > 0,$$

of this process $\xi = \{\xi(t), t \in T\}$ is the square Gaussian random process.

1.5. The estimates for the distribution of quadratic forms defined on the space of square Gaussian random variables $SG_\Xi(\Omega)$

The following lemma improves the corresponding lemma from [81].

Lemma 1.6. *Let* $\overline{\zeta}^T = (\zeta_1, \ldots, \zeta_d)$ *be the random vector,* $\zeta_i \in SG_\Xi(\Omega)$, $i = 1, \ldots, d$, *and let* A *be the* $d \times d$ *symmetric positive-dimensional matrix. Then for all* $|t| < \frac{1}{\sqrt{2}}$ *the following inequality holds*

$$E\,\mathrm{ch}\left(\sqrt{\frac{t^2 \overline{\zeta}^T A \overline{\zeta}}{E(\overline{\zeta}^T A \overline{\zeta})}}\right) \le R(\sqrt{2}|t|), \tag{1.6}$$

where $R(s) = \exp\left\{-\frac{s}{2}\right\}(1-s)^{-\frac{1}{2}}$.

Proof. Let us prove this lemma for $A = I$, where I is unit matrix, and for such vectors $\overline{\zeta}$, for which ζ_i are orthogonal, i.e. $D(\sum_{i=1}^d \lambda_i \zeta_i) = \sum_{i=1}^d \lambda_i^2 E\zeta_i^2$. Put $\sigma_i^2 = E\zeta_i^2$, $i = 1, 2, \ldots, d$. In this case from (1.4) (for $|s| < 1$) follows, that for all $\lambda_i \in R, i = 1, 2, \ldots, d$ inequality

$$E \exp\left\{\frac{s\sum_{i=1}^d \lambda_i \zeta_i}{\sqrt{2}(\sum_{i=1}^d \lambda_i^2 \sigma_i^2)^{1/2}}\right\} \le R(|s|) \tag{1.7}$$

holds true.

Let us denote

$$u = \frac{s}{\sqrt{2\sum_{i=1}^{d} \lambda_i^2 \sigma_i^2}}.$$

From (1.7) follows, that for $|u| < \left(2\sum_{i=1}^{d} \lambda_i^2\sigma_i^2\right)^{-1/2}$ we obtain

$$E\exp\left\{u\sum_{i=1}^{d}\lambda_i\zeta_i\right\} \le R\left(|u|\sqrt{2\sum_{i=1}^{d}\lambda_i^2\sigma_i^2}\right). \tag{1.8}$$

Let us define $s_i = u\lambda_i\sigma_i$. Then

$$\sum_{i=1}^{d} s_i^2 = u^2\sum_{i=1}^{d}\lambda_i^2\sigma_i^2 = \frac{s^2}{2},$$

moreover $\sum_{i=1}^{d} s_i^2 < \frac{1}{2}$. It follows from (1.8), that for all s_i for which $\sum_{i=1}^{d} s_i^2 < \frac{1}{2}$ the next inequality holds

$$E\exp\left\{\sum_{i=1}^{d} s_i\frac{\zeta_i}{\sigma_i}\right\} \le R\left(\sqrt{2\sum_{i=1}^{d} s_i^2}\right). \tag{1.9}$$

One can apply (1.9), to obtain inequality

$$E\prod_{i=1}^{d}\operatorname{ch}\left(\frac{s_i\zeta_i}{\sigma_i}\right) = E\prod_{i=1}^{d}\frac{\exp\left\{\frac{s_i\zeta_i}{\sigma_i}\right\} + \exp\left\{-\frac{s_i\zeta_i}{\sigma_i}\right\}}{2} =$$

$$= \frac{1}{2^d}E\prod_{i=1}^{d}\left(\exp\left\{\frac{s_i\zeta_i}{\sigma_i}\right\} + \exp\left\{-\frac{s_i\zeta_i}{\sigma_i}\right\}\right) = \frac{1}{2^d}\sum E\prod_{i=1}^{d}\exp\left\{\frac{s_i\zeta_i\delta_i}{\sigma_i}\right\} =$$

$$= \frac{1}{2^d}\sum E\exp\left\{\sum_{i=1}^{d}\frac{s_i\zeta_i\delta_i}{\sigma_i}\right\} \le \frac{1}{2^d}\sum R\left(\sqrt{2\sum_{i=1}^{d}s_i^2}\right) =$$

$$= R\left(\sqrt{2\sum_{i=1}^{d} s_i^2}\right).$$

where $\delta_i = \pm 1$ and sums $\sum$ are calculated by all possible δ_i, that means we have total 2^d numbers. Thus,

$$E\prod_{i=1}^{d}\operatorname{ch}\left(\frac{s_i\zeta_i}{\sigma_i}\right)\le R\left(\sqrt{2\sum_{i=1}^{d}s_i^2}\right).$$

Consider function $f(z)=\ln\operatorname{ch}\sqrt{z}$, $z>0$. $f(z)>0$ is convex function. Really, $f(0)=\ln\operatorname{ch}0=0$, $f'(z)=\frac{\operatorname{sh}\sqrt{z}}{2\sqrt{z}\operatorname{ch}\sqrt{z}}$, $f''(z)=\frac{1}{4}\frac{1-\frac{\operatorname{sh}(2\sqrt{z})}{2\sqrt{z}}}{z\left(\operatorname{ch}\sqrt{z}\right)^2}<0$, because $\operatorname{sh}(2\sqrt{z})>2\sqrt{z}$, $z>0$. Therefore, for all $z_i>0, i=1,2,\dots,d$, the inequality [51]

$$\sum_{i=1}^{d}f(z_i)\ge f(\sum_{i=1}^{d}z_i)$$

holds. This means

$$\prod_{i=1}^{d}\operatorname{ch}\sqrt{z_i}\ge\operatorname{ch}\sqrt{\sum_{i=1}^{d}z_i},\quad z_i>0.$$

Therefore, for $\sum_{i=1}^{d}s_i^2<\frac{1}{2}$ we have

$$E\operatorname{ch}\sqrt{\sum_{i=1}^{d}\frac{s_i^2\zeta_i^2}{\sigma_i^2}}\le E\prod_{i=1}^{d}\operatorname{ch}\sqrt{\frac{s_i^2\zeta_i^2}{\sigma_i^2}}=E\prod_{i=1}^{d}\operatorname{ch}\left\{\frac{|s_i\zeta_i|}{\sigma_i}\right\}=$$

$$=E\prod_{i=1}^{d}\operatorname{ch}\left\{\frac{s_i\zeta_i}{\sigma_i}\right\}\le R\left(\sqrt{2\sum_{i=1}^{d}s_i^2}\right)$$

Let us put $s_i^2=\frac{\sigma_i^2t^2}{\sum_{i=1}^{d}\sigma_i^2}$. Than from previous inequality we obtain

$$E\operatorname{ch}\left(\sqrt{\frac{t^2\sum_{i=1}^{d}\zeta_i^2}{\sum_{i=1}^{d}\sigma_i^2}}\right)\le R(\sqrt{2}t)\tag{1.10}$$

for all $|t|<\frac{1}{\sqrt{2}}$.

Let us consider the general case. Take a symmetric matrix B such, that $BB=A$, $R=\operatorname{cov}\overline{\zeta}$ and let O be the orthogonal matrix reducing BRB to the diagonal form, that is

$$OBRBO^T=D=diag(d_k^2)_{k=1}^{d}.$$

Let $\overline{\theta} = O B \overline{\zeta}$. Then

$$\overline{\theta}^T \overline{\theta} = \overline{\zeta}^T B O^T O B \overline{\zeta} = \overline{\zeta}^T A \overline{\zeta},$$

$$\mathrm{cov}\overline{\theta} = \mathrm{OBcov}\overline{\zeta}\mathrm{BO}^{\mathrm{T}} = \mathrm{D}.$$

Since $\overline{\theta}^T = (\theta_1, \ldots, \theta_d)$ is square Gaussian random vector ($\overline{\zeta}$ is square Gaussian random vector), we can to apply inequality (1.10) to it. Considering that $\overline{\theta}^T \overline{\theta} = \sum_{i=1}^d \theta_i^2 = \overline{\zeta}^T A \overline{\zeta}$, we obtain

$$\mathrm{ch}\sqrt{\frac{t^2 \overline{\theta}^T \overline{\theta}}{E \overline{\theta}^T \overline{\theta}}} = \mathrm{ch}\sqrt{\frac{t^2 \overline{\zeta}^T A \overline{\zeta}}{E \overline{\zeta}^T A \overline{\zeta}}}.$$

Lemma is proved completely. ♢

Corollary 1.2. *Assume that for* $\overline{\zeta}_n$ *and* A_n, $n \geqslant 1$ *the conditions of the lemma 1.6 are fulfilled and* $\eta = l.i.m._{n\to\infty} \overline{\zeta}_n^T A_n \overline{\zeta}_n$, $E\eta \neq 0$. *Then for all* $|t| < \frac{1}{\sqrt{2}}$ *the following inequality holds*

$$E\,\mathrm{ch}\left(\sqrt{\frac{t^2 \eta}{E\eta}}\right) \leq R(\sqrt{2}|t|).$$

Proof. Since $\eta = l.i.m._{n\longrightarrow\infty} \eta_n$, than $E\eta_n \longrightarrow E\eta$, for $n \longrightarrow \infty$, that means

$$\frac{\eta}{E\eta} = l.i.m._{n\longrightarrow\infty} \frac{\eta_n}{E\eta_n},$$

where $\eta_n = \overline{\zeta}_n^T A_n \overline{\zeta}_n$. Then exists a subsequence $\{\eta_{n_k}\}$ of the sequence $\{\eta_n\}$ such that $\eta_{n_k} \longrightarrow \eta$ for $n_k \longrightarrow \infty$ with probability 1. We will apply the Fatou lemma to obtain

$$E\,\mathrm{ch}\left(\sqrt{t^2 \frac{\eta}{E\eta}}\right) = E \liminf_{n_k\longrightarrow\infty} \mathrm{ch}\left(\sqrt{t^2 \frac{\eta_{n_k}}{E\eta_{n_k}}}\right) \leqslant$$

$$\leqslant \liminf_{n_k\longrightarrow\infty} E\,\mathrm{ch}\left(\sqrt{t^2 \frac{\eta_{n_k}}{E\eta_{n_k}}}\right) \leqslant R\left(\sqrt{2}|t|\right).$$

The theorem is proved. ♢

Lemma 1.7. *Assume that for* $\overline{\zeta}$ *and* A *the conditions of the lemma 1.6 are fulfilled and* $\eta = \overline{\zeta}^T A \overline{\zeta}$. *Then for* $x > \frac{1}{2}$ *the following inequality holds*

$$P\left\{\frac{\eta}{E\eta} > x\right\} \leq \frac{2^{\frac{1}{4}} x^{\frac{1}{4}}}{\operatorname{ch}\left(\sqrt{\frac{x}{2}} - \frac{1}{2}\right)}. \tag{1.11}$$

Proof. From the Chebyshev inequality and (1.6) follows, that for $x > 0$ and $|t| < \frac{1}{\sqrt{2}}$ we have

$$P\left\{\frac{\eta}{E\eta} > x\right\} \leq \frac{E \operatorname{ch}\sqrt{\frac{t^2\eta}{E\eta}}}{\operatorname{ch}\sqrt{t^2 x}} \leq \frac{R(\sqrt{2}|t|)}{\operatorname{ch}\sqrt{t^2 x}}$$

Let us donote $t = \frac{1}{\sqrt{2}} - \frac{1}{2\sqrt{x}}$, $x > \frac{1}{2}$. Then

$$\frac{R(\sqrt{2}t)}{\operatorname{ch}\sqrt{t^2 x}} = \frac{(2x)^{1/4} \exp\left\{\frac{1}{2\sqrt{2x}} - \frac{1}{2}\right\}}{\operatorname{ch}\left(\sqrt{\frac{x}{2}} - \frac{1}{2}\right)}.$$

Since $\exp\left\{\frac{1}{2\sqrt{2x}} - \frac{1}{2}\right\} < 1$ for $x > \frac{1}{2}$, then

$$P\left\{\frac{\eta}{E\eta} > x\right\} \leq \frac{2^{1/4} x^{1/4}}{\operatorname{ch}\left(\sqrt{\frac{x}{2}} - \frac{1}{2}\right)}.$$

Lemma is proved. ♢

Corollary 1.3. *Assume that for* $\overline{\zeta}_n$ *and* A_n, $n \geqslant 1$ *the conditions of the lemma 1.6 are fulfilled. Then the inequality (1.11) holds for*
$\eta = l.i.m._{n\to\infty} \overline{\zeta}_n^T A_n \overline{\zeta}_n$, $E\eta \neq 0$.

Proof. Corollary follows from lemma 1.7 and corollary 1.2. ♢

Lemma 1.8. *Let* $\xi_1, \xi_2, \ldots, \xi_m$, $m \geqslant 1$ *be independent normal random variables and* $E\xi_k = 0, E\xi_k^2 = \sigma_k^2, c_k = \pm 1, k = 1, \ldots, m$ *and* $s > 0$. *Then the following inequality holds*

$$\left| E \exp\left\{ i \frac{s \sum_{k=1}^m \xi_k^2 c_k}{2(\sum_{k=1}^m \sigma_k^4)^{\frac{1}{2}}} \right\} \right| \leq \frac{1}{(1+s^2)^{1/4}}.$$

Proof. It is obviously, that for ξ_k, c_k and for real valued r equality

$$E \exp\left\{ i \frac{\sum_{k=1}^m \xi_k^2 c_k}{r} \right\} = \prod_{k=1}^m E \exp\left\{ \frac{i \xi_k^2 c_k}{r} \right\} \tag{1.12}$$

holds.

Taking into account that $E\exp\{is\xi_k^2\} = (1-2is\sigma_k^2)^{-1/2}$, (1.12) can be rewritten in the form

$$E\exp\left\{i\frac{\sum_{k=1}^m \xi_k^2 c_k}{r}\right\} = \prod_{k=1}^m \left(1-2i\frac{\sigma_k^2 c_k}{r}\right)^{-\frac{1}{2}}.$$

Then

$$\left|E\exp\left\{i\frac{\sum_{k=1}^m \xi_k^2 c_k}{r}\right\}\right| = \left|\prod_{k=1}^m \left(1-2i\frac{\sigma_k^2 c_k}{r}\right)^{-\frac{1}{2}}\right| = \prod_{k=1}^m \left|1-2i\frac{\sigma_k^2 c_k}{r}\right|^{-\frac{1}{2}} =$$

$$= \prod_{k=1}^m \left(1+\left(2\frac{\sigma_k^2 c_k}{r}\right)^2\right)^{-\frac{1}{4}} = \prod_{k=1}^m \left(1+\frac{4\sigma_k^4}{r^2}\right)^{-\frac{1}{4}}.$$

Let us denote $I = \prod\limits_{k=1}^m \left(1+\frac{4\sigma_k^4}{r^2}\right)^{-\frac{1}{4}}$.

Then

$$\ln I = -\frac{1}{4}\sum_{k=1}^m \ln\left(1+\frac{4\sigma_k^4}{r^2}\right) \tag{1.13}$$

Consider the function $f(x) = \ln(1+x)$, $x > 0$. $f(x)$ is convex function ($f(0) = 0, f''(x) < 0$) and therefore

$$f(\sum_{k=1}^m x_k) \le \sum_{k=1}^m f(x_k),$$

that is

$$-\sum_{k=1}^m f(x_k) \le -f(\sum_{k=1}^m x_k)$$

for $x_k \ge 0$.

From the last inequality and (1.13) follows, that

$$\ln I \le -\frac{1}{4}\ln\left(1+\frac{4}{r^2}\sum_{k=1}^m \sigma_k^4\right)$$

$$I \le \left(1+\frac{4}{r^2}\sum_{k=1}^m \sigma_k^4\right)^{-\frac{1}{4}}.$$

Let $r = \frac{2(\sum_{k=1}^{m} \sigma_k^4)^{\frac{1}{2}}}{s}$. Then

$$I \leq (1+s^2)^{-\frac{1}{4}} \quad \text{for the real valued} \quad s,$$

which was to be proved. ◊

Theorem 1.2. *Let A be a symmetric real-valued $n \times n$ matrix, $\overline{\xi}^T = (\xi_1, \xi_2, \dots, \xi_n)$ be the random vector such that ξ_k are the normal random variables with $E\xi_k = 0$ and $D\overline{\xi}^T A\overline{\xi} > 0$. Then*

$$\left| E \exp\left\{ i \frac{s(\overline{\xi}^T A\overline{\xi} - E\overline{\xi}^T A\overline{\xi})}{\sqrt{2}\sqrt{D\overline{\xi}^T A\overline{\xi}}} \right\} \right| \leq \frac{1}{(1+s^2)^{\frac{1}{4}}},$$

for $s > 0$.

Proof. Let $\overline{\xi} = (\xi_1, \xi_2, \dots, \xi_n)^T$ be an n-dimensional Gaussian column vector, such that $E\xi_k = 0$, $k = 1, \dots, n$, and let $B = \text{cov}\overline{\xi} = \mathrm{E}\overline{\xi}\overline{\xi}^{\mathrm{T}}$ be the covariance matrix of the vector $\overline{\xi}$. Take a symmetric matrix $A = (a_{ij})_{i,j=1}^n$ with real-valued entries ($A^T = A$) and let U be the ortogonal matrix reducing $(B^{1/2})^T A B^{1/2}$ to the diagonal form. $\overline{\xi} = B^{1/2} U \overline{\gamma}$, where $\overline{\gamma} = (\gamma_1, \dots, \gamma_n)^T$ is a standard Gaussian random vector. Then

$$\overline{\xi}^T A\overline{\xi} = \overline{\gamma}^T U^T B^{1/2} A B^{1/2} U \overline{\gamma} = \overline{\gamma}^T \Lambda \overline{\gamma} = \Sigma_{k=1}^n \lambda_k \gamma_k^2,$$

where $\Lambda = diag\{\lambda_1, \dots, \lambda_n\}$ is the diagonal matrix. Since for $\lambda_k = \pm 1$ we obtain $D\left(\sum_{k=1}^n \lambda_k \gamma_k^2\right) = 2\sum_{k=1}^n \sigma_k^4 > 0$ and for $\overline{\gamma}$ holds lemma 1.8 ($E\gamma_k = 0, E\gamma_k^2 = 1, \lambda_k = \pm 1$) we will have

$$\left| E \exp\left\{ i \frac{s(\overline{\xi}^T A\overline{\xi} - E\overline{\xi}^T A\overline{\xi})}{\sqrt{2}\sqrt{D\overline{\xi}^T A\overline{\xi}}} \right\} \right| =$$

$$= \left| E \exp\left\{ i \frac{s\left(\sum_{k=1}^n \lambda_k \gamma_k^2 - E\left(\sum_{k=1}^n \lambda_k \gamma_k^2\right)\right)}{2\sqrt{\sum_{k=1}^n \sigma_k^4}} \right\} \right| \leq \frac{1}{(1+s^2)^{\frac{1}{4}}}.$$

Corollary 1.4. *Let $\zeta_i, i = 1, 2, \dots, n$ be the random variables from $SG_{\Xi}(\Omega)$. Then for $s > 0$ the next inequality holds*

$$\left| E \exp\left\{ i \frac{s}{\sqrt{2}} \frac{\zeta}{(var\zeta)^{1/2}} \right\} \right| \leqslant \frac{1}{(1+s^2)^{1/4}},$$

where $\zeta = \sum_{i=1}^{n} \lambda_i \zeta_i$, λ_i *are real numbers.*

Proof. $\zeta_i \in SG_{\Xi}(\Omega)$ therefore ζ_i can be represented as $\zeta_i = \overline{\xi}_i^T C_i \overline{\xi}_i - E\left(\overline{\xi}_i^T C_i \overline{\xi}_i\right)$, where $\overline{\xi}_i$ are Gaussian random vectors, C_i are the symmetric matrices. Let us consider vector $\overline{\xi}^T = (\overline{\xi}_1, \dots, \overline{\xi}_n)$ and the matrix

$$C = \begin{pmatrix} \lambda_1 C_1 & 0 & \dots & 0 \\ 0 & \lambda_2 C_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_n C_n \end{pmatrix}$$

In this case $\zeta = \sum_{i=1}^{n} \lambda_i \zeta_i = \overline{\xi}^T C \overline{\xi} - E\left(\overline{\xi}^T C \overline{\xi}\right)$ is square Gaussian random variable, and therefore for ζ the inequality of the theorem holds. $\diamondsuit$

Lemma 1.9. *Let* $\overline{\zeta}^T = (\zeta_1, \dots, \zeta_n)$ *be the random vector such, that* $\zeta_j \in SG_{\Xi}(\Omega)$ *and let* A *be a symmetric positive-valued matrix. Then the inequality*

$$E \exp\left\{ -\frac{u^2}{2} \frac{\overline{\zeta}^T A \overline{\zeta}}{E \overline{\zeta}^T A \overline{\zeta}} \right\} \le g(u)$$

holds true, where $g(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp\left\{-\frac{s^2}{2}\right\} \frac{1}{(1+u^2 s^2)^{\frac{1}{4}}} ds, \quad u > 0.$

Proof. Let us consider orthogonal square Gaussian random variables ζ_j, $E\zeta_j^2 = \sigma_j^2, \sigma_j^2 > 0 >$, $\lambda_j \in R^1$. Then from the theorem 1.2 and corollary 1.4 for $s > 0$ we have

$$\left| E \exp\left\{ i \frac{s}{\sqrt{2}} \frac{\sum_{j=1}^{n} \lambda_j \zeta_j}{\left(\sum_{j=1}^{n} \lambda_j^2 \sigma_j^2\right)^{\frac{1}{2}}} \right\} \right| \le \frac{1}{(1+s^2)^{\frac{1}{4}}} \tag{1.14}$$

Let us rewrite the left part in the form

$$\left| E \exp\left\{ i \frac{s}{\sqrt{2}} \frac{\sum_{j=1}^{n} \frac{\zeta_j}{\sigma_j} (\lambda_j \sigma_j)}{\left(\sum_{j=1}^{n} \lambda_j^2 \sigma_j^2\right)^{\frac{1}{2}}} \right\} \right|.$$

Define s_j as follows $s_j = s \frac{\lambda_j \sigma_j}{\left(\sum_{j=1}^{n} \lambda_j^2 \sigma_j^2\right)^{\frac{1}{2}}}$. Then $s^2 = \sum_{j=1}^{n} s_j^2$, and (1.14) can be rewritten in the form

$$\left|E\exp\left\{i\frac{1}{\sqrt{2}}\sum_{j=1}^{n}\frac{s_j\zeta_j}{\sigma_j}\right\}\right| \le \frac{1}{(1+\sum_{j=1}^{n} s_j^2)^{\frac{1}{4}}} \tag{1.15}$$

For $t_j > 0$ we will have

$$\left|\int\limits_{R^n}\dots\int E\exp\left\{i\frac{1}{\sqrt{2}}\sum_{j=1}^{n}s_j\frac{\zeta_j}{\sigma_j}\right\}\prod_{j=1}^{n}\frac{1}{\sqrt{2\pi}t_j}\exp\left\{-\frac{s_j^2}{2t_j^2}\right\}ds_1\dots ds_n\right| =$$

$$= \left|E\exp\left\{-\sum_{j=1}^{n}\frac{\zeta_j^2t_j^2}{2\sigma_j^2}\right\}\right|.$$

From (1.15) and last equality we obtain

$$\left|E\exp\left\{-\sum_{j=1}^{n}\frac{\zeta_j^2t_j^2}{2\sigma_j^2}\right\}\right| =$$

$$= \left|\int\limits_{R^n}\dots\int E\exp\left\{i\frac{1}{\sqrt{2}}\sum_{j=1}^{n}s_j\frac{\zeta_j}{\sigma_j}\right\}\prod_{j=1}^{n}\frac{1}{\sqrt{2\pi}t_j}\exp\left\{-\frac{s_j^2}{2t_j^2}\right\}ds_1\dots ds_n\right| \leqslant$$

$$\leqslant \int\limits_{R^n}\dots\int\left|E\exp\left\{i\frac{1}{\sqrt{2}}\sum_{j=1}^{n}s_j\frac{\zeta_j}{\sigma_j}\right\}\right|\prod_{j=1}^{n}\frac{1}{\sqrt{2\pi}t_j}\exp\left\{-\frac{s_j^2}{2t_j^2}\right\}ds_1\dots ds_n \leqslant$$

$$\leqslant \int\limits_{R^n}\dots\int\prod_{j=1}^{n}\left(\frac{1}{\sqrt{2\pi}t_j}\right)\exp\left\{-\frac{s_j^2}{2t_j^2}\right\}\frac{1}{(1+\sum_{j=1}^{n} s_j^2)^{\frac{1}{4}}}ds_1\dots ds_n.$$

Denote $\frac{s_j}{t_j} = u_j$. Then

$$E\exp\left\{-\sum_{j=1}^{n}\frac{\zeta_j^2t_j^2}{2\sigma_j^2}\right\} \le \int\limits_{R^n}\dots\int\left(\frac{1}{\sqrt{2\pi}}\right)^n\exp\left\{-\frac{1}{2}\sum_{j=1}^{n}u_j^2\right\}\times$$

$$\times\frac{1}{(1+\sum_{j=1}^{n} t_j^2u_j^2)^{\frac{1}{4}}}du_1\dots du_n. \tag{1.16}$$

Define t_j^2 as $t_j^2 = \sigma_j^2\frac{u^2}{\sum_{j=1}^{n}\sigma_j^2}$; $\sum_{j=1}^{n} t_j^2 = u^2$, $u > 0$.

Since $f(x) = \frac{1}{4}\ln(1+x)$ is convex function and $f(0) = 0$, then for $\alpha_i > 0$ such, that $\sum_{i=1}^n \alpha_i = 1$ and for $x_i > 0$ we have

$$\frac{1}{4}\ln(1+\sum_{i=1}^n \alpha_i x_i) \geq \sum_{i=1}^n \alpha_i(\frac{1}{4}\ln(1+x_i)).$$

$$-\frac{1}{4}\ln(1+\sum_{i=1}^n \alpha_i x_i) \leq \sum_{i=1}^n \alpha_i(-\frac{1}{4}\ln(1+x_i)).$$

Then

$$\frac{1}{(1+\sum_{i=1}^n \alpha_i x_i)^{\frac{1}{4}}} \leq \prod_{i=1}^n \frac{1}{(1+x_i)^{\frac{\alpha_i}{4}}}.$$

Whereas $\sum_{j=1}^n \frac{t_j^2}{u^2} = 1$, then

$$\frac{1}{(1+\sum_{i=1}^n t_j^2 u_j^2)^{\frac{1}{4}}} = \frac{1}{(1+\sum_{i=1}^n \frac{t_j^2}{u^2} u_j^2 u^2)^{\frac{1}{4}}} \leqslant$$

$$\leqslant \prod_{j=1}^n \frac{1}{(1+u_j^2 u^2)^{\frac{t_j^2}{4u^2}}}.$$

From the last inequality and (1.16) follows that

$$E\exp\left\{-\sum_{j=1}^n \frac{\zeta_j^2 t_j^2}{2\sigma_j^2}\right\} \leq \prod_{j=1}^n E\left(\left(\frac{1}{(1+\xi_j^2 u^2)^{\frac{1}{4}}}\right)^{\frac{t_j^2}{u^2}}\right),$$

where ξ_j are independent normally distributed random variables $N(0,1)$. Let us use the inequality $E|\xi|^\alpha \leq (E|\xi|)^\alpha, 0<\alpha<1$, to obtaine

$$E\exp\left\{-\sum_{j=1}^n \frac{\zeta_j^2 t_j^2}{2\sigma_j^2}\right\} \leq \left(E\left(\frac{1}{(1+\xi^2 u^2)^{\frac{1}{4}}}\right)\right)^{\sum_{j=1}^n \frac{t_j^2}{u^2}}$$

$$= \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty} \exp\left\{-\frac{s^2}{2}\right\}\frac{1}{(1+s^2u^2)^{\frac{1}{4}}}ds = g(u).$$

Hence,

$$E\exp\left\{-\frac{u^2}{2}\frac{\sum_{j=1}^n \zeta_j^2}{\sum_{j=1}^n \sigma_j^2}\right\} \leqslant g(u),$$

where

$$g(u) = \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} \exp\left\{-\frac{s^2}{2}\right\} \frac{1}{(1+s^2u^2)^{\frac{1}{4}}} ds.$$

Let us consider a general case. Let B be such a symmetric matrix, that $BB = A, R = \mathrm{cov}\overline{\zeta}$. Let O be the ortogonal matrix reducing BRB to the diagonal form $OBRBO^T = D = diag(d_k^2)_{k=1}^n$. Denote $\overline{\theta} = OB\overline{\zeta}$. Then

$$\overline{\theta}^T\overline{\theta} = \overline{\zeta}^T BO^T OB\overline{\zeta} = \overline{\zeta}^T A\overline{\zeta},$$

$\mathrm{cov}\overline{\theta} = \mathrm{OBcov}\overline{\zeta}\mathrm{BO^T} = \mathrm{D}$. Since $\theta_i \in SG_\Xi(\Omega), \overline{\theta}^T = (\theta_1, \ldots, \theta_n)$, then the inequality from lemma holds for $\overline{\theta}$. Therefore, $\overline{\theta}^T\overline{\theta} = \sum_{i=1}^n \theta_i^2 = \overline{\zeta}^T A\overline{\zeta}$ and

$$E\exp\left\{-\frac{u^2}{2}\frac{\overline{\zeta}^T A\overline{\zeta}}{E\overline{\zeta}^T A\overline{\zeta}}\right\} \le g(u).$$

Lemma is proved. ♢

Theorem 1.3. *Let $\overline{\zeta}^T = (\zeta_1, \ldots, \zeta_n)$ be square Gaussian random vector, $\zeta_i \in SG_\Xi(\Omega)$, $i = 1, \ldots, n$ and let A be some symmetric positive-valued matrix. Then for random variable $\eta = \overline{\zeta}^T A\overline{\zeta}$, $E\eta \neq 0$, the next inequalities hold*

$$P\left\{\frac{\eta}{E\eta} > x\right\} \ge 1 - g(u)\exp\left\{\frac{u^2x}{2}\right\}, \tag{1.17}$$

for any $u > 0$, and $0 < x < -\frac{2\ln g(u)}{u^2}$,
where $g(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp\left\{-\frac{s^2}{2}\right\} \frac{1}{(1+s^2u^2)^{\frac{1}{4}}} ds$ and

$$P\left\{\frac{\eta}{E\eta} > y\right\} \le \frac{2^{1/4}y^{1/4}}{\mathrm{ch}(\sqrt{\frac{y}{2}} - \frac{1}{2})}, \tag{1.18}$$

for $y > \frac{1}{2}$.

Proof. From the lemma 1.9 we have

$$E\exp\left\{-\frac{u^2\eta}{2E\eta}\right\} \le g(u)$$

Denote $\theta = \frac{\eta}{E\eta}$. Let $F(v)$ be distribution function of θ.

$$P\{\theta < x\} = \int\limits_0^x dF(v) = \int\limits_0^x \frac{\exp\{-\frac{u^2v}{2}\}}{\exp\{-\frac{u^2v}{2}\}} dF(v) \le$$

$$\frac{1}{\exp\{-\frac{u^2x}{2}\}} E\exp\left\{-\frac{u^2\theta}{2}\right\} \leq \frac{g(u)}{\exp\left\{-\frac{u^2x}{2}\right\}} = g(u)\exp\left\{\frac{u^2x}{2}\right\}.$$

Then $P\{\theta > x\} \geq 1 - g(u)\exp\left\{\frac{u^2x}{2}\right\}$.

Let us return to the old notation. Then

$$P\{\frac{\eta}{E\eta} > x\} \geq 1 - g(u)\exp\left\{\frac{u^2x}{2}\right\}.$$

The inequality (1.18) for $\eta = \overline{\zeta}^T A\overline{\zeta}$ was obtained in lemma 1.7. ♢

Corollary 1.5. *Assume, that for a sequence of random variables $\overline{\zeta}_m$ and for a sequence of symmetric positive-valued matrices A_m, $m \geqslant 1$, the conditions of the theorem 1.3 are fulfilled. Then theorem is true also for $\eta = l.i.m._{m\longrightarrow\infty}\overline{\zeta}_m^T A_m\overline{\zeta}_m$.*

Proof. Corollary follows from the theorem 1.3 and Fatou lemma. ♢

From the inequalities (1.17), (1.18) of the theorem 1.3 follows, that for $u > 0$, $0 < x < -\frac{2\ln g(u)}{u^2}$ and $y > \frac{1}{2}$

$$P\left\{\frac{\eta}{E\eta}\overline{\in}[x,y]\right\} \leqslant g(u)\exp\left\{\frac{u^2x}{2}\right\} + \frac{2^{\frac{1}{4}}y^{\frac{1}{4}}}{\mathrm{ch}\left(\sqrt{\frac{y}{2}} - \frac{1}{2}\right)}$$

or

$$P\left\{\frac{\eta}{E\eta} \in [x,y]\right\} \geqslant 1 - g(u)\exp\left\{\frac{u^2x}{2}\right\} - \frac{2^{\frac{1}{4}}y^{\frac{1}{4}}}{\mathrm{ch}\left(\sqrt{\frac{y}{2}} - \frac{1}{2}\right)}.$$

Let us evaluate $g(u)$:

$$g(u) = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty}\exp\left\{-\frac{s^2}{2}\right\}\frac{1}{(1+s^2u^2)^{\frac{1}{4}}}ds \leqslant$$

$$\leqslant \frac{1}{\sqrt{u}}\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty}\exp\left\{-\frac{s^2}{2}\right\}\frac{1}{\sqrt{|s|}}ds \leqslant \frac{J}{\sqrt{u}}, \quad u > 0,$$

where

$$J = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{+\infty}\exp\left\{-\frac{s^2}{2}\right\}\frac{1}{\sqrt{|s|}}ds < \infty.$$

1.6. An estimate for norm in $L_p(\mathbb{T})$ of the square Gaussian stochastic process

In the following theorem we obtain the estimate for the norm of square Gaussian stochastic processes in the space $L_p(\mathbb{T})$. This result we shall use for construction a criterion for testing hypotheses about the covariance function of Gaussian stochastic process.

Theorem 1.4. *Let $\{\mathbb{T}, \mathfrak{A}, \mu\}$ be a measurable space, where $\mathbb{T}$ is a parametric set and let $Y = \{Y(t), t \in \mathbb{T}\}$ be a square Gaussian stochastic process. Suppose that Y is a measurable process. Further, let the Lebesgue integral $\int\limits_{\mathbb{T}} (\mathbf{E}Y^2(t))^{\frac{p}{2}} d\mu(t)$ be well defined for $p \geqslant 1$. Then the integral $\int\limits_{\mathbb{T}} (Y(t))^p d\mu(t)$ exists with probability 1 and*

$$P\left\{\int\limits_{\mathbb{T}} \mid Y(t) \mid^p d\mu(t) > \varepsilon\right\} \leqslant 2\sqrt{1 + \frac{\varepsilon^{1/p}\sqrt{2}}{C_p^{\frac{1}{p}}}} \exp\left\{-\frac{\varepsilon^{\frac{1}{p}}}{\sqrt{2}C_p^{\frac{1}{p}}}\right\} \tag{1.19}$$

for all $\varepsilon \geqslant \left(\frac{p}{\sqrt{2}} + \sqrt{(\frac{p}{2}+1)p}\right)^p C_p$, where $C_p = \int\limits_{\mathbb{T}} (\mathbf{E}Y^2(t))^{\frac{p}{2}} d\mu(t)$.

Proof. Since $\max\limits_{x>0} x^\alpha e^{-x} = \alpha^\alpha e^{-\alpha}$ then $x^\alpha e^{-x} \leqslant \alpha^\alpha e^{-\alpha}$.

If ζ is a random variable from the space $SG_\Xi(\Omega)$ and $x = \frac{s}{\sqrt{2}} \cdot \frac{|\zeta|}{\sqrt{\mathbf{E}\zeta^2}}$, where $s > 0$ then

$$\mathbf{E}\left(\frac{s}{\sqrt{2}} \frac{\mid \zeta \mid}{\sqrt{\mathbf{E}\zeta^2}}\right)^\alpha \leqslant \alpha^\alpha e^{-\alpha} \cdot \mathbf{E}\exp\left\{\frac{s}{\sqrt{2}} \frac{\mid \zeta \mid}{\sqrt{\mathbf{E}\zeta^2}}\right\}$$

and

$$\mathbf{E} \mid \zeta \mid^\alpha \leqslant \left(\frac{\sqrt{2\mathbf{E}\zeta^2}}{s}\right)^\alpha \alpha^\alpha e^{-\alpha} \mathbf{E}\exp\left\{\frac{s}{\sqrt{2}} \frac{\mid \zeta \mid}{\sqrt{\mathbf{E}\zeta^2}}\right\}.$$

From the inequality (1.4) for $0 < s < 1$ we get that

$$\mathbf{E} \mid \zeta \mid^\alpha \leqslant \left(\frac{\sqrt{2\mathbf{E}\zeta^2}}{s}\right)^\alpha \alpha^\alpha e^{-\alpha} \left(\mathbf{E}\exp\left\{\frac{s}{\sqrt{2}} \frac{\zeta}{\sqrt{\mathbf{E}\zeta^2}}\right\} + \mathbf{E}\exp\left\{-\frac{s}{\sqrt{2}} \frac{\zeta}{\sqrt{\mathbf{E}\zeta^2}}\right\}\right)$$

$$\leqslant \frac{2}{\sqrt{1-s}} \left(\frac{\sqrt{2\mathbf{E}\zeta^2}}{s}\right)^\alpha \alpha^\alpha e^{-\alpha} \exp\left\{-\frac{s}{\sqrt{2}}\right\} =$$

$$= 2L_0(s) \left(\frac{\sqrt{2\mathbf{E}\zeta^2}}{s}\right)^\alpha \alpha^\alpha e^{-\alpha}, \tag{1.20}$$

where $L_0(s) = \frac{1}{\sqrt{1-s}} \left(\frac{\sqrt{2\mathbf{E}\zeta^2}}{s} \right)^{\alpha} \exp\left\{ -\frac{s}{\sqrt{2}} \right\}$.

Let $Y(t), t \in \mathbb{T}$ be a measurable square Gaussian stochastic process. Using the Chebyshev inequality we derive that for all $l \geqslant 1$

$$P\left\{ \int\limits_{\mathbb{T}} \mid Y(t) \mid^p d\mu(t) > \varepsilon \right\} \leqslant \frac{\mathbf{E}\left(\int\limits_{\mathbb{T}} \mid Y(t) \mid^p d\mu(t) \right)^l}{\varepsilon^l}.$$

Then from the generalized Minkowski inequality together with the inequality (1.20) for $l > 1$ we obtain that

$$\left(\mathbf{E}\left(\int\limits_{\mathbb{T}} \mid Y(t) \mid^p d\mu(t) \right)^l \right)^{\frac{1}{l}} \leqslant \int\limits_{\mathbb{T}} \left(\mathbf{E} \mid Y(t) \mid^{pl} \right)^{\frac{1}{l}} d\mu(t)$$

$$\leqslant \int\limits_{\mathbb{T}} (2L_0(s)(2\mathbf{E}Y^2(t))^{\frac{pl}{2}} (pl)^{pl} s^{-pl} \exp\{-pl\})^{\frac{1}{l}} d\mu(t)$$

$$= (2L_0(s))^{\frac{1}{l}} \int\limits_{\mathbb{T}} (2\mathbf{E}Y^2(t))^{\frac{p}{2}} s^{-p} (pl)^p \exp\{-p\} d\mu(t)$$

$$= (2L_0(s))^{\frac{1}{l}} 2^{\frac{p}{2}} s^{-p} (pl)^p \exp\{-p\} \int\limits_{\mathbb{T}} (\mathbf{E}Y^2(t))^{\frac{p}{2}} d\mu(t).$$

Assuming that $C_p = \int\limits_{\mathbb{T}} (\mathbf{E}Y^2(t))^{\frac{p}{2}} d\mu(t)$ we deduce that

$$\mathbf{E}\left(\int\limits_{\mathbb{T}} \mid Y(t) \mid^p d\mu(t) \right)^l \leqslant 2L_0(s) 2^{\frac{pl}{2}} (lp)^{pl} \exp\{-pl\} C_p^l s^{-pl}.$$

Hence,

$$P\left\{ \int\limits_{\mathbb{T}} \mid Y(t) \mid^p d\mu(t) > \varepsilon \right\} \leqslant 2 \cdot (2^{\frac{p}{2}})^l L_0(s) (p^p)^l (\exp\{-p\})^l C_p^l (s^{-p})^l \cdot \frac{(l^p)^l}{\varepsilon^l}$$

$$= 2L_0(s) a^l (l^p)^l,$$

where $a = \frac{2^{\frac{p}{2}} p^p C_p}{e^p s^p \varepsilon}$. That is $a^{\frac{1}{p}} = \frac{2^{\frac{1}{2}} p C_p^{\frac{1}{p}}}{e s \varepsilon^{\frac{1}{p}}}$. Let us find the minimum of the function $\psi(l) = a^l (l^p)^l$ regarding l. One can easily check that $l^* = \frac{1}{e a^{\frac{1}{p}}}$ is a point in which this function reaches its minimum.

Then

$$2L_0(s)\psi(l^*) = 2L_0(s)a^{\frac{1}{ea^{\frac{1}{p}}}} \cdot \left(\frac{1}{ea^{\frac{1}{p}}}\right)^{p\cdot\frac{1}{ea^{\frac{1}{p}}}} = 2L_0(s)a^{\frac{1}{ea^{\frac{1}{p}}}} \cdot a^{-\frac{1}{ea^{\frac{1}{p}}}} \cdot e^{-\frac{p}{ea^{\frac{1}{p}}}}$$
$$= 2L_0(s)\exp\left\{-\frac{pes\varepsilon^{\frac{1}{p}}}{2^{\frac{1}{2}}peC_p^{\frac{1}{p}}}\right\} = 2L_0(s)\exp\left\{-\frac{s\varepsilon^{\frac{1}{p}}}{2^{\frac{1}{2}}C_p^{\frac{1}{p}}}\right\}$$
$$= \frac{2}{\sqrt{1-s}}\exp\left\{-s\left(\frac{1}{2}+\frac{\varepsilon^{1/p}}{2^{\frac{1}{2}}C_p^{\frac{1}{p}}}\right)\right\}.$$

In turn, minimizing $\theta(s) = \frac{2}{\sqrt{1-s}}\exp\left\{-s\left(\frac{1}{2}+\frac{\varepsilon^{1/p}}{2^{\frac{1}{2}}C_p^{\frac{1}{p}}}\right)\right\}$ in s, we deduce $s^* = 1 - \frac{1}{1+\frac{\sqrt{2}\varepsilon^{1/p}}{C_p^{1/p}}}$. Thus

$$\theta(s^*) = 2\sqrt{1+\frac{\varepsilon^{1/p}\sqrt{2}}{C_p^{\frac{1}{p}}}}\exp\left\{-\frac{\varepsilon^{\frac{1}{p}}}{\sqrt{2}C_p^{\frac{1}{p}}}\right\}.$$

From the fact that $l^* \geqslant 1$ it follows that inequality (1.19) holds if $\frac{1}{ea^{\frac{1}{p}}} = \frac{s\varepsilon^{1/p}}{\sqrt{2}pC_p^{1/p}} \geqslant 1$. Substituting in this expression the value of s^* we obtain the following inequality $\varepsilon^{2/p} \geqslant pC_p^{1/p}(C_p^{1/p} + \sqrt{2}\varepsilon^{1/p})$. Solving this inequality with respect to ε and taking into account that $\varepsilon > 0$ we deduce that inequality (1.19) holds when $\varepsilon \geqslant \left(\frac{p}{\sqrt{2}} + \sqrt{(\frac{p}{2}+1)p}\right)^p C_p$. The theorem is proved. ♢

Chapter 2
The construction of the model of Gaussian stochastic processes with a certain accuracy and reliability.

Stochastic processes are widely used in various fields of science. With the help of stochastic processes can be described many phenomena in the environment. In order to effectively study of all necessary qualitative and quantitative properties and characteristics of the process in the theory of stochastic processes was decided to construct their models. During the twentieth century a number of simulation methods have been developed, among them the method of minimal transformation, canonical representations, autoregression, and others like that. However, in 1978 Mikhailov in [96] proposed a somewhat new approach to the construction of models. This method took the name of the method of partition and randomization of the spectrum. In the paper [131], this method was modified and applied for the construction of models of Gaussian nonstationary stochastic processes and fields. The advantage of this method is that the constructed models are sub-Gaussian. In addition, with this method, the covariance functions of the models and the covariance functions of the processes are almost identical.

The first part of this chapter contains the construction of model of the Gaussian stochastic process. In addition to constructing models of Gaussian stochastic processes, in this chapter we also investigated the accuracy and reliability of these models in the different functional spaces. The accuracy and reliability of the constructed models are mainly investigated in the papers by Kozachenko and his students. The results presented in this chapter were published in the papers [131] and [85].

2.1. Constructing a model of Gaussian stochastic process

Let $\{\Omega, \mathbf{B}, \mathcal{P}\}$ be a standart, fixed probability space, $\mathbb{T}$ be a parametric set. Let $\xi = \{\xi(t), t \in \mathbb{T}\}$ be a zero-mean real-valued Gaussian stochastic process. The covariance function of the process is defined as

$$R(t,s) = \int\limits_0^\infty g(t,\lambda)g(s,\lambda)dF(\lambda),$$

where $F(\lambda)$ is a distribution function. According to the Karhunen theorem [40], the process ξ can be represented as follows

$$\xi(t) = \int_0^\infty g(t,\lambda) d\eta(\lambda), \tag{2.1}$$

where $\eta(\lambda)$ is a Gaussian process with independent increments, such that $\mathsf{E}(\eta(b) - \eta(c))^2 = F(b) - F(c), b > c$, and $\mathsf{E}\eta(\lambda) = 0$.

Let $L > 0$ be a given real number. We consider such partition $\Lambda = \{\lambda_0, ..., \lambda_N\}$ of the set $[0, \infty]$ that $\lambda_0 = 0, \lambda_k < \lambda_{k+1}, \lambda_{N-1} = L, \lambda_N = +\infty$. For this partition we can write

$$\xi(t) = \sum_{k=0}^{N-1} \int_{\lambda_k}^{\lambda_{k+1}} g(t,\lambda) d\eta(\lambda)$$

As a model for the process ξ we consider

$$\xi_\Lambda(t) = \sum_{k=0}^{N-1} \eta_k g(t, \zeta_k), \tag{2.2}$$

where η_k and ζ_k are independent random variables, η_k are such Gaussian random variables that $\mathsf{E}\eta_k = 0$, $\mathsf{E}\eta_k^2 = F(\lambda_{k+1}) - F(\lambda_k) = b_k^2$; ζ_k, $k = 0, ..., N-2$ are random variables taking values on the segments $[\lambda_k; \lambda_{k+1}]$, $\zeta_{N-1} = L$ and if $b_k^2 > 0$, then

$$F_k(\lambda) = P\{\zeta_k < \lambda\} = \frac{F(\lambda) - F(\lambda_k)}{F(\lambda_{k+1}) - F(\lambda_k)}.$$

If $b_k^2 = 0$, then $\zeta_k = 0$ with probability one. For the sake of simplicity we assume that $b_k^2 > 0, k = 0, 1, ..., N$.

This model is a zero-mean process

$$\mathsf{E}\xi_\Lambda(t) = \mathsf{E}\sum_{k=0}^{N-1} \eta_k g(t, \zeta_k) = \sum_{k=0}^{N-1} \mathsf{E}\eta_k \mathsf{E}g(t, \zeta_k) = 0.$$

Covariance function of the process $\xi_\Lambda(t)$ is almost the same as covariance function of the process $\xi(t)$, namely at a certain choice of Λ, covariance function of the process $\xi_\Lambda(t)$ can be made arbitrarily close to the covariance function of $\xi(t)$.

Putting $\eta_k = \int_{\lambda_k}^{\lambda_{k+1}} d\eta(\lambda)$ we consider the following difference

$$\eta_\Lambda(t) = \xi(t) - \xi_\Lambda(t) = \sum_{k=0}^{N-1} \int_{\lambda_k}^{\lambda_{k+1}} g(t,\lambda) d\eta(\lambda) - \sum_{k=0}^{N-1} \int_{\lambda_k}^{\lambda_{k+1}} g(t,\zeta_k) d\eta(\lambda) =$$

$$= \sum_{k=0}^{N-1} \int_{\lambda_k}^{\lambda_{k+1}} \left(g(t,\lambda) - g(t,\zeta_k)\right) d\eta(\lambda) \quad (2.3)$$

Let the following condition hold for the function $g(t, \lambda)$

$$\mid g(t,\lambda) - g(t,u) \mid \leqslant S(\mid u - \lambda \mid) \cdot Z(t), \quad (2.4)$$

where $Z(t)$, $t \in \mathbb{T}$ is some continuous function and $S(\lambda), \lambda \in \mathbb{R}$ monotone increases, such that $S(\lambda) \to 0$ as $\lambda \to 0$.

Lemma 2.1. *Let condition (2.4) holds for a function $g(t, \lambda)$. Then we have*

$$E\left(\int_{\lambda_k}^{\lambda_{k+1}} \left(g(t,\lambda) - g(t,\zeta_k)\right) d\eta(\lambda)\right)^{2m+1} = 0,$$

$$E\left(\int_{\lambda_k}^{\lambda_{k+1}} \left(g(t,\lambda) - g(t,\zeta_k)\right) d\eta(\lambda)\right)^{2m} \leqslant$$

$$\leqslant \frac{(2m)!}{2^m \cdot m!} Z^{2m}(t) E\left(\int_{\lambda_k}^{\lambda_{k+1}} S^2(|\lambda - \zeta_k|) dF(\lambda)\right)^m$$

Proof. Since for a zero-mean Gaussian random variable ξ it is

$$\mathsf{E}\xi = 0, \mathsf{E}\xi^{2m+1} = 0, \mathsf{E}\xi^{2k} = \frac{(2k)!}{2^k \cdot k!}\sigma^{2k}$$

and the random variables ζ_k are independent of $\eta(\lambda)$, then by the Fubini's theorem (E_{ζ_k} is a conditional expectation with respect to ζ_k):

$$\mathsf{E}\left(\int_{\lambda_k}^{\lambda_{k+1}} \left(g(t,\lambda) - g(t,\zeta_k)\right) d\eta(\lambda)\right)^{2m} =$$

$$= \mathsf{E}\mathsf{E}_{\zeta_k}\left(\int_{\lambda_k}^{\lambda_{k+1}} \left(g(t,\lambda) - g(t,\zeta_k)\right) d\eta(\lambda)\right)^{2m} =$$

$$= \frac{(2m)!}{2^m \cdot m!} \mathsf{E} \left(\int_{\lambda_k}^{\lambda_{k+1}} |g(t,\lambda) - g(t,\zeta_k)|^2 dF(\lambda) \right)^m \leqslant$$

$$\leqslant \frac{(2m)!}{2^m \cdot m!} \mathsf{E} \left(\int_{\lambda_k}^{\lambda_{k+1}} S^2(|\lambda - \zeta_k|) Z^2(t) dF(\lambda) \right)^m =$$

$$= \frac{(2m)!}{2^m \cdot m!} Z^{2m}(t) \mathsf{E} \left(\int_{\lambda_k}^{\lambda_{k+1}} S^2(|\lambda - \zeta_k|) dF(\lambda) \right)^m ,$$

which finishes the proof. ◇

Theorem 2.1. *The stochastic process $\xi(t) - \xi_\Lambda(t)$ is sub-Gaussian and the following inequality holds*

$$\tau(\xi(t) - \xi_\Lambda(t)) \leqslant$$

$$\leqslant Z(t) \left[\sum_{k=0}^{N-2} b_k^2 \sup_{m \geqslant 1} \left(ES^{2m}(|\zeta_k - \zeta_k^*|) \right)^{\frac{1}{m}} + \int_L^\infty S^2(|\lambda - L|) dF(\lambda) \right]^{\frac{1}{2}} ,$$

where $b_k^2 = F(\lambda_{k+1}) - F(\lambda_k)$ and ζ_k^* are random variables independent of ζ_k but with the same distribution as ζ_k.

Proof. Using the Lemma 2.1 for $k \leqslant N-2$ we obtain

$$\tau^2 \left(\int_{\lambda_k}^{\lambda_{k+1}} (g(t,\lambda) - g(t,\zeta_k)) d\eta(\lambda) \right) \leqslant \theta^2 \left(\int_{\lambda_k}^{\lambda_{k+1}} (g(t,\lambda) - g(t,\zeta_k)) d\eta(\lambda) \right) \leqslant$$

$$\leqslant \sup_{m \geqslant 1} b_k^2 Z^2(t) \left(\mathsf{E} \left(\int_{\lambda_k}^{\lambda_{k+1}} S^2(|\lambda - \zeta_k|) dF_k(\lambda) \right)^m \right)^{\frac{1}{m}} =$$

$$= \sup_{m \geqslant 1} b_k^2 Z^2(t) \left(\int_{\lambda_k}^{\lambda_{k+1}} \left(\int_{\lambda_k}^{\lambda_{k+1}} S^2(|\lambda - u|) dF_k(\lambda) \right)^m dF_k(u) \right)^{\frac{1}{m}} .$$

In the case when $k = N-1$ we have

$$\tau^2\left(\int\limits_L^\infty (g(t,\lambda)-g(t,L))d\eta(\lambda)\right) \leqslant \theta^2\left(\int\limits_L^\infty (g(t,\lambda)-g(t,L))d\eta(\lambda)\right) \leqslant$$

$$\leqslant \sup_{m\geqslant 1}\left[\frac{2^m m!}{(2m)!}\mathsf{E}\left(\int\limits_L^\infty (g(t,\lambda)-g(t,L))d\eta(\lambda)\right)^{2m}\right]^{\frac{1}{m}} \leqslant$$

$$\leqslant \sup_{m\geqslant 1}\left[\left(\int\limits_L^\infty |g(t,\lambda)-g(t,L)|^2 dF(\lambda)\right)^m\right]^{\frac{1}{m}} \leqslant Z^2(t)\int\limits_L^\infty S^2(|\lambda-L|)dF(\lambda).$$

Lemma 1.3 implies that $\int\limits_{\lambda_k}^{\lambda_{k+1}} (g(t,\lambda)-g(t,\zeta_k))d\eta(\lambda)$ are sub-Gaussian random variables.

Since the terms in the sum (2.3) for different k are independent, so from the last equality we have

$$\tau^2(\xi(t)-\xi_\Lambda(t)) \leqslant$$

$$\leqslant Z^2(t)\sum_{k=0}^{N-2} b_k^2 \sup_{m\geqslant 1}\left(\int\limits_{\lambda_k}^{\lambda_{k+1}}\left(\int\limits_{\lambda_k}^{\lambda_{k+1}} S^2(|\lambda-u|)dF_k(\lambda)\right)^m dF_k(u)\right)^{\frac{1}{m}} +$$

$$+Z^2(t)\int\limits_L^\infty S^2(|\lambda-L|)dF(\lambda).$$

Then, from the Fubini's theorem and the Lyapunov inequality we obtain

$$\tau(\xi(t)-\xi_\Lambda(t)) \leqslant$$

$$\leqslant Z(t)\left[\sum_{k=0}^{N-2}\sup_{m\geqslant 1} b_k^2\left(\int\limits_{\lambda_k}^{\lambda_{k+1}}\left(\int\limits_{\lambda_k}^{\lambda_{k+1}} S^2(|\lambda-u|)dF_k(\lambda)\right)^m dF_k(u)\right)^{\frac{1}{m}} +\right.$$

$$\left.+\int\limits_L^\infty S^2(|\lambda-L|)dF(\lambda)\right]^{\frac{1}{2}} =$$

$$= Z(t)\left[\sum_{k=0}^{N-2}\sup_{m\geqslant 1} b_k^2\left(\mathsf{E}_{\zeta_k^*}\left(\mathsf{E}_{\zeta_k} S^2(|\zeta_k-\zeta_k^*|)\right)^m\right)^{\frac{1}{m}} +\right.$$

$$+\int_{L}^{\infty} S^2(|\lambda - L|)dF(\lambda)\Bigg]^{\frac{1}{2}} \leqslant$$

$$\leqslant Z(t)\left[\sum_{k=0}^{N-2} \sup_{m\geqslant 1} b_k^2 \left(\mathsf{E}_{\zeta_k^*}\mathsf{E}_{\zeta_k} S^{2m}(|\zeta_k - \zeta_k^*|)\right)^{\frac{1}{m}} + \int_{L}^{\infty} S^2(|\lambda - L|)dF(\lambda)\right]^{\frac{1}{2}} \leqslant$$

$$\leqslant Z(t)\left[\sum_{k=0}^{N-2} b_k^2 \sup_{m\geqslant 1} \left(\mathsf{E} S^{2m}(|\zeta_k - \zeta_k^*|)\right)^{\frac{1}{m}} + \int_{L}^{\infty} S^2(|\lambda - L|)dF(\lambda)\right]^{\frac{1}{2}},$$

which is the desired statement. ♢

Corollary 2.1. *If for all $\lambda, u \in \mathbb{R}_+$ there exists an absolute constant $C > 0$ so, that*

$$\sup_{t\in\mathbb{T}} |g(t,\lambda) - g(t,u)| \leqslant C,$$

then we have

$$\tau(\xi(t) - \xi_\Lambda(t)) \leqslant$$

$$\leqslant Z(t)\left[\sum_{k=0}^{N-2} b_k^2 \sup_{m\geqslant 1} \left(ES^{2m}(|\zeta_k - \zeta_k^*|)\right)^{\frac{1}{m}} + C^2(F(+\infty) - F(L))\right]^{\frac{1}{2}},$$

where b_k^2 and ζ_k^* remain the same as in the previous Theorem 2.1.

Example 2.1. Let covariance function of stochastic process ξ have the following form

$$R(t,s) = \int_{0}^{\infty} \cos t\lambda \cos s\lambda dF(\lambda).$$

i.e. $g(t,\lambda) = cos(t\lambda)$. Then $\xi(t) = \int_0^\infty \cos t\lambda d\eta(\lambda)$ is a zero-mean real-valued Gaussian stochastic process, where $\eta(\lambda)$ is a Gaussian process with independent increments, $\mathbf{E}(\eta(b) - \eta(c))^2 = F(b) - F(c), b > c$. $\mathbf{E}\eta(\lambda) = 0$.

Consider the following straightforward estimate:

$$|\cos t\lambda - \cos tu|^2 = \left|2\sin\frac{t(\lambda-u)}{2}\sin\frac{t(\lambda+u)}{2}\right|^2 \leqslant$$

$$\leqslant \left|2\sin\frac{t(u-\lambda)}{2}\right|^2 \leqslant 2^{2(1-\alpha)}t^{2\alpha}|u-\lambda|^{2\alpha},$$

where $0 < \alpha \leqslant 1$.

By virtue of Theorem 2.1 and Corollary 2.1 and taking into account that the functions $Z(t) = 2^{(1-\alpha)}t^\alpha$, $S(\lambda) = \lambda^\alpha$, while $C = 2$ we obtain the following inequality

$$\tau^2(\xi(t) - \xi_\Lambda(t)) \leqslant 2^{2(1-\alpha)}t^{2\alpha}\sum_{k=0}^{N-2} b_k^2 \mid \lambda_{k+1} - \lambda_k \mid^{2\alpha} + 4(F(+\infty) - F(L)).$$

Example 2.2. Consider the covariance function of stochastic process ξ which have the following form

$$R(t,s) = \int_0^\infty J_l(t\lambda)J_l(s\lambda)dF(\lambda),$$

i.e. $g(t,\lambda) = J_l(t\lambda)$, where $J_l(t\lambda) = \frac{1}{\pi}\int_0^\pi \cos(l\varphi - t\lambda\sin\varphi)d\varphi$ is the integral representation of the Bessel functions of the first kind.

Then $\xi(t) = \int_0^\infty J_l(t\lambda)d\eta(\lambda)$ is a zero-mean real-valued Gaussian stochastic process, where $\eta(\lambda)$ is a Gaussian process with independent increments, $\mathbf{E}(\eta(b) - \eta(c))^2 = F(b) - F(c), b > c$. $\mathbf{E}\eta(\lambda) = 0$.

Let us find the estimate for the squared difference

$$\Delta_J(\lambda, u) = |J_l(t\lambda) - J_l(tu)|^2.$$

By direct calculations we get

$$\Delta_J(\lambda,u) = \frac{1}{\pi^2}\left|\int_0^\pi 2\sin\left(\frac{2l\varphi - t(\lambda+u)\sin\varphi}{2}\right)\sin\left(\frac{t(u-\lambda)\sin\varphi}{2}\right)d\varphi\right|^2 \leqslant$$

$$\leqslant \frac{1}{\pi^2}\int_0^\pi \left|2\sin\frac{t(u-\lambda)\sin\varphi}{2}\right|^2 d\varphi \leqslant$$

$$\leqslant \frac{1}{\pi^2}\int_0^{\pi} t^2|u-\lambda|^2\sin^2\varphi d\varphi = \frac{t^2}{\pi}|u-\lambda|^2.$$

Applying Theorem 2.1 and Corollary 2.1 and having in mind that $Z(t) = \frac{t}{\sqrt{\pi}}$, $S(\lambda)=\lambda$ and $C=1$, we arrive at the following inequality

$$\tau^2(\xi(t)-\xi_\Lambda(t)) \leqslant \frac{1}{\pi}t^2\sum_{k=0}^{N-2} b_k^2|\lambda_{k+1}-\lambda_k|^2 + (F(+\infty)-F(L)).$$

2.2. Accuracy and reliability the model for Gaussian stochastic process in space $L_p(\mathbb{T}), p \geqslant 1$

Definition 2.1. [72] A stochastic process $\xi_\Lambda(t)$ approximates the process $\xi(t)$ with reliability $(1-\delta), 0<\delta<1$ and accuracy $\varepsilon>0$ in $L_p(\mathbb{T})$, if the partition Λ is such that the following inequality holds

$$\mathbf{P}\left\{\left(\int_0^T |\ \xi(t)-\xi_\Lambda(t)\ |^p\ dt\right)^{\frac{1}{p}} > \varepsilon\right\} \leqslant \delta.$$

Theorem 2.2. *[53] Suppose that $\xi = \{\xi(t), t\in\mathbb{T}\}$ is a sub-Gaussian stochastic process, $\mathbb{E}\xi(t)=0$, $\tau^2(t)=\tau^2(\xi(t))=\mathbb{E}(\xi(t))^2$. Suppose there exists an integral $\int_T \left(\mathbb{E}(\xi(t))^2\right)^{\frac{p}{2}} dt<\infty$, $p\geqslant 1$. Then the integral $\int_{\mathbb{T}} |\ \xi(t)\ |^p$ $dt<\infty$, exists with probability 1 and for all ε such that $\varepsilon > c_p^{\frac{1}{p}} p^{\frac{1}{2}}$, where $c_p=\int_{\mathbb{T}}\left(\mathbb{E}(\xi(t))^2\right)^{\frac{p}{2}} dt$ the inequality holds true*

$$\mathbf{P}\left\{\|\ \xi(t)\ \|_{L_p} > \varepsilon\right\} \leqslant 2\exp\left\{-\frac{\varepsilon^2}{2c_p^{\frac{2}{p}}}\right\}.$$

Theorem 2.3. *Suppose that the partition Λ in the model $\xi_\Lambda(t)$ is such that*

$$\int_{\mathbb{T}} (\tau(\xi(t)-\xi_\Lambda(t)))^p\, dt \leqslant \frac{\varepsilon^p}{\max\left(p^{\frac{p}{2}}, \left(2\ln\frac{2}{\delta}\right)^{\frac{p}{2}}\right)}.$$

Then this model approximates the Gaussian process $\xi(t)$ with accuracy $\varepsilon>0$ and reliability $1-\delta, 0<\delta<1$ in the space $L_p(\mathbb{T}), p\geqslant 1$.

Proof. If $\varepsilon > \left(\int\limits_{\mathbb{T}} (\tau(\xi(t) - \xi_\Lambda(t)))^p \, dt\right)^{\frac{1}{p}} \cdot p^{\frac{1}{2}}$, then according to the Theorem 2.2 and Definition 2.1 we have

$$\mathbf{P}\left\{\| \, \xi(t) - \xi_\Lambda(t) \, \|_{L_p} > \varepsilon\right\} \leqslant 2 \exp\left\{-\frac{\varepsilon^2}{2c_p^{\frac{2}{p}}}\right\} \leqslant \delta,$$

where $c_p = \int\limits_{\mathbb{T}} (\tau(\xi(t) - \xi_\Lambda(t)))^p \, dt$.

And then the last inequality is true if the following condition holds

$$\int\limits_{T} (\tau(\xi(t) - \xi_\Lambda(t)))^p \, dt \leqslant \frac{\varepsilon^p}{\left(2\ln\frac{2}{\delta}\right)^{\frac{p}{2}}},$$

which finishes the proof. ♢

Example 2.3. Let $F(\lambda)$ in Example 2.1 be such that $F(+\infty) = 1$, $F(L) = 1 - \frac{1}{1+L^\alpha}$, $0 \leqslant \alpha \leqslant 1$, $\mathbb{T} = [0, T]$.

With the aid of the Corollary 2.1 we get

$$\tau^2(\xi(t) - \xi_\Lambda(t)) \leqslant 2^{2(1-\alpha)} t^{2\alpha} \sum_{k=0}^{N-2} b_k^2 |\lambda_{k+1} - \lambda_k|^{2\alpha} + 4(F(+\infty) - F(L)).$$

Letting $|\lambda_{k+1} - \lambda_k| = \frac{L}{N-1}$ we conclude

$$\tau^2(\xi(t) - \xi_\Lambda(t)) \leqslant 2^{2(1-\alpha)} t^{2\alpha} F(L) \left(\frac{L}{N-1}\right)^{2\alpha} + 4(F(+\infty) - F(L)) \leqslant$$

$$\leqslant 4\left(\frac{tL}{2(N-1)}\right)^{2\alpha} + \frac{4}{1+L^\alpha},$$

hence

$$\tau(\xi(t) - \xi_\Lambda(t)) \leqslant 2\left[\left(\frac{tL}{2(N-1)}\right)^{2\alpha} + \frac{1}{1+L^\alpha}\right]^{\frac{1}{2}}.$$

Next, minimize $y_1(L) = 2\left[\left(\frac{tL}{2(N-1)}\right)^{2\alpha} + \frac{1}{1+L^\alpha}\right]^{\frac{1}{2}}$ with respect to L; it follows that argument minimum is $y_1(L_0)$,

$$L_0 = \frac{1}{2^{\frac{1}{3\alpha}}} \left(\frac{2(N-1)}{t}\right)^{\frac{2}{3}}$$

.

Then

$$\tau(\xi(t)-\xi_\Lambda(t)) \leqslant 2\left[\left(\frac{t}{2(N-1)}\right)^{2\alpha} \frac{1}{\sqrt[3]{4}} \left(\frac{2(N-1)}{t}\right)^{\frac{4\alpha}{3}} + \right.$$

$$\left. +\sqrt[3]{2}\left(\frac{t}{2(N-1)}\right)^{\frac{2\alpha}{3}}\right]^{\frac{1}{2}} = \frac{2\sqrt{3}}{\sqrt[3]{2}} \left(\frac{t}{2(N-1)}\right)^{\frac{\alpha}{3}}.$$

Hence

$$\int_0^T (\tau(\xi(t)-\xi_\Lambda(t)))^p dt \leqslant \left(\frac{2\sqrt{3}}{\sqrt[3]{2}}\right)^p \left(\frac{1}{2(N-1)}\right)^{\frac{\alpha p}{3}} \int_0^T t^{\frac{\alpha p}{3}} dt =$$

$$= \left(\frac{2\sqrt{3}}{\sqrt[3]{2}}\right)^p \left(\frac{1}{2(N-1)}\right)^{\frac{\alpha p}{3}} \frac{T^{\frac{\alpha p}{3}+1}}{\frac{\alpha p}{3}+1}.$$

Hence, by Theorem 2.3, the inequality

$$\int_{\mathbb{T}} (\tau(\xi(t)-\xi_\Lambda(t)))^p \, dt \leqslant \frac{\varepsilon^p}{\max\left(p^{\frac{p}{2}}, \left(2\ln\frac{2}{\delta}\right)^{\frac{p}{2}}\right)}$$

follows, when N satisfies

$$N \geqslant \frac{1}{2}\left(\frac{T^{\frac{\alpha p}{3}+1}(2\sqrt{3})^p \cdot \max\left(p^{\frac{p}{2}}, \left(\ln\frac{2}{\delta}\right)^{\frac{p}{2}}\right)}{\sqrt[3]{2^p}\left(\frac{\alpha p}{3}+1\right)\varepsilon^p}\right)^{\frac{3}{\alpha p}} + 1. \qquad (2.5)$$

Thus the model $\xi_\Lambda(t)$ approximate process $\xi(t)$ with reliability $1-\delta, 0 < \delta < 1$ and accuracy $\varepsilon > 0$ in the space $L_p(\mathbb{T})$, if the relationship (2.5) holds.

Example 2.4. Let $\xi_\Lambda(t) = \sum\limits_{k=0}^{N-1} \eta_k \cos(t\zeta_k)$ is the model from the Example 2.1 and let the partition Λ is such that $|\lambda_{k+1}-\lambda_k| = \frac{L}{N-1}$, when $0 \leqslant \alpha \leqslant 1$

we have

$$\tau^2(\xi(t)-\xi_\Lambda(t)) \leqslant 2^{2(1-\alpha)}t^{2\alpha}F(L)\left(\frac{L}{N-1}\right)^{2\alpha}+4(F(+\infty)-F(L)).$$

Namely

$$\int_0^T(\tau(\xi(t)-\xi_\Lambda(t)))^p dt \leqslant$$

$$\leqslant \int_0^T\left(2^{2(1-\alpha)}t^{2\alpha}F(L)\left(\frac{L}{N-1}\right)^{2\alpha}+4(F(+\infty)-F(L))\right)^{\frac{p}{2}}dt \leqslant$$

$$\leqslant \int_0^T\left(2^{p(1-\alpha)}D_p(F(L))^{\frac{p}{2}}\left(\frac{L}{N-1}\right)^{p\alpha}t^{p\alpha}+4^{\frac{p}{2}}D_p(F(+\infty)-F(L))^{\frac{p}{2}}\right)dt =$$

$$= 2^{p(1-\alpha)}D_p(F(L))^{\frac{p}{2}}\left(\frac{L}{N-1}\right)^{p\alpha}\frac{T^{p\alpha+1}}{p\alpha+1}+4^{\frac{p}{2}}D_pT(F(+\infty)-F(L))^{\frac{p}{2}},$$

where $D_p = \begin{cases} 1, & \text{as } 0<\frac{p}{2}\leqslant 1, \\ 2^{\frac{p}{2}-1}, & \text{as } \frac{p}{2}>1 \end{cases}$.

The inequality from Theorem 2.3 holds, when N satisfies

$$N \geqslant \frac{2^{\frac{1-\alpha}{\alpha}}D_p^{\frac{1}{p\alpha}}L\cdot T^{1+\frac{1}{p\alpha}}\,(F(L))^{\frac{1}{2\alpha}}}{(1+p\alpha)^{\frac{1}{p\alpha}}\left(\frac{\varepsilon^p}{\max\left(p^{\frac{p}{2}},\left(2\ln\frac{2}{\delta}\right)^{\frac{p}{2}}\right)}-4^{\frac{p}{2}}D_pT(F(+\infty)-F(L)^{\frac{p}{2}}\right)^{\frac{1}{p\alpha}}}.$$

Let in the Gaussian stochastic process $\xi(t)=\int\limits_0^\infty\cos t\lambda d\eta(\lambda)$ function$F(\lambda)$ defined as follows $F(\lambda)=1-\frac{1}{1+\lambda^3}$, then using for the N previous inequality when $p=4$, $\alpha=1$, $T=1$, $\delta=0,01$ and $\varepsilon=0,06$ we get

$$N \geqslant \sqrt{2}L\frac{\left(\left(1-\frac{1}{1+L^3}\right)\ln 200\right)^{\frac{1}{2}}}{\left(0,0000324-320\left(\frac{\ln 200}{1+L^3}\right)^2\right)^{\frac{1}{4}}}.$$

Using the software package Mathematica, we find that the minimum of this function with respect to L equal to $N(29,746)=1458,486$, it is easy to see in the next graph.

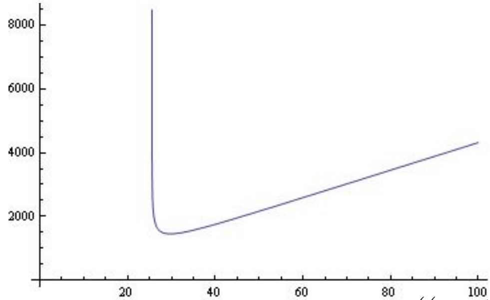

Figure 2.1. Graph of $N(L)=\sqrt{2}L\frac{\left(\left(1-\frac{1}{1+L^3}\right)\ln 200\right)^{\frac{1}{2}}}{\left(0{,}0000324-320\left(\frac{\ln 200}{1+L^3}\right)^2\right)^{\frac{1}{4}}}$.

Namely, selecting the minimal partition $N=1459$, we can construct a model $\xi_\Lambda(t)$ that approximate the process $\xi(t)$ with reliability $0,99$ and accuracy $0,06$ in the space $L_4([0,1])$.

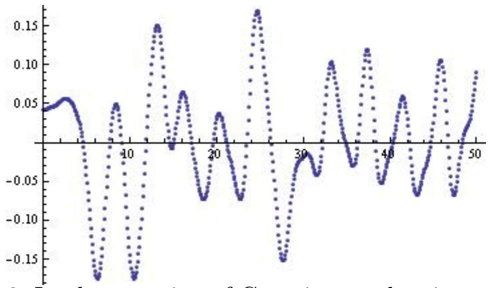

Figure 2.2. Implementation of Gaussian stochastic process, $F(\lambda)=1-\frac{1}{1+\lambda^3}$.

If for this Gaussian stochastic process we choose $F(\lambda)=1-\frac{1}{1+\lambda^5}$, and $p=2$, $\alpha=1$, $T=1$, $\delta=0,01$, $\varepsilon=0,06$, than the inequality for N will have the following form

$$N\geqslant\sqrt{\frac{2}{3}}L\left(\frac{\left(1-\frac{1}{1+L^5}\right)\ln 200}{0,0036-\frac{8\ln 200}{1+L^5}}\right)^{\frac{1}{2}}$$

In this case, minimizing the previous function with respect to L, using the software package Mathematica, we deduce that $N(8,3751)=310,405$.

Therefore, selecting the minimal partition $N=311$, we can construct

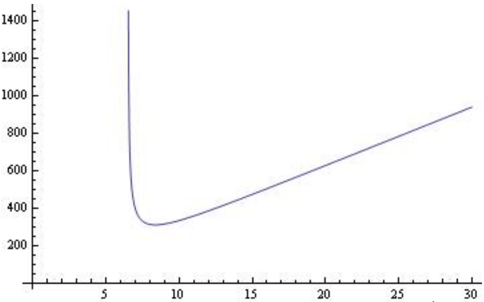

Figure 2.3. Graph of $N(L) = \sqrt{\frac{2}{3}} L \left(\frac{\left(1 - \frac{1}{1+L^5}\right) \ln 200}{0{,}0036 - \frac{8 \ln 200}{1+L^5}} \right)^{\frac{1}{2}}$.

a model $\xi_\Lambda(t)$ that approximate the process $\xi(t)$ with reliability $0,99$ and accuracy $0,06$ in the space $L_2([0,1])$.

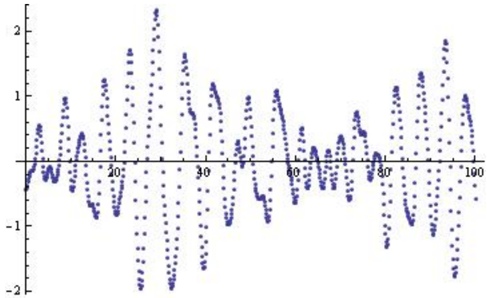

Figure 2.4. Implementation of Gaussian stochastic process, $F(\lambda) = 1 - \frac{1}{1+\lambda^5}$.

2.3. Accuracy and reliability of a model of Gaussian stochastic process in $C(\mathbb{T})$

Let $\mathbb{T} = [0, T]$ be a parametric set. Let $\xi = \{\xi(t), t \in \mathbb{T}\}$ be a zero-mean real-valued Gaussian stochastic process. Let the image of the process represented by (2.1), and the model $\xi_\Lambda(t)$ defined in (2.2).

We assume that $\eta_\Lambda(t) = \xi(t) - \xi_\Lambda(t)$ represented similarly as in (2.3).

For any $t, s \in \mathbb{T}$ we consider the following difference

$$\eta_\Lambda(t) - \eta_\Lambda(s) = \sum_{k=0}^{N-1} \int_{\lambda_k}^{\lambda_{k+1}} (g(t,\lambda) - g(t,\zeta_k) - g(s,\lambda) + g(s,\zeta_k))\, d\eta(\lambda). \tag{2.6}$$

Assume that for $g(t,\lambda)$ the following conditions hold

$$| g(t,\lambda) - g(t,u) | \leqslant S(| \lambda - u |) \cdot Z(t), \tag{2.7}$$

$$| g(t,\lambda) - g(t,u) - g(s,\lambda) + g(s,u) | \leqslant S_1(| \lambda - u |) \cdot | Z_1(t) - Z_1(s) |, \tag{2.8}$$

where $Z(t)$ and $Z_1(t)$ are some continuous functions, $S(\lambda), \lambda \in \mathbb{R}$ is a monotonically increasing function, such that $S(\lambda) \to 0$ when $\lambda \to 0$.

Example 2.5. Let $g(t,\lambda)$ be continuous, twice differentiable function on t and on λ and let $C(t) = \sup\limits_{y \in [0,\Lambda]} \left| \frac{\partial g(t,y)}{\partial y} \right| < \infty$.

Consider the following difference

$$|g(t,\lambda) - g(t,u)| = \left| \int\limits_u^\lambda \frac{\partial g(t,y)}{\partial y} dy \right| \leqslant \int\limits_u^\lambda \left| \frac{\partial g(t,y)}{\partial y} \right| dy \leqslant |\lambda - u| \cdot C(t).$$

Therefore, if we choose $Z(t) = C(t)$, then the function $g(t,\lambda)$ will satisfy the condition (2.7).

We will show that if there is such $C_1(T,L)$, which depends only from T and L, and such that

$$\left| \frac{\partial^2 g(t,\lambda)}{\partial t \partial \lambda} \right| \leqslant C_1(T,L), \tag{2.9}$$

then the function $g(t,\lambda)$ satisfy the condition (2.8). Consider $\Delta = \{s \leqslant l \leqslant t, u \leqslant y \leqslant \lambda\}$, where $s,t \in [0,T]$, $\lambda, u \in [0,L]$. For definiteness we assume that $t > s$, $u > \lambda$. Then from the properties of multiple integrals and (2.9) the following condition holds

$$|g(t,\lambda) - g(t,u) - g(s,\lambda) + g(s,u)| =$$
$$= \left| \int\limits_\Delta \int \frac{\partial^2 g(l,y)}{\partial l \partial y} dl dy \right| \leqslant C_1(T,L)|t-s||u-\lambda|.$$

Example 2.6. If $g(t,\lambda) = e^{-(t+v)^2}$, $0 < t \leqslant T$, $0 < v \leqslant L$, then one can easy to show (similarly as in the Example 2.5), that in this case the condition (2.7) holds.

We will prove the validity of condition (2.8)

$$|g(t,\lambda)-g(t,u)-g(s,\lambda)+g(s,u)| = \left|\int\limits_u^\lambda \frac{\partial g(t,v)}{\partial v}dv - \int\limits_u^\lambda \frac{\partial g(s,v)}{\partial v}dv\right| \leqslant$$
$$\leqslant \int\limits_u^\lambda \left|\frac{\partial g(t,v)}{\partial v} - \frac{\partial g(s,v)}{\partial v}\right| dv.$$

We will estimate the integrand expression

$$\left|\frac{\partial g(t,v)}{\partial v} - \frac{\partial g(s,v)}{\partial v}\right| = 2\left|e^{-(t+v)^2}(t+v) - e^{-(s+v)^2}(s+v)\right| =$$
$$= 2\left|\left(e^{-(t+v)^2}(t+v) - e^{-(s+v)^2}(t+v)\right) +\right.$$
$$\left. + \left(e^{-(s+v)^2}(t+v) - e^{-(s+v)^2}(s+v)\right)\right| =$$
$$= 2\left|(t+v)\left(e^{-(t+v)^2} - e^{-(s+v)^2}\right) + e^{-(s+v)^2}(t-s)\right| \leqslant$$
$$\leqslant 2\left((t+v)e^{-(t+v)^2}\left|1 - e^{-\left((s+v)^2-(t+v)^2\right)}\right| + e^{-(s+v)^2}|t-s|\right) \leqslant$$
$$\leqslant 2\left((t+v)e^{-(t+v)^2}\left|(s+v)^2-(t+v)^2\right| + e^{-(s+v)^2}|t-s|\right) \leqslant$$
$$\leqslant 2\left((t+v)|t-s|(s+t+2v) + |t-s|\right) \leqslant 2|t-s|\left(2(T+L)^2+1\right).$$

Hence,

$$|\, g(t,\lambda)-g(t,u)-g(s,\lambda)+g(s,u)\, | \leqslant C_1(T,L)|\lambda-u|\cdot|t-s|,$$

where $C_1(T,L) = 2\left(2(T+L)^2+1\right)$.

In the following Lemma, we will be find estimates of moments, which will be used later to assess the accuracy of the simulation.

Lemma 2.2. *If for the function $g(t,\lambda)$ the condition (2.8) holds, then for $m = 0,1,\ldots$ the following relationships hold*

$$\mathbb{E}\left(\int\limits_{\lambda_k}^{\lambda_{k+1}} \left(g(t,\lambda)-g(t,\zeta_k)-g(s,\lambda)+g(s,\zeta_k)\right) d\eta(\lambda)\right)^{2m+1} = 0,$$

$$\mathbb{E}\left(\int\limits_{\lambda_k}^{\lambda_{k+1}} (g(t,\lambda) - g(t,\zeta_k) - g(s,\lambda) + g(s,\zeta_k))\, d\eta(\lambda)\right)^{2m} \leqslant$$

$$\leqslant \frac{(2m)!}{2^m \cdot m!}\mathbb{E}\left(\int\limits_{\lambda_k}^{\lambda_{k+1}} S_1^2(\mid \lambda - \zeta_k \mid) \mid Z_1(t) - Z_1(s) \mid^2 dF(\lambda)\right)^m .$$

Proof. Since for a zero-mean Gaussian random variable ξ $\mathbb{E}\xi = 0, \mathbb{E}\xi^{2k+1} = 0, \mathbb{E}\xi^{2k} = \frac{(2k)!}{2^k \cdot k!}\sigma^{2k}$ and the random variables ζ_k are independent of $\eta(\lambda)$, then by the Fubini's theorem and taking into account the condition (2.8) we obtoine ($\mathbb{E}_{\zeta_k}$ is a conditional expectation with respect to ζ_k):

$$\mathbb{E}\left(\int\limits_{\lambda_k}^{\lambda_{k+1}} (g(t,\lambda) - g(t,\zeta_k) - g(s,\lambda) + g(s,\zeta_k))\, d\eta(\lambda)\right)^{2m} \leqslant$$

$$\leqslant \frac{(2m)!}{2^m \cdot m!}\mathbb{E}\left(\int\limits_{\lambda_k}^{\lambda_{k+1}} (g(t,\lambda) - g(t,\zeta_k) - g(s,\lambda) + g(s,\zeta_k))^2\, dF(\lambda)\right)^m \leqslant$$

$$\leqslant \frac{(2m)!}{2^m \cdot m!}\mathbb{E}\left(\int\limits_{\lambda_k}^{\lambda_{k+1}} S_1^2(\mid \lambda - \zeta_k \mid) \mid Z_1(t) - Z_1(s) \mid^2 dF(\lambda)\right)^m . \quad \diamondsuit$$

We obtain the estimates for supremum of norm of the stochastic process. These estimates we will be used for research of the conditions of selecting partition L such that the constructed model will be approximated the Gaussian process with a given accuracy and reliability.

Denote $\sigma_0 = \sup\limits_{0\leqslant t\leqslant T} \tau(\eta_\Lambda(t))$ and $\sigma(h) = \sup\limits_{|t-s|\leqslant h} \tau\left(\eta_\Lambda(t) - \eta_\Lambda(s)\right)$.

Theorem 2.4. *Let $\eta_\Lambda(t)$ be defined as in (2.3) and let*

$$\int\limits_L^\infty S^2(\mid \lambda - L \mid) dF(\lambda) < \infty.$$

Then stochastic process $\eta_\Lambda(t)$ is sub-Gaussian and the following inequality holds

$$\sigma_0 \leqslant \left(\sum_{k=0}^{N-2} S^2(\mid \lambda_{k+1} - \lambda_k \mid)\,(F(\lambda_{k+1}) - F(\lambda_k)) +\right.$$

$$+\int\limits_L^{\infty} S^2(|\lambda - L|)dF(\lambda)\Bigg)^{\frac{1}{2}} \cdot \sup_{0\leqslant t\leqslant T}|Z(t)|.$$

Proof. From Lemma 2.1 it follows that the conditions of Lemma 1.3 hold for $\int\limits_{\lambda_k}^{\lambda_{k+1}}(g(t,\lambda)-g(t,\zeta_k))d\eta(\lambda)$, that's why $\int\limits_{\lambda_k}^{\lambda_{k+1}}(g(t,\lambda)-g(t,\zeta_k))d\eta(\lambda) \in Sub(\Omega)$ for all $k = 0, 2, ..., N-1$ and the following inequality takes place

$$\tau\left(\int\limits_{\lambda_k}^{\lambda_{k+1}}(g(t,\lambda)-g(t,\zeta_k))d\eta(\lambda)\right) \leqslant \theta\left(\int\limits_{\lambda_k}^{\lambda_{k+1}}(g(t,\lambda)-g(t,\zeta_k))d\eta(\lambda)\right) =$$

$$= \sup_{m\geqslant 1}\left[\frac{2^m \cdot m!}{(2m)!}\mathbb{E}\left(\int\limits_{\lambda_k}^{\lambda_{k+1}}(g(t,\lambda)-g(t,\zeta_k))d\eta(\lambda)\right)^{2m}\right]^{\frac{1}{2m}}.$$

Using the Lemma 2.1 we obtain the following inequality

$$\mathbb{E}\left(\int\limits_{\lambda_k}^{\lambda_{k+1}}(g(t,\lambda)-g(t,\zeta_k))d\eta(\lambda)\right)^{2m} \leqslant$$

$$\leqslant \frac{(2m)!}{2^m \cdot m!}|Z(t)|^{2m}\mathbb{E}\left(\int\limits_{\lambda_k}^{\lambda_{k+1}} S^2(|\lambda-\zeta_k|)dF(\lambda)\right)^m \leqslant$$

$$\leqslant \frac{(2m)!}{2^m \cdot m!}b_k^{2m}|Z(t)|^{2m}\int\limits_{\lambda_k}^{\lambda_{k+1}}\left(\int\limits_{\lambda_k}^{\lambda_{k+1}} S^2(|\lambda-u|)dF_k(\lambda)\right)^m dF_k(u) \leqslant$$

$$\leqslant \frac{(2m)!}{2^m \cdot m!}|Z(t)|^{2m}S^{2m}(|\lambda_{k+1}-\lambda_k|)\left(F(\lambda_{k+1})-F(\lambda_k)\right)^m.$$

Then

$$\tau\left(\int\limits_{\lambda_k}^{\lambda_{k+1}}(g(t,\lambda)-g(t,\zeta_k))d\eta(\lambda)\right) \leqslant$$

$$\leqslant \sup_{m\geqslant 1}\left[|Z(t)|^{2m}S^{2m}(|\lambda_{k+1}-\lambda_k|)\left(F(\lambda_{k+1})-F(\lambda_k)\right)^m\right]^{\frac{1}{2m}} =$$

$$= |Z(t)|S(|\lambda_{k+1}-\lambda_k|)\left(F(\lambda_{k+1})-F(\lambda_k)\right)^{\frac{1}{2}}.$$

Since $\int\limits_{\lambda_k}^{\lambda_{k+1}} (g(t,\lambda) - g(t,\zeta_k))d\eta(\lambda),\ k = 1,2,...,N-1$ are independent, then by Lemma 1.2 we obtained

$$\tau^2\left(\sum_{k=0}^{N-1}\int\limits_{\lambda_k}^{\lambda_{k+1}} (g(t,\lambda) - g(t,\zeta_k))d\eta(\lambda)\right) \leqslant \sum_{k=0}^{N-1}\tau^2\left(\int\limits_{\lambda_k}^{\lambda_{k+1}} (g(t,\lambda) - g(t,\zeta_k))d\eta(\lambda)\right).$$

Hence,

$$\tau^2(\eta_\Lambda(t)) \leqslant \sum_{k=0}^{N-2}|Z(t)|^2 S^2(|\ \lambda_{k+1}-\lambda_k\ |)\,(F(\lambda_{k+1})-F(\lambda_k)) + \\ + \int\limits_L^\infty |Z(t)|^2 S^2(|\ \lambda - L\ |)dF(\lambda).$$

Namely

$$\tau(\eta_\Lambda(t)) \leqslant |Z(t)|\left(\sum_{k=0}^{N-2} S^2(|\ \lambda_{k+1}-\lambda_k\ |)\,(F(\lambda_{k+1})-F(\lambda_k)) + \right. \\ \left. + \int\limits_L^\infty S^2(|\ \lambda - L\ |)dF(\lambda)\right)^{\frac{1}{2}}.$$

Therefore

$$\sigma_0 \leqslant \left(\sum_{k=0}^{N-2} S^2(|\ \lambda_{k+1}-\lambda_k\ |)\,(F(\lambda_{k+1})-F(\lambda_k)) + \right. \\ \left. + \int\limits_L^\infty S^2(|\ \lambda - L\ |)dF(\lambda)\right)^{\frac{1}{2}} \sup_{0\leqslant t\leqslant T}|Z(t)|. \quad \diamondsuit$$

Corollary 2.2. *Let the conditions of Theorem 2.4 hold and let the split* $\Lambda = \{\lambda_0, ..., \lambda_N\}$ *of the set* $[0,\infty)$ *be such that* $\lambda_0 = 0$, $\lambda_k < \lambda_{k+1}$, $\lambda_{N-1} = L$, $\lambda_N = \infty$, $\lambda_{k+1} - \lambda_k = \frac{L}{N-1}$, *then*

$$\sigma_0 \leqslant \left(S^2\left(\frac{L}{N-1}\right)F(L) + \int\limits_L^\infty S^2(|\ \lambda - L\ |)dF(\lambda)\right)^{\frac{1}{2}} \cdot \sup_{0\leqslant t\leqslant T}|Z(t)|.$$

Theorem 2.5. *Let $\eta_\Lambda(t)$ be a stochastic process defined as in (2.3) and let*

$$\int\limits_L^\infty S_1^2(\mid \lambda - L \mid)dF(\lambda) < \infty.$$

Then the following inequality holds

$$\sigma(h) \leqslant \left(\sum_{k=0}^{N-2} S_1^2(\mid \lambda_{k+1} - \lambda_k \mid)(F(\lambda_{k+1}) - F(\lambda_k)) + \right.$$

$$\left. + \int\limits_L^\infty S_1^2(\mid \lambda - L \mid)dF(\lambda)\right)^{\frac{1}{2}} \cdot \sup_{|t-s|\leqslant h} \mid Z_1(t) - Z_1(s) \mid .$$

Proof. From Lemma 2.2 it follows that the conditions of Lemma 1.3 hold for $\int\limits_{\lambda_k}^{\lambda_{k+1}} (g(t,\lambda) - g(t,\zeta_k) - g(s,\lambda) + g(s,\zeta_k))d\eta(\lambda)$, that's why

$$\int\limits_{\lambda_k}^{\lambda_{k+1}} (g(t,\lambda) - g(t,\zeta_k) - g(s,\lambda) + g(s,\zeta_k))d\eta(\lambda) \in Sub(\Omega).$$

Since $\int\limits_{\lambda_k}^{\lambda_{k+1}} (g(t,\lambda) - g(t,\zeta_k) - g(s,\lambda) + g(s,\zeta_k))d\eta(\lambda)$, $k = 0, 2, ..., N-1$ are independent, then by Lemma 1.2 and condition (2.6) we obtain

$$\tau^2(\eta_\Lambda(t) - \eta_\Lambda(s)) \leqslant$$

$$\leqslant \sum_{k=0}^{N-1} \tau^2 \left(\int\limits_{\lambda_k}^{\lambda_{k+1}} (g(t,\lambda) - g(t,\zeta_k) - g(s,\lambda) + g(s,\zeta_k))d\eta(\lambda) \right) \leqslant$$

$$\leqslant \sum_{k=0}^{N-1} \theta^2 \left(\int\limits_{\lambda_k}^{\lambda_{k+1}} (g(t,\lambda) - g(t,\zeta_k) - g(s,\lambda) + g(s,\zeta_k))d\eta(\lambda) \right).$$

Hence, from Lemma 2.2

$$\mathbb{E}\left(\int\limits_{\lambda_k}^{\lambda_{k+1}} (g(t,\lambda) - g(t,\zeta_k) - g(s,\lambda) + g(s,\zeta_k))d\eta(\lambda) \right)^{2m} \leqslant$$

$$\leqslant \frac{(2m)!}{2^m \cdot m!}\mathbb{E}\left(\int\limits_{\lambda_k}^{\lambda_{k+1}} S_1^2(|\lambda-\zeta_k|)|Z_1(t)-Z_1(s)|^2 dF(\lambda)\right)^m \leqslant$$

$$\leqslant \frac{(2m)!}{2^m \cdot m!} b_k^{2m}(Z_1(t)-Z_1(s))^{2m} \int\limits_{\lambda_k}^{\lambda_{k+1}} \left(\int\limits_{\lambda_k}^{\lambda_{k+1}} S_1^2(|\lambda-u|)dF_k(\lambda)\right)^m dF_k(u) \leqslant$$

$$\leqslant \frac{(2m)!}{2^m \cdot m!}|Z_1(t)-Z_1(s)|^{2m} \cdot S_1^{2m}(|\lambda_{k+1}-\lambda_k|)(F(\lambda_{k+1})-F(\lambda_k))^m.$$

Then

$$\tau^2\left(\int\limits_{\lambda_k}^{\lambda_{k+1}} (g(t,\lambda)-g(t,\zeta_k)-g(s,\lambda)+g(s,\zeta_k))d\eta(\lambda)\right) \leqslant$$

$$\leqslant \sup_{m\geqslant 1}\left[|Z_1(t)-Z_1(s)|^{2m} S_1^{2m}(|\lambda_{k+1}-\lambda_k|)(F(\lambda_{k+1})-F(\lambda_k))^m\right]^{\frac{1}{m}} =$$

$$= |Z_1(t)-Z_1(s)|^2 S_1^2(|\lambda_{k+1}-\lambda_k|)(F(\lambda_{k+1})-F(\lambda_k)).$$

Hence,

$$\tau^2\left(\eta_\Lambda(t)-\eta_\Lambda(s)\right) \leqslant$$

$$\leqslant |Z_1(t)-Z_1(s)|^2\left(\sum_{k=0}^{N-2} S_1^2(|\lambda_{k+1}-\lambda_k|)(F(\lambda_{k+1})-F(\lambda_k)) + \int\limits_L^\infty S_1^2(|\lambda-L|)dF(\lambda)\right).$$

Namely,

$$\sigma(h) \leqslant \left(\sum_{k=0}^{N-2} S_1^2(|\lambda_{k+1}-\lambda_k|)(F(\lambda_{k+1})-F(\lambda_k)) + \int\limits_L^\infty S_1^2(|\lambda-L|)dF(\lambda)\right)^{\frac{1}{2}} \sup_{|t-s|\leqslant h}|Z_1(t)-Z_1(s)|. \quad \diamondsuit$$

Corollary 2.3. *Let the conditions of Theorem 2.5 hold and let the split* $\Lambda = \{\lambda_0, ..., \lambda_N\}$ *of the set* $[0,\infty)$ *be such that* $\lambda_0 = 0, \lambda_k < \lambda_{k+1}, \lambda_{N-1} = L, \lambda_N = \infty$, $\lambda_{k+1}-\lambda_k = \frac{L}{N-1}$, *then*

$$\sigma(h) \leqslant \left(S_1^2\left(\frac{L}{N-1}\right) F(L) + \int\limits_L^\infty S_1^2(|\lambda - L|) dF(\lambda) \right)^{\frac{1}{2}} \times$$
$$\times \sup_{|t-s|\leqslant h} |Z_1(t) - Z_1(s)|.$$

It is obvious that for any ε one can find such L and N that

$$\left(S_1^2\left(\frac{L}{N-1}\right) F(L) + \int\limits_L^\infty S_1^2(|\ \lambda - L\ |) dF(\lambda) \right)^{\frac{1}{2}} \leqslant \varepsilon.$$

Further, we investigate conditions and estimations convergence of the model by probability in the space $C(\mathbb{T})$. These conditions make it possible to construct a model which approximates stochastic Gaussian process with a given accuracy and reliability.

Definition 2.2. [72] Stochastic process $\xi_\Lambda(t)$ approaches Gaussian process $\xi(t)$ with given reliability $1-\beta, 0<\beta<1$ and accuracy $\delta>0$ in $C(\mathbb{T})$, if there exists split L, such that inequality holds

$$P\left\{ \sup_{t\in\mathbf{T}} |\xi(t) - \xi_\Lambda(t)| > \delta \right\} \leqslant \beta.$$

From the Theorem 2.4 it follows that $\eta_\Lambda(t) \in Sub(\Omega), t \in \mathbb{T}$ then pseudometric ρ generated by the process $\eta_\Lambda(t)$ on $\mathbb{T}$ is as follows

$$\rho(t,s) = \tau(\eta_\Lambda(t) - \eta_\Lambda(s)), t, s \in \mathbb{T}$$

Definition 2.3. [19] A set $Q \subset \mathbf{T}$ is called an ε-net in the set $\mathbb{T}$ with respect to the pseudometric ρ if for any point $x \in \mathbb{T}$ there exists at least one point $y \in Q$ such that $\rho(x,y) \leqslant \varepsilon$.

Definition 2.4. [19] If there exist a finite ε-covering of a set $\mathbf{T}$, then $N_\rho(\mathbf{T},\varepsilon)$ denotes the smallest number of elements an ε-covering of this set. We put $N_\rho(\mathbf{T},\varepsilon) = +\infty$ if there exists no finite ε-covering of the set $\mathbf{T}$. The function $N_\rho(\mathbf{T},\varepsilon)$, $\varepsilon>0$, called the metric massiveness of the set $\mathbf{T}$ with respect to the pseudometric ρ.

Note that $N_\rho(\mathbb{T},\varepsilon)$ coincides with the number of points in a minimum of ε-covering of the set $\mathbb{T}$.

Definition 2.5. [19] Suppose that

$$H_\rho(\mathbb{T},\varepsilon) = \begin{cases} \ln N_\rho(\mathbb{T},\varepsilon), & \text{if } N_\rho(\mathbb{T},\varepsilon) < +\infty, \\ +\infty, & \text{if } N_\rho(\mathbb{T},\varepsilon) = \infty. \end{cases}$$

The function $H_\rho(\mathbb{T},\varepsilon)$, $\varepsilon > 0$, is called the metric entropy of the set $\mathbb{T}$ with respect to the pseudometric ρ.

For simplicity, denote

$$N(\varepsilon) = N_\rho(\mathbb{T},\varepsilon),\ H(\varepsilon) = H_\rho(\mathbb{T},\varepsilon),$$

where $N_\rho(\mathbf{T},\varepsilon)$ and $H_\rho(\mathbf{T},\varepsilon)$ are metric massiveness and metric entropy of the set $\mathbf{T}$ with respect to the pseudometric ρ. Further, let $r = (r(\upsilon), \upsilon \geqslant 1)$ be a nonnegative monotone nondecreasing function such that $r(\exp\{\upsilon\}), \upsilon \geqslant 1$ is convex and $r(\upsilon) \to \infty$ as $\upsilon \to \infty$.

Consider the integral

$$\widehat{I}_r(u) = \int_0^u r(N(\varepsilon))d\varepsilon,\ u > 0,$$

called the entropy integral.

Theorem 2.6. *Suppose that $X = (X(t), t \in \mathbb{T})$ is sub-Gaussian stochastic process. Let $\varepsilon_0 = \sup_{t\in\mathbb{T}} \tau(X(t)) < \infty$, $(\mathbb{T},\rho)$ is separable space and the process X is separable process on $(\mathbb{T},\rho)$ and let $\widehat{I}_r(\theta\varepsilon_0) < \infty$, then for all $\lambda > 0$ holds*

$$\mathbb{E}\exp\left\{\lambda \sup_{t\in\mathbb{T}} |X(t)|\right\} \leqslant 2\hat{Q}(\lambda),$$

where

$$\hat{Q}(\lambda) = \inf_{0<\theta<1} \exp\left\{\frac{(\lambda\varepsilon_0)^2}{2(1-\theta)^2}\right\} \cdot r^{(-1)}\left(\frac{\widehat{I}_r(\theta\varepsilon_0)}{\theta\varepsilon_0}\right).$$

Furthermore, for all $\theta \in (0,1)$ and $u > 0$

$$P\left\{\sup_{t\in\mathbf{T}} |X(t)| \geqslant u\right\} \leqslant 2\hat{A}(u,\theta),$$

where

$$\hat{A}(u,\theta) = \exp\left\{-\frac{(u(1-\theta))^2}{2\varepsilon_0^2}\right\} \cdot r^{(-1)}\left(\frac{\widehat{I}_r(\theta\varepsilon_0)}{\theta\varepsilon_0}\right).$$

The Theorem 2.6 is a particular case of the Theorem 4.4 from the book [19].

Denote $C(h) = \sup\limits_{|t-s|\leqslant h} |Z_1(t) - Z_1(s)|$. Let $C(h)$ be such function to which there exists an inverse function.

Theorem 2.7. *Let in model $\xi_\Lambda(t)$ split Λ be such that when $\delta > 0, \theta \in (0,1)$ the following relationship takes place*

$$2\exp\left\{-\frac{(\delta(1-\theta))^2}{2\varepsilon_0^2}\right\} r^{(-1)}\left(\frac{\tilde{I}_r(\theta\varepsilon_0)}{\theta\varepsilon_0}\right) \leqslant \beta,$$

where $\varepsilon_0 = \sup\limits_{t\in T} \tau(\eta_\Lambda(t)) = \sigma_0, \eta_\Lambda(t) = \xi(t) - \xi_\Lambda(t)$ and let

$$\widehat{I}_r(\theta\varepsilon_0) \leqslant \int\limits_0^{\theta\varepsilon_0} r\left(\frac{T}{2C^{(-1)}(V)} + 1\right) d\varepsilon = \tilde{I}_r(\theta\varepsilon_0),$$

$$\tilde{I}_r(\theta\varepsilon_0) < \infty,$$

where $V = \varepsilon\left(S_1^2\left(\frac{L}{N-1}\right)F(L) + \int\limits_L^\infty S_1^2(|\lambda - L|)dF(\lambda)\right)^{-\frac{1}{2}}$, $C^{(-1)}(x), x > 0$ is the function inverse to $C(\cdot)$. Then the model $\xi_\Lambda(t)$ approximates $\xi(t)$ with a given reliability $1-\beta, 0<\beta<1$ and accuracy $\delta > 0$ in the space $C(\mathbb{T})$.

Proof. Since for the metric massiveness it is true that $N(\varepsilon) \leqslant \frac{T}{2\sigma^{(-1)}(\varepsilon)} + 1$, so using this fact and Theorem 2.6 we obtain that for the sub-Gaussian process $\eta_\Lambda(t)$ the following inequality holds

$$\mathbf{P}\left\{\sup_{t\in T} |\eta_\Lambda(t)| > \delta\right\} \leqslant 2\exp\left\{-\frac{(\delta(1-\theta))^2}{2\varepsilon_0^2}\right\} \cdot r^{(-1)}\left(\frac{\widehat{I}_r(\theta\varepsilon_0)}{\theta\varepsilon_0}\right),$$

where

$$\widehat{I}_r(\theta\varepsilon_0) = \int\limits_0^{\theta\varepsilon_0} r(N(\varepsilon))d\varepsilon \leqslant \int\limits_0^{\theta\varepsilon_0} r\left(\frac{T}{2\sigma^{(-1)}(\varepsilon)} + 1\right) d\varepsilon < \infty$$

From Corollary 2.3 it follows that

$$\sigma(h) = \sup_{|t-s|\leqslant h} \tau(\eta_\Lambda(t) - \eta_\Lambda(s)) \leqslant$$

$$\leqslant C(h) \cdot \left(S_1^2 \left(\frac{L}{N-1} \right) F(L) + \int_L^\infty S_1^2(\mid \lambda - L \mid) dF(\lambda) \right)^{\frac{1}{2}}.$$

One can readily show that the function inverse to estimation of $\sigma(h)$ is

$$\sigma^{(-1)}(h) = C^{(-1)} \left(h \left(S_1^2 \left(\frac{L}{N-1} \right) F(\Lambda) + \int_L^\infty S_1^2(|\lambda - L|) dF(\lambda) \right)^{-\frac{1}{2}} \right).$$

Then

$$\widehat{I}_r(\theta\varepsilon_0) \leqslant \int_0^{\theta\varepsilon_0} r\left(\frac{T}{2C^{(-1)}(V)} + 1 \right) d\varepsilon = \tilde{I}_r(\theta\varepsilon_0),$$

where $V = \varepsilon \left(S_1^2 \left(\frac{L}{N-1} \right) F(L) + \int_L^\infty S_1^2(\mid \lambda - L \mid) dF(\lambda) \right)^{-\frac{1}{2}}$. Estimate for $\widehat{I}_r(\theta\varepsilon_0)$ one can make arbitrarily small by appropriate selection of L and N.

Namely there exists such split Λ and $\beta \in (0,1)$ that the following condition holds

$$2\exp\left\{ -\frac{(\delta(1-\theta))^2}{2\varepsilon_0^2} \right\} r^{(-1)} \left(\frac{\tilde{I}_r(\theta\varepsilon_0)}{\theta\varepsilon_0} \right) \leqslant \beta.$$

Then by Definition 2.2 we get that the model $\xi_\Lambda(t)$ approximates $\xi(t)$ with a given reliability $1-\beta, 0 < \beta < 1$ and accuracy $\delta > 0$ in the space $C(\mathbb{T})$. ♢

Example 2.7. Assume that $r(u) = u^\alpha, \alpha > 0$ and let $\eta_\Lambda(t) = (\eta_\Lambda(t), t \in \mathbb{T})$ be a sub-Gaussian process satisfying $\int_0^{\varepsilon_0} N^\alpha(\varepsilon) d\varepsilon < \infty$, where $N(\varepsilon)$ is metric massiveness. Then $r^{(-1)}(x) = x^{\frac{1}{\alpha}}, x > 0$.

Hence,

$$\mathbf{P}\left\{ \sup_{t\in T} \mid \eta_\Lambda(t) \mid \geqslant u \right\} \leqslant 2\exp\left\{ \frac{u^2(1-\theta)^2}{2\varepsilon_0^2} \right\} \left(\frac{1}{\theta\varepsilon_0} \int_0^{\theta\varepsilon_0} N^\alpha(\varepsilon) d\varepsilon \right)^{\frac{1}{\alpha}}.$$

for all $\theta \in (0,1)$ and $u > 0$.

Fix $\varepsilon_0 < \frac{u}{\sqrt{2}}$ and put $\theta = 1 - \sqrt{1 - \frac{2\varepsilon_0^2}{u^2}}$.

Now one can easily calculate that

$$\exp\left\{\frac{u^2(1-\theta)^2}{2\varepsilon_0^2}\right\}\left(\frac{1}{\theta\varepsilon_0}\int\limits_0^{\theta\varepsilon_0} N^\alpha(\varepsilon)d\varepsilon\right)^{\frac{1}{\alpha}} \leqslant$$

$$\leqslant e\left(\int\limits_0^{\varepsilon_0} N^\alpha(\varepsilon)d\varepsilon\right)^{\frac{1}{\alpha}}\left(\frac{u}{\varepsilon_0}\right)^{\frac{2}{\alpha}}\exp\left\{-\frac{1}{2}\left(\frac{u}{\varepsilon_0}\right)^2\right\}.$$

Hence when $x > \sqrt{2}$ we have

$$\mathbf{P}\left\{\sup_{t\in T} \mid \eta_\Lambda(t) \mid \geqslant x\varepsilon_0\right\} \leqslant 2e\left(\int\limits_0^{\varepsilon_0} N^\alpha(\varepsilon)d\varepsilon\right)^{\frac{1}{\alpha}} x^{\frac{2}{\alpha}}\exp\left\{-\frac{x^2}{2}\right\}.$$

Using the estimates for $\sigma^{(-1)}(h)$ and $N(\varepsilon)$ we obtain

$$\mathbf{P}\left\{\sup_{t\in T} \mid \eta_\Lambda(t) \mid \geqslant x\varepsilon_0\right\} \leqslant 2e\left(\int\limits_0^{\varepsilon_0}\left(\frac{T}{2C^{(-1)}(V)}+1\right)^\alpha d\varepsilon\right)^{\frac{1}{\alpha}} x^{\frac{2}{\alpha}}\exp\left\{-\frac{x^2}{2}\right\},$$

where $V = \varepsilon\left(S_1^2\left(\frac{L}{N-1}\right)F(L) + \int\limits_L^\infty S_1^2(\mid \lambda - L \mid)dF(\lambda)\right)^{-\frac{1}{2}}$.

Assume that $C(h) = Dh^\rho$, $D \in \mathbb{R}$, and $0 < \rho \leqslant 1$, $\rho > \alpha$ then

$$\sigma(h) \leqslant Dh^\rho \cdot \left(S_1^2\left(\frac{L}{N-1}\right)F(L) + \int\limits_L^\infty S_1^2(\mid \lambda - L \mid)dF(\lambda)\right)^{\frac{1}{2}}.$$

Therefore,

$$\sigma^{(-1)}(h) = \left(h\left(D\left(S_1^2\left(\frac{L}{N-1}\right)F(L) + \int\limits_L^\infty S_1^2(\mid \lambda - L \mid)dF(\lambda)\right)^{\frac{1}{2}}\right)^{-1}\right)^{\frac{1}{\rho}}. \quad (2.10)$$

When $\sigma^{(-1)}(h)$ is defined as in (2.10), then

$$\mathbf{P}\left\{\sup_{t\in\mathbb{T}} \mid \eta_\Lambda(t) \mid \geqslant x\varepsilon_0\right\} \leqslant$$

$$\leqslant 2ex^{\frac{2}{\alpha}} \exp\left\{-\frac{x^2}{2}\right\}\left(\int_0^{\varepsilon_0}\left(\frac{T}{2}\varepsilon^{-\frac{1}{\rho}}\left(D\left(S_1^2\left(\frac{L}{N-1}\right)F(L)+\right.\right.\right.\right.$$

$$\left.\left.\left.\left.+\int_L^\infty S_1^2(\mid \lambda - L\mid)dF(\lambda)\right)^{\frac{1}{2}}\right)^{\frac{1}{\rho}}+1\right)^{\alpha}d\varepsilon\right)^{\frac{1}{\alpha}} \leqslant$$

$$\leqslant 2ex^{\frac{2}{\alpha}} \exp\left\{-\frac{x^2}{2}\right\}\left(\int_0^{\varepsilon_0}\left(\left(\left(\frac{T}{2}\right)^{\alpha}\left(D\left(S_1^2\left(\frac{L}{N-1}\right)F(L)+\right.\right.\right.\right.\right.$$

$$\left.\left.\left.\left.\left.+\int_L^\infty S_1^2(\mid \lambda - L\mid)dF(\lambda)\right)^{\frac{1}{2}}\right)^{\frac{\alpha}{\rho}}\varepsilon^{-\frac{\alpha}{\rho}}+1\right)d\varepsilon\right)^{\frac{1}{\alpha}} =$$

$$= 2ex^{\frac{2}{\alpha}} \exp\left\{-\frac{x^2}{2}\right\}\left(\frac{T^\alpha}{2^\alpha\left(1-\frac{\alpha}{\rho}\right)}\left(D\left(S_1^2\left(\frac{L}{N-1}\right)F(L)+\right.\right.\right.$$

$$\left.\left.\left.+\int_L^\infty S_1^2(\mid \lambda - u\mid)dF(\lambda)dF(u)\right)^{\frac{1}{2}}\right)^{\frac{\alpha}{\rho}}\cdot\varepsilon_0^{1-\frac{\alpha}{\rho}}+\varepsilon_0\right)^{\frac{1}{\alpha}}.$$

Hence,

$$\mathbf{P}\left\{\sup_{t\in T} \mid \eta_\Lambda(t) \mid > \delta\right\} \leqslant 2\left(\frac{\delta}{\varepsilon_0}\right)^{\frac{2}{\alpha}} \exp\left\{1-\frac{\delta^2}{2\varepsilon_0^2}\right\}\left(\frac{T^\alpha}{2^\alpha\left(1-\frac{\alpha}{\rho}\right)}\cdot\varepsilon_0^{1-\frac{\alpha}{\rho}}\times\right.$$

$$\left.\times\left(D\left(S_1^2\left(\frac{L}{N-1}\right)F(L)+\int_L^\infty S_1^2(\mid \lambda - L\mid)dF(\lambda)\right)^{\frac{1}{2}}\right)^{\frac{\alpha}{\rho}}+\varepsilon_0\right)^{\frac{1}{\alpha}}.$$

Put $S_1(u) = C\cdot u^\nu$ and

$$F(\lambda) = \begin{cases} 1-\frac{1}{\lambda^\gamma}, & \lambda > 2,\\ 0, & \lambda < 2,\end{cases}$$

where $\nu > 0$, $\gamma > 2\nu$, then

$$\mathbf{P}\left\{\sup_{t\in T} \mid \eta_\Lambda(t) \mid > \delta\right\} \leqslant 2\left(\frac{\delta}{\varepsilon_0}\right)^{\frac{2}{\alpha}} \exp\left\{1-\frac{\delta^2}{2\varepsilon_0^2}\right\} \times$$
$$\times \left(\frac{T^\alpha}{2^\alpha\left(1-\frac{\alpha}{\rho}\right)}\left(D\left(C^2\left(\frac{L}{N-1}\right)^{2\nu}+2C^2 c_\nu \frac{L^{2\nu-\gamma}(\gamma-\nu)}{\gamma-2\nu}\right)^{\frac{1}{2}}\right)^{\frac{\alpha}{\rho}} \times\right.$$
$$\left.\times \varepsilon_0^{1-\frac{\alpha}{\rho}}+\varepsilon_0\right)^{\frac{1}{\alpha}},$$

where $c_\nu = \begin{cases} 1, & \text{as } 0 < 2\nu \leqslant 1, \\ 2^{2\nu-1}, & \text{as } 2\nu > 1 \end{cases}$.

The $\left(D\left(C^2\left(\frac{L}{N-1}\right)^{2\nu}+2C^2 c_\nu \frac{L^{2\nu-\gamma}(\gamma-\nu)}{\gamma-2\nu}\right)^{\frac{1}{2}}\right)^{\frac{\alpha}{\rho}}$ is minimized at

$$\hat{L} = \left(\frac{(N-1)^{2\nu} c_\nu \gamma(\nu-\gamma)}{\nu(2\nu-\gamma)}\right)^{\frac{1}{2\gamma}}.$$

Then from the Theorem 2.7 we obtain that inequality

$$2\exp\left\{-\frac{(\delta(1-\theta))^2}{2\varepsilon_0^2}\right\} r^{(-1)}\left(\frac{\widehat{I_r}(\theta\varepsilon_0)}{\theta\varepsilon_0}\right) \leqslant \beta,$$

holds when N satisfied condition

$$N \geqslant \left(\frac{\beta^\alpha}{2^\alpha\left(\frac{\delta}{\varepsilon_0}\right)^2 \exp\left\{\alpha\left(1-\frac{\delta^2}{2\varepsilon_0^2}\right)\right\}}-\varepsilon_0\right)^{\frac{\rho\gamma}{\alpha\nu(\nu-\gamma)}} \cdot \left(\frac{2\left(1-\frac{\alpha}{\rho}\right)^{\frac{1}{\alpha}}}{T}\right)^{\frac{\rho\gamma}{\nu(\nu-\gamma)}} \times$$
$$\times\left(DC\varepsilon_0^{\frac{\rho}{\alpha}-1}\left(1+\frac{\nu}{\gamma-\nu}\right)^{\frac{1}{2}}\right)^{\frac{\gamma}{\nu(\gamma-\nu)}} \cdot \left(\frac{\nu(\gamma-2\nu)}{c_\nu\gamma(\gamma-\nu)}\right)^{\frac{1}{2(\nu-\gamma)}}+1,$$

where $0 < \alpha \leqslant 1$, $0 < \rho \leqslant 1$, $\rho > \alpha$, $\nu > 0$, $\gamma > 2\nu$, $C \in \mathbb{R}$, $D \in \mathbb{R}$ and $c_\nu = \begin{cases} 1, & \text{as } 0 < 2\nu \leqslant 1, \\ 2^{2\nu-1}, & \text{as } 2\nu > 1 \end{cases}$.

Hence, model $\xi_\Lambda(t)$ approximates process $\xi(t)$ with a given reliability $1-\beta, 0 < \beta < 1$ and accuracy $\delta > 0$ in the space $C([0,T])$, if for N the last relationship holds.

Chapter 3
Construction of models of Gaussian stochastic fields and homogeneous and isotropic stochastic fields with the required accuracy and reliability in different functional spaces.

As we noted in the previous chapter, simulation is an effective means of studying various characteristics of phenomena in the environment. There are many such phenomena that depend not only on some random factor and on time. Therefore, by a stochastic process we cannot describe every phenomenon. For these phenomena can be used a stochastic fields.

One of the most important problems of the theory of stochastic processes and stochastic fields is the problem of modeling and approximating the processes and fields. There are known several methods for constructing models of stochastic processes. The most popular for stationary processes is the method of splitting and randomization of the spectrum developed by G. A. Mikhailov and his collaborators (see [97], [98], [99], [100]). M. I. Yadrenko and his coauthors used different methods (see [139], [140], [141], [142], [143]). When constructing models of stochastic processes and random fields, it is important to know how close are approximating models and correspondi-ng processes and fields in some metrics. A number of papers by Yu. V. Kozachenko and his collaborators is devoted to constructing models of stochastic fields with a given reliability and accuracy (see [55], [64], [67], [71], [85]).

In this chapter, the same method as in the previous chapter is used to construct a model of stochastic field. We used the representation of a homogeneous and isotropic stochastic field proposed by Yadrenko in the book [141]. An important task in the simulation of stochastic fields is to evaluate the probability of deviating the model of stochastic field from this field, for example, in the uniform metric (see [67], [85]) or in the space $L_p(T)$ (see [67], [133], [132]). In this chapter, these results are submitted in the section 3.2 and section 3.6. We used the representation of the stochastic field that contains Bessel functions of the first kind. We did not find the necessary estimates for the Bessel functions of the first kind in the literature (although they may have already been obtained somewhere); therefore, these estimates were obtained by us and are presented in the section 3.3.

Both for the processes and for the fields we investigate the accuracy and reliability of the constructed models, in particular in the spaces $C(T)$ and $L_p(T)$. Kozachenko and his students considered similar tasks in the papers [1], [58] [64], [66], [71], [72]. The results obtained in this Chapter were published in the papers [132], [133] and [134].

3.1. Construction of the model of Gaussian stochastic field

Consider the space $\mathbb{R}^d$ with the metric $\rho(\boldsymbol{t},\boldsymbol{s})$, $\boldsymbol{t}^T=(t_1,\dots,t_d)$, $\boldsymbol{s}^T=(s_1,\dots,s_d)$, where $\rho(\boldsymbol{t},\boldsymbol{s})=\max\limits_{i=\overline{1,d}}|t_i-s_i|$. Let $\mathbb{T}$ be a set in the form $\mathbb{T}=\left\{\boldsymbol{t}:-A\leqslant t_i\leqslant A, i=\overline{1,d}\right\}$, where $A>0$ a number. Let $\xi=\{\xi(\boldsymbol{t}),\boldsymbol{t}\in\mathbb{T}\}$ be a zero-mean Gaussian stochastic field, covariance function of which allows images

$$R(\boldsymbol{t},\boldsymbol{s})=\mathbf{E}g(\boldsymbol{t},\boldsymbol{\lambda})g(\boldsymbol{s},\boldsymbol{\lambda})=\int_{-\infty}^{+\infty}\!\!\cdots\int g(\boldsymbol{t},\boldsymbol{\lambda})g(\boldsymbol{s},\boldsymbol{\lambda})dF(\boldsymbol{\lambda}).$$

where $F(\boldsymbol{\lambda})$ is continuous distribution function. Let $\{\mathbb{R}^d,\mathfrak{A},\mu(\cdot)\}$ be a measurable space, where $\mathfrak{A}$ is Borel σ-algebra, $\mu(\boldsymbol{\lambda})$ is a finite measure generated by the function $F(\boldsymbol{\lambda})$.

According to the Karhunen theorem field $\xi(\boldsymbol{t})$ can be represented as follows

$$\xi(\boldsymbol{t})=\int_{-\infty}^{+\infty}\!\!\cdots\int g(\boldsymbol{t},\boldsymbol{\lambda})d\eta(\boldsymbol{\lambda}),$$

where $\eta(A_1)$ is random measure that subordinated to the measure μ such that $\mathbf{E}(\eta(A_1)\eta(A_2))=\mu(A_1\bigcap A_2)$, $A_1,A_2\in\mathfrak{A}$. Let Λ be some measurable set of the space $\mathbb{R}^d$ and we consider such partition $\Lambda=\{\Delta_1,\dots,\Delta_N\}$ of the space, that $\Delta_i\bigcap\Delta_j=\varnothing, i\neq j$.

As a model we consider

$$\xi_N(\boldsymbol{t})=\sum_{k=1}^{N}\eta_k g(\boldsymbol{t},\boldsymbol{\zeta}_k),$$

where $\eta_k=\int\limits_{\Delta_k}d\eta(\boldsymbol{\lambda})$ are such Gaussian random variables that $\mathbf{E}\eta_k=0$, $\mathbf{E}\eta_k^2=\mu(\Delta_k)=b_k^2$ and $\boldsymbol{\zeta}_k\in\mathbb{R}^d$ are independent random variables being independent of η_k and taking values on Δ_k with cumulative distribution function

$$\mathbf{P}\left\{\boldsymbol{\zeta}_k\in A\right\}=\frac{\mu(A\bigcap\Delta_k)}{\mu(\Delta_k)}:=\mu_k(A).$$

The model $\xi_N(\boldsymbol{t})$ is zero-mean stochastic field:

$$\mathbf{E}\xi_N(\boldsymbol{t}) = \mathbf{E}\sum_{k=1}^{N} \eta_k g(\boldsymbol{t}, \boldsymbol{\zeta}_k) = \sum_{k=1}^{N} \mathbf{E}\eta_k \mathbf{E} g(\boldsymbol{t}, \boldsymbol{\zeta}_k) = 0.$$

The covariance function of the $\xi_N(\boldsymbol{t})$ almost coincides with the covariance function of stochastic feld $\xi(\boldsymbol{t})$. Namely, by the certain choice of the partition Λ the covariance function of $\xi_N(\boldsymbol{t})$ can be made an arbitrarily close to the covariance function of $\xi(\boldsymbol{t})$.

Let the following conditions hold for the function $g(\boldsymbol{t}, \boldsymbol{\lambda})$

$$g(\boldsymbol{t}, \boldsymbol{\lambda}) \leqslant C(\boldsymbol{t}),$$

$$|g(\boldsymbol{t}, \boldsymbol{\lambda}) - g(\boldsymbol{t}, \boldsymbol{u})| \leqslant S\left(|\boldsymbol{\lambda} - \boldsymbol{u}|\right) \cdot Z(\boldsymbol{t}),$$

where $Z(\boldsymbol{t})$ is limited on the compact, $S(x), x \in \mathbb{R}$ is continuous, monotonically nondecreasing function.

Lemma 3.1. *Stochastic field $\xi_N(\boldsymbol{t})$ is sub-Gaussian stochastic field.*

Proof. Consider $\mathbf{E}\left(\eta_k g(\boldsymbol{t}, \boldsymbol{\zeta}_k)\right)$. Since for a zero-mean Gaussian random variable η we have $\mathbf{E}\eta^{2m+1} = 0$, $\mathbf{E}\eta^{2m} = \frac{(2m)!}{2^m \cdot m!}\sigma^{2m}$, then

$$\mathbf{E}\left(\eta_k g(\boldsymbol{t}, \boldsymbol{\zeta}_k)\right)^{2m+1} = 0,$$
$$\mathbf{E}\left(\eta_k g(\boldsymbol{t}, \boldsymbol{\zeta}_k)\right)^{2m} = \mathbf{E}(\eta_k)^{2m} \mathbf{E}(g(\boldsymbol{t}, \boldsymbol{\zeta}_k))^{2m} \leqslant \frac{(2m)!}{2^m \cdot m!}\left(b_k \cdot C(\boldsymbol{t})\right)^{2m} < \infty.$$

In that $\sup\limits_{m \geqslant 1}\left[\frac{(2m)!}{2^m \cdot m!}\mathbf{E}\left(\eta_k g(\boldsymbol{t}, \boldsymbol{\zeta}_k)\right)^{2m}\right]^{\frac{1}{2m}} < \infty$, then it follows from Lemma 1.3 that $\eta_k g(\boldsymbol{t}, \boldsymbol{\zeta}_k)$ are sub-Gaussian stochastic fields. That is why $\xi_N(\boldsymbol{t}) = \sum\limits_{k=1}^{N} \eta_k g(\boldsymbol{t}, \boldsymbol{\zeta}_k)$ is sub-Gaussian stochastic field. ♢

Let

$$\chi_N(\boldsymbol{t}) = \xi(\boldsymbol{t}) - \xi_N(\boldsymbol{t}) = \sum_{k=1}^{N} \int_{\Delta_k} \left(g(\boldsymbol{t}, \boldsymbol{\lambda}) - g(\boldsymbol{t}, \boldsymbol{\zeta}_k)\right) d\eta(\boldsymbol{\lambda}) \tag{3.1}$$

Lemma 3.2. *The following relationships hold true*

$$\mathbf{E}\left(\int_{\Delta_k} \left(g(\boldsymbol{t}, \boldsymbol{\lambda}) - g(\boldsymbol{t}, \boldsymbol{\zeta}_k)\right) d\eta(\boldsymbol{\lambda})\right)^{2m+1} = 0,$$

$$\mathbf{E}\left(\int\limits_{\Delta_k}(g(\boldsymbol{t},\boldsymbol{\lambda})-g(\boldsymbol{t},\boldsymbol{\zeta}_k))\,d\eta(\boldsymbol{\lambda})\right)^{2m}\leqslant$$

$$\leqslant\frac{(2m)!}{2^m\cdot m!}Z^{2m}(\boldsymbol{t})\mathbf{E}\left(\int\limits_{\Delta_k}S^2\left(|\boldsymbol{\lambda}-\boldsymbol{\zeta}_k|\right)d\mu(\lambda)\right)^m.$$

Proof. Since for a zero-mean Gaussian random variable η it is $\mathbf{E}\boldsymbol{\eta}^{2m+1}=0$, $\mathbf{E}\boldsymbol{\eta}^{2m}=\frac{(2m)!}{2^m\cdot m!}\sigma^{2m}$ and the random variables ζ_k are independent of $\eta(\boldsymbol{\lambda})$, then by the Fubini's theorem ($\mathbf{E}_{\boldsymbol{\zeta_k}}$ - is a conditional expectation with respect to $\boldsymbol{\zeta_k}$):

$$\mathbf{E}\left(\int\limits_{\Delta_k}(g(\boldsymbol{t},\boldsymbol{\lambda})-g(\boldsymbol{t},\boldsymbol{\zeta}_k))\,d\eta(\boldsymbol{\lambda})\right)^{2m}=$$

$$=\mathbf{E}\mathbf{E}_{\boldsymbol{\zeta}_k}\left(\int\limits_{\Delta_k}(g(\boldsymbol{t},\boldsymbol{\lambda})-g(\boldsymbol{t},\boldsymbol{\zeta}_k))\,d\eta(\boldsymbol{\lambda})\right)^{2m}=$$

$$=\frac{(2m)!}{2^m\cdot m!}\mathbf{E}\left(\int\limits_{\Delta_k}|g(\boldsymbol{t},\boldsymbol{\lambda})-g(\boldsymbol{t},\boldsymbol{\zeta}_k)|^2\,d\mu(\boldsymbol{\lambda})\right)^m\leqslant$$

$$\leqslant\frac{(2m)!}{2^m\cdot m!}\mathbf{E}\left(\int\limits_{\Delta_k}S^2\left(|\boldsymbol{\lambda}-\boldsymbol{\zeta}_k|\right)Z^2(\boldsymbol{t})d\mu(\boldsymbol{\lambda})\right)^m=$$

$$=\frac{(2m)!}{2^m\cdot m!}Z^{2m}(t)\mathbf{E}\left(\int\limits_{\Delta_k}S^2\left(|\boldsymbol{\lambda}-\boldsymbol{\zeta}_k|\right)d\mu(\boldsymbol{\lambda})\right)^m,$$

which finishes the proof. ♢

Theorem 3.1. *The following inequality holds*

$$\tau\left(\xi(\boldsymbol{t})-\xi_N(\boldsymbol{t})\right)\leqslant Z(\boldsymbol{t})\left(\sum_{k=1}^N b_k^2\sup_{m\geqslant1}\left(\mathbf{E}S^{2m}\left(|\boldsymbol{\zeta}_k-\boldsymbol{\zeta}_k^*|\right)\right)^{\frac{1}{m}}\right)^{\frac{1}{2}},\qquad(3.2)$$

where $b_k^2=\mu(\Delta_k)$, $\boldsymbol{\zeta}_k^*,\boldsymbol{\zeta}_k$ *are independent and* $\boldsymbol{\zeta}_k^*$ *have the same distributions as* $\boldsymbol{\zeta}_k$.

Proof. It follows from Lemma 1.3 that

$$\tau^2 \left(\int\limits_{\Delta_k} (g(\boldsymbol{t}, \boldsymbol{\lambda}) - g(\boldsymbol{t}, \boldsymbol{\zeta}_k))\, d\eta(\boldsymbol{\lambda}) \right) \leqslant \theta^2 \left(\int\limits_{\Delta_k} (g(\boldsymbol{t}, \boldsymbol{\lambda}) - g(\boldsymbol{t}, \boldsymbol{\zeta}_k))\, d\eta(\boldsymbol{\lambda}) \right) =$$

$$= \sup_{m \geqslant 1} \left(\frac{(2m)!}{2^m \cdot m!} \mathbf{E} \left(\int\limits_{\Delta_k} (g(\boldsymbol{t}, \boldsymbol{\lambda}) - g(\boldsymbol{t}, \boldsymbol{\zeta}_k))\, d\eta(\boldsymbol{\lambda}) \right)^{2m} \right)^{\frac{1}{m}} \leqslant$$

Applying Lemma 3.2, we obtain

$$\leqslant \sup_{m \geqslant 1} \left(b_k^2 Z^2(\boldsymbol{t}) \left(\mathbf{E} \left(\int\limits_{\Delta_k} S^2 \left(|\boldsymbol{\lambda} - \boldsymbol{\zeta}_k| \right) d\mu_k(\boldsymbol{\lambda}) \right)^m \right)^{\frac{1}{m}} \right) =$$

$$= \sup_{m \geqslant 1} \left(b_k^2 Z^2(\boldsymbol{t}) \left(\int\limits_{\Delta_k} \left(\int\limits_{\Delta_k} S^2 \left(|\boldsymbol{\lambda} - \boldsymbol{u}| \right) d\mu_k(\boldsymbol{\lambda}) \right)^m d\mu_k(\boldsymbol{u}) \right)^{\frac{1}{m}} \right),$$

Since the terms in the sum (3.3) for different k are independent, so from the last equality we have

$$\tau^2(\xi(\boldsymbol{t}) - \xi_N(\boldsymbol{t})) \leqslant$$

$$\leqslant Z^2(\boldsymbol{t}) \sum_{k=1}^{N} b_k^2 \sup_{m \geqslant 1} \left(\int\limits_{\Delta_k} \left(\int\limits_{\Delta_k} S^2 \left(|\boldsymbol{\lambda} - \boldsymbol{u}| \right) d\mu_k(\boldsymbol{\lambda}) \right)^m d\mu_k(\boldsymbol{u}) \right)^{\frac{1}{m}}.$$

Then, from the Fubini's theorem and the Lyapunov inequality we obtain

$$\tau(\xi(\boldsymbol{t}) - \xi_N(\boldsymbol{t})) \leqslant$$

$$\leqslant Z(\boldsymbol{t}) \left(\sum_{k=1}^{N} \sup_{m \geqslant 1} b_k^2 \left(\int\limits_{\Delta_k} \left(\int\limits_{\Delta_k} S^2 \left(|\boldsymbol{\lambda} - \boldsymbol{u}| \right) d\mu_k(\boldsymbol{\lambda}) \right)^m d\mu_k(\boldsymbol{u}) \right)^{\frac{1}{m}} \right)^{\frac{1}{2}} =$$

$$= Z(\boldsymbol{t}) \left(\sum_{k=1}^{N} \sup_{m \geqslant 1} b_k^2 \left(\mathbf{E}_{\boldsymbol{\zeta}_k^*} \left(\mathbf{E}_{\boldsymbol{\zeta}_k} S^2 \left(|\boldsymbol{\zeta}_k - \boldsymbol{\zeta}_k^*| \right) \right)^m \right)^{\frac{1}{m}} \right)^{\frac{1}{2}} \leqslant$$

$$\leqslant Z(\boldsymbol{t}) \left(\sum_{k=1}^{N} \sup_{m \geqslant 1} b_k^2 \left(\mathbf{E}_{\boldsymbol{\zeta}_k^*} \mathbf{E}_{\boldsymbol{\zeta}_k} S^{2m} \left(|\boldsymbol{\zeta}_k - \boldsymbol{\zeta}_k^*| \right) \right)^{\frac{1}{m}} \right)^{\frac{1}{2}} \leqslant$$

$$\leqslant Z(\boldsymbol{t}) \left(\sum_{k=1}^{N} b_k^2 \sup_{m \geqslant 1} \left(\mathbf{E} S^{2m} \left(|\boldsymbol{\zeta}_k - \boldsymbol{\zeta}_k^*| \right) \right)^{\frac{1}{m}} \right)^{\frac{1}{2}},$$

which is the desired statement. ♢

Remark 3.1. It is obviously that 3.1 makes sense only if the function $S(\cdot)$ such that the right side of the inequality (3.2) is finite.

3.2. The accuracy of modeling of Gaussian fields in $L_p(\mathbb{T}), p \geqslant 1$

Definition 3.1. [67] Let $\{\mathbb{T}, \mathfrak{B}, \mu\}$ be a measurable space. A stochastic field $\hat{X}(t)$ approximates the field $X(t)$ with reliability $(1-\delta), 0 < \delta < 1$ and accuracy $\varepsilon > 0$ in $L_p(\mathbb{T})$, if there exists a partition, such that the following inequality holds

$$\mathbf{P} \left\{ \left(\int_{\mathbb{T}} | X(t) - \hat{X}(t) |^p \, d\mu(t) \right)^{\frac{1}{p}} > \varepsilon \right\} \leqslant \delta.$$

Theorem 3.2. *Let $X = \{X(t), t \in \mathbb{T}\}$ be sub-Gaussian stochastic field, $\mathbb{E}X(t) = 0$, $\tau^2(t) = \tau^2(X(t))$. Suppose that there exists an integral $\int_{\mathbb{T}} (\tau(t))^p \, d\mu(t) < \infty$, $p \geqslant 1$. Then the integral $\int_{\mathbb{T}} | X(t) |^p \, d\mu(t) < \infty$, exists with probability 1 and for all ε satisfying $\varepsilon > c_p^{\frac{1}{p}} p^{\frac{1}{2}}$, where $c_p = \int_{\mathbb{T}} (\tau(t))^p \, d\mu(t)$ we have*

$$\mathbf{P} \left\{ \| X(t) \|_{L_p} > \varepsilon \right\} \leqslant 2 \exp \left\{ -\frac{\varepsilon^2}{2 c_p^{\frac{2}{p}}} \right\}.$$

The Theorem 3.2 is a particular case of the Theorem 2.1 from the [53].

Theorem 3.3. *Suppose that the partition Λ in the model $\xi_N(\boldsymbol{t})$ is such, that*

$$\int_{\mathbb{T}} (\tau(\xi(\boldsymbol{t}) - \xi_N(\boldsymbol{t})))^p \, d\boldsymbol{t} \leqslant \frac{\varepsilon^p}{\max \left(p^{\frac{p}{2}}, \left(2 \ln \frac{2}{\delta}\right)^{\frac{p}{2}} \right)}.$$

Then this model approximates the Gaussian stochastic field $\xi(\boldsymbol{t})$ with reliability $1-\delta$, $0<\delta<1$ and accuracy $\varepsilon>0$ in the space $L_p(\mathbb{T})$.

Proof. If $\varepsilon > \left(\int\limits_{\mathbb{T}} \left(\tau\left(\xi(\boldsymbol{t})-\xi_N(\boldsymbol{t})\right)\right)^p d\boldsymbol{t}\right)^{\frac{1}{p}} \cdot p^{\frac{1}{2}}$, then according to Theorem 3.2 and Definition 3.1 we have

$$\mathbf{P}\left\{\|\xi(\boldsymbol{t})-\xi_N(\boldsymbol{t})\|_{L_p} > \varepsilon\right\} \leqslant 2\exp\left\{-\frac{\varepsilon^2}{2c_p^{\frac{2}{p}}}\right\} \leqslant \delta,$$

де $c_p = \int\limits_{\mathbb{T}} \left(\tau\left(\xi(\boldsymbol{t})-\xi_N(\boldsymbol{t})\right)\right)^p d\boldsymbol{t}$.

Accordingly, the last estimate is valid when

$$\int\limits_{\mathbb{T}} (\tau(\xi(\boldsymbol{t})-\xi_N(\boldsymbol{t})))^p\, d\boldsymbol{t} \leqslant \frac{\varepsilon^p}{\max\left(p^{\frac{p}{2}}, \left(2\ln\frac{2}{\delta}\right)^{\frac{p}{2}}\right)}.$$

The proof is completed. ♢

Remark 3.2. Using Theorem 3.1 and Theorem 3.3 it is clear that the model $\xi_N(\boldsymbol{t})$ will be approximate field $\xi(\boldsymbol{t})$ with reliability $1-\delta$, $0<\delta<1$ and accuracy $\varepsilon>0$ in the space $L_p(\mathbb{T})$ if the following relationship holds

$$\left(\sum_{k=1}^{N} b_k^2 \sup_{m\geqslant 1}\left(\mathbf{E}S^{2m}\left(|\boldsymbol{\zeta}_k-\boldsymbol{\zeta}_k^*|\right)\right)^{\frac{1}{m}}\right)^{\frac{p}{2}} \int\limits_{\mathbb{T}} (Z(\boldsymbol{t}))^p\, d\boldsymbol{t} \leqslant \frac{\varepsilon^p}{\max\left(p^{\frac{p}{2}}, \left(2\ln\frac{2}{\delta}\right)^{\frac{p}{2}}\right)}.$$

Example 3.1. Consider the space $\mathbb{R}^2$. Let $\mathbb{T}$ be a parametric set in the following form $\mathbb{T} = \{\boldsymbol{t} : -A \leqslant t_i \leqslant A, i=1,2\}$, where $A>0$ is some number. Consider the Gaussian stochastic field with the following covariance function

$$R(\boldsymbol{t},\boldsymbol{s}) = \int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} \cos\langle \boldsymbol{t},\boldsymbol{\lambda}\rangle \cos\langle \boldsymbol{s},\boldsymbol{\lambda}\rangle dF(\boldsymbol{\lambda}),$$

where $F(\boldsymbol{\lambda})$ is continuous distribution function, $\mu(\boldsymbol{\lambda})$ is finite measure generated by the function $F(\boldsymbol{\lambda})$.

Then $\xi(\boldsymbol{t}) = \int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} \cos\langle \boldsymbol{t},\boldsymbol{\lambda}\rangle d\eta(\boldsymbol{\lambda})$ is a real-valued Gaussian centered stochastic field, where $\eta(A_1)$ is random measure subordinated to the measure μ such that $\mathbf{E}(\eta(A_1)\eta(A_2)) = \mu(A_1 \bigcap A_2)$, $A_1, A_2 \in \mathfrak{A}$.

Let $\Lambda = \{\Delta_1, \ldots, \Delta_{N^2+1}\}$ be a partition of the space $\mathbb{R}^2$, such that $\Delta_{N^2+1} = \{|\lambda_1| > L \text{ or } |\lambda_2| > L\}, L > 1$, and $\Delta_i, i = \overline{1, N^2}$ is a partition of the square $\overline{\Delta_{N^2+1}}$ on N^2 squares with the length of sides in $\frac{2L}{N}$ and $\Delta_i \bigcap \Delta_j = \varnothing, i \neq j$. Then the model of this field can be represented in the following form

$$\xi_N(\boldsymbol{t}) = \sum_{k=1}^{N^2+1} \eta_k \cos\langle \boldsymbol{t}, \boldsymbol{\zeta}_k \rangle,$$

where $\eta_k = \int\limits_{\Delta_k} d\eta(\boldsymbol{\lambda})$ are Gaussian random variables such that $\mathbf{E}\eta_k = 0$, $\mathbf{E}\eta_k^2 = \mu(\Delta_k) = b_k^2$, and $\boldsymbol{\zeta}_k \in \mathbb{R}^2$ are independent random variables being independent of η_k and and taking values on Δ_k with cumulative distribution function

$$\mathbf{P}\{\boldsymbol{\zeta}_k \in A\} = \frac{\mu(A \bigcap \Delta_k)}{\mu(\Delta_k)} := \mu_k(A).$$

It's easy to check that the model is centered field. Now we estimate the following expression

$$\begin{aligned} |\cos\langle \boldsymbol{t}, \boldsymbol{\lambda} \rangle - \cos\langle \boldsymbol{t}, \boldsymbol{u} \rangle|^2 &= \left| 2 \sin \frac{\langle \boldsymbol{t}, \boldsymbol{\lambda} - \boldsymbol{u} \rangle}{2} \sin \frac{\langle \boldsymbol{t}, \boldsymbol{\lambda} + \boldsymbol{u} \rangle}{2} \right|^2 \leqslant \\ &\leqslant \left| 2 \sin \frac{\langle \boldsymbol{t}, \boldsymbol{\lambda} - \boldsymbol{u} \rangle}{2} \right|^2 = 4 \sin^2 \frac{\langle \boldsymbol{t}, \boldsymbol{\lambda} - \boldsymbol{u} \rangle}{2} \leqslant \\ &\leqslant 4 \frac{\|\boldsymbol{t}\|^{2\alpha} \cdot \|\boldsymbol{\lambda} - \boldsymbol{u}\|^{2\alpha}}{2^{2\alpha}} = 2^{2(1-\alpha)} \|\boldsymbol{t}\|^{2\alpha} \cdot \|\boldsymbol{\lambda} - \boldsymbol{u}\|^{2\alpha}, \end{aligned}$$

where $0 < \alpha < 1$.

Applying Lemma 3.2 for all $\Delta_k, k = \overline{1, N^2}$ and having in mind that

$$Z(\boldsymbol{t}) = 2^{(1-\alpha)} \|\boldsymbol{t}\|^{\alpha}, S(|\boldsymbol{\lambda}|) = \|\lambda\|^{\alpha}.$$

we arrive at the following inequalities

$$\mathbf{E} \left(\int\limits_{\Delta_k} (\cos\langle \boldsymbol{t}, \boldsymbol{\lambda} \rangle - \cos\langle \boldsymbol{t}, \boldsymbol{\zeta}_k \rangle) \, d\eta(\boldsymbol{\lambda}) \right)^{2m+1} = 0,$$

$$\mathbf{E}\left(\int\limits_{\Delta_k} (\cos\langle \boldsymbol{t}, \boldsymbol{\lambda}\rangle - \cos\langle \boldsymbol{t}, \boldsymbol{\zeta}_k\rangle)\, d\eta(\boldsymbol{\lambda})\right)^{2m} \leqslant$$

$$\leqslant \frac{(2m)!}{2^m \cdot m!} 2^{2m(1-\alpha)} \|\boldsymbol{t}\|^{2m\alpha} \int\limits_{\Delta_k} \left(\int\limits_{\Delta_k} \|\boldsymbol{\lambda} - \boldsymbol{u}\|^{2\alpha}\, d\mu(\boldsymbol{\lambda})\right)^m d\mu_k(\boldsymbol{u}),$$

where $0 < \alpha < 1$.

By the Theorem 3.1 for the first N^2 terms we get

$$\tau^2\left(\int\limits_{\Delta_k} (\cos\langle \boldsymbol{t}, \boldsymbol{\lambda}\rangle - \cos\langle \boldsymbol{t}, \boldsymbol{\zeta}_k\rangle)\, d\eta(\boldsymbol{\lambda})\right) \leqslant$$

$$\leqslant 2^{2(1-\alpha)} \|\boldsymbol{t}\|^{2\alpha} b_k^2 \sup_{m\geqslant 1} \left(\int\limits_{\Delta_k} \left(\int\limits_{\Delta_k} \|\boldsymbol{\lambda} - \boldsymbol{u}\|^{2\alpha}\, d\mu_k(\boldsymbol{\lambda})\right)^m d\mu_k(\boldsymbol{u})\right)^{\frac{1}{m}}.$$

And for $k = N^2 + 1$ we obtain the following condition

$$b_{N^2+1}^2 \left(\int\limits_{\Delta_{N^2+1}} \left(\int\limits_{\Delta_{N^2+1}} (\cos\langle \boldsymbol{t}, \boldsymbol{\lambda}\rangle - \cos\langle \boldsymbol{t}, \boldsymbol{\zeta}_k\rangle)\, d\mu_{N^2+1}(\boldsymbol{\lambda})\right)^m d\mu_{N^2+1}(\boldsymbol{u})\right)^{\frac{1}{m}} \leqslant$$

$$\leqslant 4\left(\int\limits_{\Delta_{N^2+1}} \left(\int\limits_{\Delta_{N^2+1}} d\mu_{N^2+1}(\boldsymbol{\lambda})\right)^m d\mu(\boldsymbol{u})\right)^{\frac{1}{m}} \leqslant 4\mu\left(\Delta_{N^2+1}\right).$$

Let the measure μ of the space be less than one, then

$$\begin{aligned}\tau^2\left(\xi(\boldsymbol{t}) - \xi_N(\boldsymbol{t})\right) &\leqslant 2^{2(1-\alpha)} \|\boldsymbol{t}\|^{2\alpha} \sum_{k=1}^{N^2} b_k^2 \left(\frac{2L}{N}\right)^{4\alpha} + 4\mu\left(\Delta_{N^2+1}\right) \leqslant \\ &\leqslant 2^{2(1-\alpha)} \|\boldsymbol{t}\|^{2\alpha} \mu(\overline{\Delta_{N^2+1}}) \left(\frac{2L}{N}\right)^{4\alpha} + 4\mu\left(\Delta_{N^2+1}\right) \leqslant \\ &\leqslant 4\left(\frac{\|\boldsymbol{t}\| \cdot 4L^2}{2N^2}\right)^{2\alpha} + 4\mu\left(\Delta_{N^2+1}\right).\end{aligned} \tag{3.3}$$

Hence

$$\int\limits_{\mathbb{T}} \left(\tau\left(\xi(\boldsymbol{t}) - \xi_N(\boldsymbol{t})\right)\right)^p d\boldsymbol{t} \leqslant \int\limits_{\mathbb{T}} \left(4\left(\frac{\|\boldsymbol{t}\| \cdot 4L^2}{2N^2}\right)^{2\alpha} + 4\mu\left(\Delta_{N^2+1}\right)\right)^{\frac{p}{2}} d\boldsymbol{t} \leqslant$$

$$\leqslant \left(4\left(\frac{\sqrt{2}A \cdot 4L^2}{2N^2}\right)^{2\alpha} + 4\mu\left(\Delta_{N^2+1}\right)\right)^{\frac{p}{2}} \cdot \int\limits_{\mathbb{T}} d\boldsymbol{t} =$$

$$= \left(4\left(\frac{2\sqrt{2}A \cdot L^2}{N^2}\right)^{2\alpha} + 4\mu\left(\Delta_{N^2+1}\right)\right)^{\frac{p}{2}} \cdot (2A)^2.$$

We will choose such L that

$$\mu\left(\Delta_{N^2+1}\right) \leqslant \frac{\varepsilon^2}{8 \cdot (2A)^{\frac{4}{p}} \max\left(p, 2\ln\frac{2}{\delta}\right)},$$

where $0 < \delta < 1$, $\varepsilon > 0$.

Hence, by Theorem 3.3 the inequality

$$\int\limits_{\mathbb{T}} \left(\tau(\xi(\boldsymbol{t}) - \xi_N(\boldsymbol{t}))\right)^p d\boldsymbol{t} \leqslant \frac{\varepsilon^p}{\max\left(p^{\frac{p}{2}}, \left(2\ln\frac{2}{\delta}\right)^{\frac{p}{2}}\right)},$$

follows, when N satisfies

$$N \geqslant \max(Z_1, Z_2),$$

where

$$Z_1 = \frac{2^{\frac{3p\alpha+4p+4}{4p\alpha}} L \cdot A^{\frac{p\alpha+2}{2p\alpha}} \left(\ln\frac{2}{\delta}\right)^{\frac{1}{4\alpha}}}{\varepsilon^{\frac{1}{2\alpha}}}, Z_2 = \frac{2^{\frac{3p\alpha+3p+4}{4p\alpha}} L \cdot A^{\frac{p\alpha+2}{2p\alpha}} p^{\frac{1}{4\alpha}}}{\varepsilon^{\frac{1}{2\alpha}}}.$$

Thus the model $\xi_N(\boldsymbol{t})$ approximate field $\xi(\boldsymbol{t})$ with reliability $1-\delta$, $0 < \delta < 1$ and accuracy $\varepsilon > 0$ in the space $L_p(\mathbb{T})$ under previous condition.

Example 3.2. Let the field and its model are the same as in Example 3.1. Then from the inequality (3.3), we have

$$\tau^2\left(\xi(\boldsymbol{t}) - \xi_N(\boldsymbol{t})\right) \leqslant 2^{2(1-\alpha)} \|\boldsymbol{t}\|^{2\alpha} \mu\left(\overline{\Delta_{N^2+1}}\right) \left(\frac{2L}{N}\right)^{4\alpha} + 4\mu\left(\Delta_{N^2+1}\right).$$

From this evaluation and Theorem 3.3 it follows that the model $\xi_N(\boldsymbol{t})$

approximate field $\xi(\boldsymbol{t})$ with reliability $1-\delta$, $0<\delta<1$ and accuracy $\varepsilon>0$ in the space $L_p(\mathbb{T})$, if for N the following inequality holds $N \geqslant \max(Z_3, Z_4)$, where

$$Z_3 = \left(\frac{2^{\frac{4+3p+3p\alpha}{2p\alpha}} A^{1+\frac{2}{p\alpha}} L^2 \left(\ln \frac{2}{\delta}\right)^{\frac{1}{2\alpha}} \left(\mu\left(\overline{\Delta_{N^2+1}}\right)\right)^{\frac{1}{2\alpha}}}{\left(\varepsilon^2 - 8(2A)^{\frac{4}{p}} \mu\left(\Delta_{N^2+1}\right) \ln \frac{2}{\delta}\right)^{\frac{1}{2\alpha}}} \right)^{\frac{1}{2}},$$

$$Z_4 = \left(\frac{2^{\frac{4+2p+3p\alpha}{2p\alpha}} A^{1+\frac{2}{p\alpha}} L^2 p^{\frac{1}{2\alpha}} \left(\mu\left(\overline{\Delta_{N^2+1}}\right)\right)^{\frac{1}{2\alpha}}}{\left(\varepsilon^2 - 4(2A)^{\frac{4}{p}} \mu\left(\Delta_{N^2+1}\right) p\right)^{\frac{1}{2\alpha}}} \right)^{\frac{1}{2}}.$$

Let $\mu([-\lambda_1, \lambda_1] \times [-\lambda_2, \lambda_2]) = (1-e^{-\lambda_1})(1-e^{-\lambda_2})$, $p=2$, $\alpha=1$, $A=1$, $\delta = 0,01$ and $\varepsilon = 0,06$ then we obtained that $N(6,075) = 161,4968$.

So, if we choose the minimum partition $N = 162$, then we can constructed the model $\xi_N(\boldsymbol{t})$ of Gaussian stochastic field $\xi(\boldsymbol{t})$.

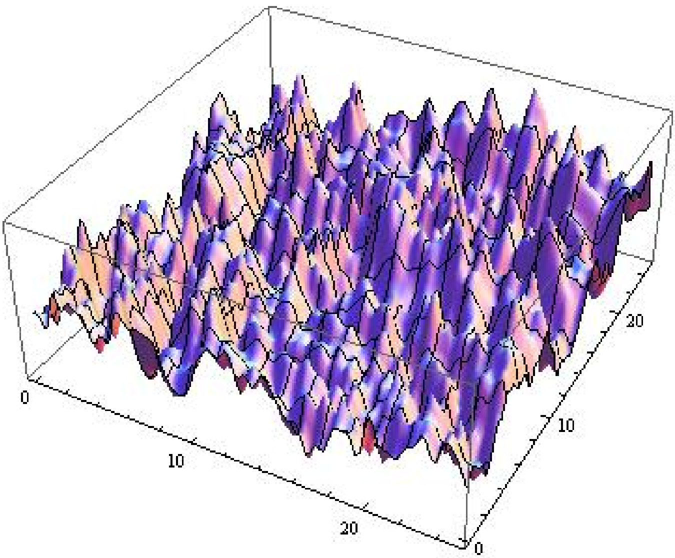

Figure 3.1. Model of Gaussian stochastic field in the space $L_2(\mathbb{T})$.

3.3. Estimates of Bessel functions of the first kind

In this section, we found new estimates for Bessel functions of the first kind. Also we considered differences Bessel functions with different arguments. The estimates for the difference between two and four functions were obtained.

It is known that

$$J_k(u) = \frac{1}{\pi}\int\limits_0^{\pi} \cos(k\varphi - u\sin\varphi)d\varphi, k = \overline{1,\infty}$$

is the integral representation of the Bessel functions of the first kind (see [6]).

Lemma 3.3. *For all* $0 < \alpha \leqslant 1$,

$$\mid J_k(u) \mid\leqslant 2^{1-\alpha} \mid u \mid^{\alpha} \pi^{\alpha}\frac{1}{k^{\alpha}}.$$

Proof. We have

$$|J_k(u)| = \frac{1}{\pi}\left|\int\limits_0^{\pi} \cos(k\varphi - u\sin\varphi)d\varphi\right| = \frac{1}{\pi}\left|\int\limits_0^{\pi} \cos(k\varphi)\cos(u\sin\varphi)d\varphi + \right.$$

$$\left. + \int\limits_0^{\pi} \sin(k\varphi)\sin(u\sin\varphi)d\varphi\right| = \frac{1}{\pi}\left|I_1 + I_2\right| \leqslant \frac{1}{\pi}\left(\mid I_1 \mid + \mid I_2 \mid\right).$$

Since the integrand in the integral I_1 is an even and periodic function with period 2π, we can transform I_1 as follows

$$I_1 = \int\limits_0^{\pi} \cos(k\varphi)\cos(u\sin\varphi)d\varphi = \frac{1}{2}\int\limits_{-\pi}^{\pi} \cos(k\varphi)\cos(u\sin\varphi)d\varphi =$$

$$= \frac{1}{2}\int\limits_{-\pi}^{\pi} \cos\left(k\left(\varphi + \frac{\pi}{k}\right)\right)\cos\left(u\sin\left(\varphi + \frac{\pi}{k}\right)\right)d\varphi = \frac{1}{2}\int\limits_{-\pi}^{\pi} \cos(k\varphi + \pi)\times$$

$$\times\cos\left(u\sin\left(\varphi + \frac{\pi}{k}\right)\right)d\varphi = -\frac{1}{2}\int\limits_{-\pi}^{\pi} \cos(k\varphi)\cos\left(u\sin\left(\varphi + \frac{\pi}{k}\right)\right)d\varphi.$$

Then the integral I_1 is written as follows

$$I_1 = -\frac{1}{4}\int\limits_{-\pi}^{\pi} \cos(k\varphi)\cos\left(u\sin\left(\varphi+\frac{\pi}{k}\right)\right)d\varphi + \frac{1}{4}\int\limits_{-\pi}^{\pi}\cos(k\varphi)\cos(u\sin\varphi)d\varphi$$

We are going to obtain a bound for $|I_1|$. Indeed,

$$|I_1| = \left|-\frac{1}{4}\int\limits_{-\pi}^{\pi} \cos(k\varphi)\cos\left(u\sin\left(\varphi+\frac{\pi}{k}\right)\right)d\varphi + \frac{1}{4}\int\limits_{-\pi}^{\pi}\cos(k\varphi)\cos(u\sin\varphi)d\varphi\right| \leqslant$$

$$\leqslant \frac{1}{4}\int\limits_{-\pi}^{\pi}\left|\cos(k\varphi)\left(\cos(u\sin\varphi)-\cos\left(u\sin\left(\varphi+\frac{\pi}{k}\right)\right)\right)\right|d\varphi = \frac{1}{4}\int\limits_{-\pi}^{\pi}|\cos(k\varphi)\times$$

$$\times 2\sin\left(\frac{u(\sin(\varphi+\frac{\pi}{k})-\sin\varphi)}{2}\right)\cdot\sin\left(\frac{u(\sin(\varphi+\frac{\pi}{k})+\sin\varphi)}{2}\right)\Bigg|d\varphi \leqslant$$

$$\leqslant \frac{1}{2}\int\limits_{-\pi}^{\pi}\left|\sin\left(\frac{u(\sin(\varphi+\frac{\pi}{k})-\sin\varphi)}{2}\right)\right|d\varphi \leqslant \frac{1}{2}\int\limits_{-\pi}^{\pi}\left|\frac{u}{2}\right|^{\alpha}\Bigg|\sin\left(\varphi+\frac{\pi}{k}\right)-$$

$$-\sin\varphi\Bigg|^{\alpha}d\varphi = \frac{1}{2}\int\limits_{-\pi}^{\pi}\left|\frac{u}{2}\right|^{\alpha}\left|2\cos\left(\frac{2\varphi+\frac{\pi}{k}}{2}\right)\cdot\sin\frac{\pi}{2k}\right|^{\alpha}d\varphi \leqslant \frac{1}{2}\int\limits_{-\pi}^{\pi}|u|^{\alpha}\times$$

$$\times\left|\sin\frac{\pi}{2k}\right|^{\alpha}d\varphi = |u|^{\alpha}\cdot\pi\left|\sin\frac{\pi}{2k}\right|^{\alpha} \leqslant \pi\cdot|\,u\,|^{\alpha}\left(\frac{\pi}{2k}\right)^{\alpha}.$$

For the integral I_2,

$$I_2 = \int\limits_{0}^{\pi}\sin(k\varphi)\sin(u\sin\varphi)d\varphi = \frac{1}{2}\int\limits_{-\pi}^{\pi}\sin(k\varphi)\sin(u\sin\varphi)\,d\varphi =$$

$$= \frac{1}{2}\int\limits_{-\pi}^{\pi}\sin\left(k\left(\varphi+\frac{\pi}{k}\right)\right)\sin\left(u\sin\left(\varphi+\frac{\pi}{k}\right)\right)d\varphi = \frac{1}{2}\int\limits_{-\pi}^{\pi}\sin(k\varphi-\pi))\times$$

$$\times\sin\left(u\sin\left(\varphi+\frac{\pi}{k}\right)\right)d\varphi = -\frac{1}{2}\int\limits_{-\pi}^{\pi}\sin(k\varphi)\sin\left(u\sin\left(\varphi+\frac{\pi}{k}\right)\right)d\varphi.$$

As in the case of I_1, the integral I_2 is transformed to the following form

$$I_2 = -\frac{1}{4}\int\limits_{-\pi}^{\pi}\sin(k\varphi)\sin\left(u\sin\left(\varphi+\frac{\pi}{k}\right)\right)d\varphi + \frac{1}{4}\int\limits_{-\pi}^{\pi}\sin(k\varphi)\sin(u\sin\varphi)d\varphi.$$

Then $|I_2|$ admits the following bound

$$|I_2| \leqslant \pi \cdot |u|^\alpha \left(\frac{\pi}{2k}\right)^\alpha,$$

whence

$$|J_k(u)| = \frac{1}{\pi} \mid I_1 + I_2 \mid \leqslant \frac{1}{\pi}\left(\pi \cdot \mid u \mid^\alpha \left(\frac{\pi}{2k}\right)^\alpha + \pi \cdot \mid u \mid^\alpha \left(\frac{\pi}{2k}\right)^\alpha\right) =$$
$$= 2^{1-\alpha} \cdot \mid u \mid^\alpha \pi^\alpha \cdot \frac{1}{k^\alpha},$$

where $0 < \alpha \leqslant 1$. ♢

Lemma 3.4. *For all* $0 < \alpha \leqslant 1$

$$\mid J_k(t\lambda) - J_k(tu) \mid \leqslant 4^{1-\alpha} t^\alpha \mid \lambda - u \mid^\alpha \pi^\alpha \cdot \frac{1}{k^\alpha}\left(1 + \frac{t^\alpha \mid \lambda + u \mid^\alpha}{2^\alpha}\right).$$

Proof. Using the integrals I_1 and I_2 evaluated in the proof of Lemma 3.3, we get

$$|J_k(t\lambda) - J_k(tu)| = \frac{1}{\pi}\left|\left(-\frac{1}{4}\int\limits_{-\pi}^{\pi} \cos(k\varphi)\cos\left(t\lambda \sin\left(\varphi + \frac{\pi}{k}\right)\right) d\varphi \, + \right.\right.$$
$$+ \frac{1}{4}\int\limits_{-\pi}^{\pi} \cos(k\varphi)\cos(t\lambda\sin\varphi) d\varphi + \frac{1}{4}\int\limits_{-\pi}^{\pi} \cos(k\varphi)\cos\left(tu\sin\left(\varphi + \frac{\pi}{k}\right)\right) d\varphi -$$
$$- \frac{1}{4}\int\limits_{-\pi}^{\pi} \cos(k\varphi)\cos(tu\sin\varphi) d\varphi\Bigg) + \left(-\frac{1}{4}\int\limits_{-\pi}^{\pi} \sin(k\varphi)\sin\left(t\lambda\sin\left(\varphi + \frac{\pi}{k}\right)\right) d\varphi \, + \right.$$
$$+ \frac{1}{4}\int\limits_{-\pi}^{\pi} \sin(k\varphi)\sin(t\lambda\sin\varphi) d\varphi + \frac{1}{4}\int\limits_{-\pi}^{\pi} \sin(k\varphi)\sin\left(tu\sin\left(\varphi + \frac{\pi}{k}\right)\right) d\varphi -$$
$$\left.\left. - \frac{1}{4}\int\limits_{-\pi}^{\pi} \sin(k\varphi)\sin(tu\sin\varphi) d\varphi\right)\right| = \frac{1}{\pi} \mid S_1 + S_2 \mid \leqslant \frac{1}{\pi}\left(\mid S_1 \mid + \mid S_2 \mid\right).$$

Then we find a bound for $|S_1|$

$$|S_1| = \Bigg| -\frac{1}{4}\int_{-\pi}^{\pi} \cos(k\varphi)\cos\left(t\lambda\sin\left(\varphi+\frac{\pi}{k}\right)\right)d\varphi + \frac{1}{4}\int_{-\pi}^{\pi}\cos(k\varphi)\cos(t\lambda\sin\varphi)d\varphi +$$

$$+\frac{1}{4}\int_{-\pi}^{\pi}\cos(k\varphi)\cos\left(tu\sin\left(\varphi+\frac{\pi}{k}\right)\right)d\varphi - \frac{1}{4}\int_{-\pi}^{\pi}\cos(k\varphi)\cos(tu\sin\varphi)d\varphi \,|\leqslant$$

$$\leqslant \frac{1}{4}\int_{-\pi}^{\pi}|\cos(k\varphi)|\Big|(\cos(t\lambda\sin\varphi)-\cos(tu\sin\varphi)) - \left(\cos\left(t\lambda\sin\left(\varphi+\frac{\pi}{k}\right)\right) -\right.$$

$$\left.-\ \cos\left(tu\sin\left(\varphi+\frac{\pi}{k}\right)\right)\right)\Big|\,d\varphi = \frac{1}{2}\int_{-\pi}^{\pi}|\cos(k\varphi)|\cdot\Big|\sin\frac{t(u+\lambda)\sin\varphi}{2}\ \times$$

$$\times\sin\frac{t(u-\lambda)\sin\varphi}{2} - \sin\frac{t(u+\lambda)\sin(\varphi+\frac{\pi}{k})}{2}\sin\frac{t(u-\lambda)\sin(\varphi+\frac{\pi}{k})}{2}\Big|\,d\varphi =$$

$$= \frac{1}{2}\int_{-\pi}^{\pi}|\cos(k\varphi)|\Big|\sin\frac{t(u+\lambda)\sin\varphi}{2}\left(\sin\frac{t(u-\lambda)\sin\varphi}{2}-\right.$$

$$\left.-\sin\frac{t(u-\lambda)\sin(\varphi+\frac{\pi}{k})}{2}\right) + \sin\frac{t(u-\lambda)\sin(\varphi+\frac{\pi}{k})}{2}\left(\sin\frac{t(u+\lambda)\sin\varphi}{2}-\right.$$

$$\left.-\sin\frac{t(u+\lambda)\sin(\varphi+\frac{\pi}{k})}{2}\right)\Big|\,d\varphi = \int_{-\pi}^{\pi}|\cos(k\varphi)|\,\Big|\sin\frac{t(u+\lambda)\sin\varphi}{2}\times$$

$$\times\cos\frac{t(u-\lambda)(\sin\varphi+\sin(\varphi+\frac{\pi}{k}))}{4}\sin\frac{t(u-\lambda)(\sin\varphi-\sin(\varphi+\frac{\pi}{k}))}{4}+$$

$$+\cos\frac{t(u+\lambda)(\sin\varphi+\sin(\varphi+\frac{\pi}{k}))}{4}\sin\frac{t(u+\lambda)(\sin\varphi-\sin(\varphi+\frac{\pi}{k}))}{4}\times$$

$$\times\sin\frac{t(u-\lambda)\sin(\varphi+\frac{\pi}{k})}{2}\Big|\,d\varphi \leqslant \int_{-\pi}^{\pi}\left(\left|\sin\frac{t(u-\lambda)(\sin\varphi-\sin(\varphi+\frac{\pi}{k}))}{4}\right|+\right.$$

$$\left.+\left|\sin\frac{t(u-\lambda)\sin(\varphi+\frac{\pi}{k})}{2}\sin\frac{t(u+\lambda)(\sin\varphi-\sin(\varphi+\frac{\pi}{k}))}{4}\right|\right)d\varphi\leqslant$$

$$\leqslant 2\pi\left(\frac{t^\alpha\mid\lambda-u\mid^\alpha\left(\frac{\pi}{2k}\right)^\alpha}{2^\alpha}+\frac{t^\alpha\mid\lambda-u\mid^\alpha t^\alpha\mid\lambda+u\mid^\alpha\left(\frac{\pi}{2k}\right)^\alpha}{2^\alpha\cdot 2^\alpha}\right)=$$

$$=2\pi\left(\frac{t}{2}\right)^\alpha\mid\lambda-u\mid^\alpha\left(\frac{\pi}{2k}\right)^\alpha\left(1+\frac{t^\alpha\mid\lambda+u\mid^\alpha}{2^\alpha}\right).$$

Similarly, we obtain a bound for $|S_2|$

$$
\begin{aligned}
|S_2| = &\left| -\frac{1}{4}\int\limits_{-\pi}^{\pi} \sin(k\varphi)\sin\left(t\lambda\sin\left(\varphi+\frac{\pi}{k}\right)\right)d\varphi + \frac{1}{4}\int\limits_{-\pi}^{\pi}\sin(k\varphi)\sin(t\lambda\sin\varphi)d\varphi + \right.\\
&+\frac{1}{4}\int\limits_{-\pi}^{\pi}\sin(k\varphi)\sin\left(tu\sin\left(\varphi+\frac{\pi}{k}\right)\right)d\varphi - \frac{1}{4}\int\limits_{-\pi}^{\pi}\sin(k\varphi)\sin(tu\sin\varphi)d\varphi \,|\leqslant\\
&\leqslant \frac{1}{4}\int\limits_{-\pi}^{\pi}|\sin(k\varphi)|\cdot\left|(\sin(t\lambda\sin\varphi)-\sin(tu\sin\varphi)) - \left(\sin\left(t\lambda\sin\left(\varphi+\frac{\pi}{k}\right)\right)-\right.\right.\\
&\left.\left.-\ \sin\left(tu\sin\left(\varphi+\frac{\pi}{k}\right)\right)\right)\right|d\varphi = \frac{1}{2}\int\limits_{-\pi}^{\pi}|\sin(k\varphi)|\cdot\left|\cos\frac{t(\lambda+u)\sin\varphi}{2}\times\right.\\
&\left.\times\sin\frac{t(\lambda-u)\sin\varphi}{2} - \cos\frac{t(\lambda+u)\sin(\varphi+\frac{\pi}{k})}{2}\sin\frac{t(\lambda-u)\sin(\varphi+\frac{\pi}{k})}{2}\right|d\varphi =\\
&=\frac{1}{2}\int\limits_{-\pi}^{\pi}|\sin(k\varphi)|\left|\cos\frac{t(\lambda+u)\sin\varphi}{2}\left(\sin\frac{t(\lambda-u)\sin\varphi}{2}-\right.\right.\\
&\left.-\sin\frac{t(\lambda-u)\sin(\varphi+\frac{\pi}{k})}{2}\right)+\sin\frac{t(\lambda-u)\sin(\varphi+\frac{\pi}{k})}{2}\left(\cos\frac{t(\lambda+u)\sin\varphi}{2}-\right.\\
&\left.\left.-\cos\frac{t(\lambda+u)\sin(\varphi+\frac{\pi}{k})}{2}\right)\right|d\varphi\leqslant\int\limits_{-\pi}^{\pi}|\sin(k\varphi)|\left(\left|\cos\frac{t(\lambda+u)\sin\varphi}{2}\times\right.\right.\\
&\left.\times\cos\frac{t(\lambda-u)(\sin\varphi+\sin(\varphi+\frac{\pi}{k}))}{4}\sin\frac{t(\lambda-u)(\sin\varphi-\sin(\varphi+\frac{\pi}{k}))}{4}\right|+\\
&+\left|\sin\frac{t(\lambda+u)(\sin(\varphi+\frac{\pi}{k})-\sin\varphi)}{4}\sin\frac{t(\lambda+u)(\sin\varphi+\sin(\varphi+\frac{\pi}{k}))}{4}\times\right.\\
&\left.\left.\times\sin\frac{t(\lambda-u)\sin(\varphi+\frac{\pi}{k})}{2}\right|\right)d\varphi\leqslant\int\limits_{-\pi}^{\pi}\left(\left|\sin\frac{t(\lambda-u)(\sin\varphi-\sin(\varphi+\frac{\pi}{k}))}{4}\right|+\right.\\
&\left.+\left|\sin\frac{t(\lambda-u)\sin(\varphi+\frac{\pi}{k})}{2}\sin\frac{t(\lambda+u)(\sin(\varphi+\frac{\pi}{k})-\sin\varphi)}{4}\right|\right)d\varphi\leqslant\\
&\leqslant 2\pi\left(\frac{t^\alpha\mid\lambda-u\mid^\alpha\left(\frac{\pi}{2k}\right)^\alpha}{2^\alpha}+\frac{t^\alpha\mid\lambda-u\mid^\alpha t^\alpha\mid\lambda+u\mid^\alpha\left(\frac{\pi}{2k}\right)^\alpha}{2^\alpha\cdot2^\alpha}\right)=\\
&=2\pi\left(\frac{t}{2}\right)^\alpha\mid\lambda-u\mid^\alpha\left(\frac{\pi}{2k}\right)^\alpha\left(1+\frac{t^\alpha\mid\lambda+u\mid^\alpha}{2^\alpha}\right).
\end{aligned}
$$

Then

$$|J_k(t\lambda) - J_k(tu)| \leqslant 2\left(\frac{t}{2}\right)^{\alpha}|\lambda - u|^{\alpha}\left(\frac{\pi}{2k}\right)^{\alpha}\left(1 + \frac{t^{\alpha}|\lambda + u|^{\alpha}}{2^{\alpha}}\right) +$$
$$+ 2\left(\frac{t}{2}\right)^{\alpha}|\lambda - u|^{\alpha}\left(\frac{\pi}{2k}\right)^{\alpha}\left(1 + \frac{t^{\alpha}|\lambda + u|^{\alpha}}{2^{\alpha}}\right) =$$
$$= 4^{1-\alpha}\left(\frac{\pi \cdot t|\lambda - u|}{k}\right)^{\alpha}\left(1 + \frac{t^{\alpha}|\lambda + u|^{\alpha}}{2^{\alpha}}\right). \qquad \diamondsuit$$

Lemma 3.5. *For all* $0 < \alpha \leqslant 1$ *and* $0 < \beta \leqslant 1$

$$|J_k(t\lambda) - J_k(s\lambda)| \leqslant 4^{1-\alpha}\pi^{\alpha} \cdot \frac{1}{k^{\alpha}}\left(\lambda^{\alpha}|s-t|^{\alpha} + \lambda^{\alpha+\beta}|s-t|^{\beta}|s+t|^{\alpha}\right).$$

Proof. Substituting the expressions for the integrals I_1 and I_2 from the proof of the Lemma 3.3, we have

$$|J_k(t\lambda) - J_k(s\lambda)| = \frac{1}{\pi}\left|\left(-\frac{1}{4}\int_{-\pi}^{\pi}\cos(k\varphi)\cos\left(t\lambda\sin\left(\varphi + \frac{\pi}{k}\right)\right)d\varphi + \right.\right.$$
$$+\frac{1}{4}\int_{-\pi}^{\pi}\cos(k\varphi)\cos(t\lambda\sin\varphi)d\varphi + \frac{1}{4}\int_{-\pi}^{\pi}\cos(k\varphi)\cos\left(s\lambda\sin\left(\varphi + \frac{\pi}{k}\right)\right)d\varphi -$$
$$\left.-\frac{1}{4}\int_{-\pi}^{\pi}\cos(k\varphi)\cos(s\lambda\sin\varphi)d\varphi\right) + \left(-\frac{1}{4}\int_{-\pi}^{\pi}\sin(k\varphi)\sin\left(t\lambda\sin\left(\varphi + \frac{\pi}{k}\right)\right)d\varphi + \right.$$
$$+\frac{1}{4}\int_{-\pi}^{\pi}\sin(k\varphi)\sin(t\lambda\sin\varphi)d\varphi + \frac{1}{4}\int_{-\pi}^{\pi}\sin(k\varphi)\sin\left(s\lambda\sin\left(\varphi + \frac{\pi}{k}\right)\right)d\varphi -$$
$$\left.\left.-\frac{1}{4}\int_{-\pi}^{\pi}\sin(k\varphi)\sin(s\lambda\sin\varphi)d\varphi\right)\right| = \frac{1}{\pi}|S_1 + S_2| \leqslant \frac{1}{\pi}\left(|S_1| + |S_2|\right).$$

Now we estimate $|S_1|$

$$|S_1| = \frac{1}{4}\left|\int_{-\pi}^{\pi}\cos(k\varphi)\left[\left(\cos(t\lambda\sin\varphi) - \cos(s\lambda\sin\varphi)\right) - \left(\cos\left(t\lambda\sin\left(\varphi + \frac{\pi}{k}\right)\right) - \right.\right.\right.$$
$$\left.\left.\left.-\cos\left(s\lambda\sin\left(\varphi + \frac{\pi}{k}\right)\right)\right)\right]d\varphi\right| \leqslant \frac{1}{2}\int_{-\pi}^{\pi}|\cos(k\varphi)| \cdot \left|\sin\frac{\lambda(s+t)\sin\varphi}{2} \times \right.$$
$$\left.\times\sin\frac{\lambda(s-t)\sin\varphi}{2} - \sin\frac{\lambda(s+t)\sin(\varphi + \frac{\pi}{k})}{2}\sin\frac{\lambda(s-t)\sin(\varphi + \frac{\pi}{k})}{2}\right|d\varphi =$$

$$= \frac{1}{2} \int\limits_{-\pi}^{\pi} |\cos(k\varphi)| \left| \sin \frac{\lambda(s+t)\sin\varphi}{2} \left(\sin \frac{\lambda(s-t)\sin\varphi}{2} - \right.\right.$$

$$\left. - \sin \frac{\lambda(s-t)\sin(\varphi + \frac{\pi}{k})}{2} \right) + \sin \frac{\lambda(s-t)\sin(\varphi+\frac{\pi}{k})}{2} \left(\sin \frac{\lambda(s+t)\sin\varphi}{2} - \right.$$

$$\left.\left. - \sin \frac{\lambda(s+t)\sin(\varphi+\frac{\pi}{k})}{2} \right) \right| d\varphi = \int\limits_{-\pi}^{\pi} |\cos(k\varphi)| \cdot \left| \sin \frac{\lambda(s+t)\sin\varphi}{2} \times \right.$$

$$\times \cos \frac{\lambda(s-t)(\sin\varphi + \sin(\varphi+\frac{\pi}{k}))}{4} \sin \frac{\lambda(s-t)(\sin\varphi - \sin(\varphi+\frac{\pi}{k}))}{4} +$$

$$+ \cos \frac{\lambda(s+t)(\sin\varphi + \sin(\varphi+\frac{\pi}{k}))}{4} \sin \frac{\lambda(s+t)(\sin\varphi - \sin(\varphi+\frac{\pi}{k}))}{4} \times$$

$$\left. \times \sin \frac{\lambda(s-t)\sin(\varphi+\frac{\pi}{k})}{2} \right| d\varphi \leqslant \int\limits_{-\pi}^{\pi} \left(\left| \sin \frac{\lambda(s-t)(\sin\varphi - \sin(\varphi+\frac{\pi}{k}))}{4} \right| + \right.$$

$$\left. + \left| \sin \frac{\lambda(s-t)\sin(\varphi+\frac{\pi}{k})}{2} \sin \frac{\lambda(s+t)(\sin\varphi - \sin(\varphi+\frac{\pi}{k}))}{4} \right| \right) d\varphi \leqslant$$

$$\leqslant 2\pi \left(\frac{\lambda^\alpha |s-t|^\alpha \left(\frac{\pi}{2k}\right)^\alpha}{2^\alpha} + \frac{\lambda^\beta |s-t|^\beta \lambda^\alpha |s+t|^\alpha \left(\frac{\pi}{2k}\right)^\alpha}{2^\beta \cdot 2^\alpha} \right).$$

A bound for $|S_2|$ is obtained similarly.

$$|S_2| = \frac{1}{4} \left| \int\limits_{-\pi}^{\pi} \sin(k\varphi) \left[(\sin(t\lambda\sin\varphi) - \sin(s\lambda\sin\varphi)) - \left(\sin\left(t\lambda \sin\left(\varphi + \frac{\pi}{k}\right)\right) - \right.\right.\right.$$

$$\left.\left.\left. - \sin\left(s\lambda\sin\left(\varphi+\frac{\pi}{k}\right)\right)\right)\right] d\varphi \right| \leqslant \frac{1}{2} \int\limits_{-\pi}^{\pi} |\sin(k\varphi)| \cdot \left| \cos \frac{\lambda(t+s)\sin\varphi}{2} \times \right.$$

$$\left. \times \sin \frac{\lambda(t-s)\sin\varphi}{2} - \cos \frac{\lambda(t+s)\sin(\varphi+\frac{\pi}{k})}{2} \sin \frac{\lambda(t-s)\sin(\varphi+\frac{\pi}{k})}{2} \right| d\varphi =$$

$$= \frac{1}{2} \int\limits_{-\pi}^{\pi} |\sin(k\varphi)| \left| \cos \frac{\lambda(t+s)\sin\varphi}{2} \left(\sin \frac{\lambda(t-s)\sin\varphi}{2} - \right.\right.$$

$$\left. - \sin \frac{\lambda(t-s)\sin(\varphi+\frac{\pi}{k})}{2} \right) + \sin \frac{\lambda(t-s)\sin(\varphi+\frac{\pi}{k})}{2} \left(\cos \frac{\lambda(t+s)\sin\varphi}{2} - \right.$$

$$\left.\left. - \cos \frac{\lambda(t+s)\sin(\varphi+\frac{\pi}{k})}{2} \right) \right| d\varphi \leqslant \int\limits_{-\pi}^{\pi} |\sin(k\varphi)| \cdot \left(\left| \cos \frac{\lambda(t+s)\sin\varphi}{2} \times \right.\right.$$

$$\times \cos \frac{\lambda(t-s)(\sin\varphi + \sin(\varphi+\frac{\pi}{k}))}{4} \sin \frac{\lambda(t-s)(\sin\varphi - \sin(\varphi+\frac{\pi}{k}))}{4}\Bigg| +$$

$$+\left|\sin \frac{\lambda(t+s)(\sin\varphi + \sin(\varphi+\frac{\pi}{k}))}{4} \sin \frac{\lambda(t+s)(\sin(\varphi+\frac{\pi}{k}) - \sin\varphi)}{4}\times\right.$$

$$\left.\times \sin \frac{\lambda(t-s)\sin(\varphi+\frac{\pi}{k})}{2}\right|\Bigg) d\varphi \leqslant \int\limits_{-\pi}^{\pi} \left(\left|\sin \frac{\lambda(t-s)(\sin\varphi - \sin(\varphi+\frac{\pi}{k}))}{4}\right| +\right.$$

$$\left. + \left|\sin \frac{\lambda(t-s)\sin(\varphi+\frac{\pi}{k})}{2} \sin \frac{\lambda(t+s)(\sin(\varphi+\frac{\pi}{k}) - \sin\varphi)}{4}\right|\right) d\varphi \leqslant$$

$$\leqslant 2\pi \left(\frac{\lambda^\alpha |s-t|^\alpha \left(\frac{\pi}{2k}\right)^\alpha}{2^\alpha} + \frac{\lambda^\beta |s-t|^\beta \lambda^\alpha |s+t|^\alpha \left(\frac{\pi}{2k}\right)^\alpha}{2^\beta \cdot 2^\alpha} \right).$$

Then

$$|J_k(t\lambda) - J_k(s\lambda)| \leqslant \frac{1}{\pi} \left[2\pi \left(\frac{\lambda^\alpha |s-t|^\alpha \left(\frac{\pi}{2k}\right)^\alpha}{2^\alpha} + \frac{\lambda^\beta |s-t|^\beta \lambda^\alpha |s+t|^\alpha \left(\frac{\pi}{2k}\right)^\alpha}{2^\beta \cdot 2^\alpha} \right) + \right.$$

$$\left. + 2\pi \left(\frac{\lambda^\alpha |s-t|^\alpha \left(\frac{\pi}{2k}\right)^\alpha}{2^\alpha} + \frac{\lambda^\beta |s-t|^\beta \lambda^\alpha |s+t|^\alpha \left(\frac{\pi}{2k}\right)^\alpha}{2^\beta \cdot 2^\alpha} \right) \right] =$$

$$= 4^{1-\alpha} \pi^\alpha \frac{1}{k^\alpha} \left(\lambda^\alpha |s-t|^\alpha + \lambda^{\alpha+\beta} |s-t|^\beta |s+t|^\alpha \right). \qquad \diamondsuit$$

Lemma 3.6. *For all* $0 < \alpha \leqslant 1$ *holds*

$$|J_k(t\lambda) - J_k(tu) - J_k(s\lambda) + J_k(su)| \leqslant 2 \cdot 4^{1-\alpha} |\lambda - u|^\alpha |s-t|^\alpha \left(\frac{\pi}{2k}\right)^\alpha \times$$

$$\times \left(1 + \frac{|\lambda+u|^\alpha |s-t|^\alpha}{4^\alpha} + \frac{|t+s|^\alpha (\lambda^\alpha + 2u^\alpha)}{2^\alpha} + \frac{|t+s|^{2\alpha} u^\alpha |\lambda+u|^\alpha}{4^\alpha \cdot 2^\alpha} \right).$$

Proof. Since

$$J_k(t\lambda) = \frac{1}{\pi} \left(-\frac{1}{4} \int\limits_{-\pi}^{\pi} \cos(k\varphi) \cos\left(t\lambda \sin\left(\varphi + \frac{\pi}{k}\right)\right) d\varphi + \frac{1}{4} \int\limits_{-\pi}^{\pi} \cos(k\varphi) \times \right.$$

$$\times \cos(t\lambda \sin\varphi) d\varphi - \frac{1}{4} \int\limits_{-\pi}^{\pi} \sin(k\varphi) \sin\left(t\lambda \sin\left(\varphi + \frac{\pi}{k}\right)\right) d\varphi +$$

$$+\frac{1}{4}\int_{-\pi}^{\pi}\sin(k\varphi)\sin(t\lambda\sin\varphi)d\varphi\Bigg),$$

we conclude that

$$|J_k(t\lambda)-J_k(tu)-J_k(s\lambda)+J_k(su)|=\frac{1}{\pi}\Bigg|\frac{1}{4}\int_{-\pi}^{\pi}\cos(k\varphi)\left(\cos(t\lambda\sin\varphi)\right.-$$
$$-\cos\left(t\lambda\sin\left(\varphi+\frac{\pi}{k}\right)\right)-\cos(tu\sin\varphi)+\cos\left(tu\sin\left(\varphi+\frac{\pi}{k}\right)\right)-$$
$$-\cos(s\lambda\sin\varphi)+\cos\left(s\lambda\sin\left(\varphi+\frac{\pi}{k}\right)\right)+\cos(su\sin\varphi)-$$
$$\left.-\cos\left(su\sin\left(\varphi+\frac{\pi}{k}\right)\right)\right)d\varphi+\frac{1}{4}\int_{-\pi}^{\pi}\sin(k\varphi)\left(\sin(t\lambda\sin\varphi)\right.-$$
$$-\sin\left(t\lambda\sin\left(\varphi+\frac{\pi}{k}\right)\right)-\sin(tu\sin\varphi)+\sin\left(tu\sin\left(\varphi+\frac{\pi}{k}\right)\right)-$$
$$-\sin(s\lambda\sin\varphi)+\sin\left(s\lambda\sin\left(\varphi+\frac{\pi}{k}\right)\right)+\sin(su\sin\varphi)-$$
$$\left.-\sin\left(su\sin\left(\varphi+\frac{\pi}{k}\right)\right)\right)d\varphi\Bigg|=\frac{1}{\pi}|K_1+K_2|\leqslant\frac{1}{\pi}\left(|K_1|+|K_2|\right).$$

Then we find a bound for $|K_1|$

$$|K_1|=\frac{1}{4}\Bigg|\int_{-\pi}^{\pi}\cos(k\varphi)\Bigg[\left(\cos(t\lambda\sin\varphi)-\cos(s\lambda\sin\varphi)\right)-\left(\cos\left(t\lambda\sin\left(\varphi+\frac{\pi}{k}\right)\right)-\right.$$
$$\left.-\cos\left(s\lambda\sin\left(\varphi+\frac{\pi}{k}\right)\right)\right)-\left(\cos(tu\sin\varphi)-\cos(su\sin\varphi)\right)+$$
$$+\left(\cos\left(tu\sin\left(\varphi+\frac{\pi}{k}\right)\right)-\cos\left(su\sin\left(\varphi+\frac{\pi}{k}\right)\right)\right)\Bigg]d\varphi\Bigg|\leqslant\frac{1}{4}\int_{-\pi}^{\pi}|\cos(k\varphi)|\times$$
$$\times\left|2\sin\frac{\lambda(t+s)\sin\varphi}{2}\sin\frac{\lambda(s-t)\sin\varphi}{2}-2\sin\frac{\lambda(t+s)\sin(\varphi+\frac{\pi}{k})}{2}\times\right.$$
$$\times\sin\frac{\lambda(s-t)\sin(\varphi+\frac{\pi}{k})}{2}-2\sin\frac{u(t+s)\sin\varphi}{2}\sin\frac{u(s-t)\sin\varphi}{2}+$$
$$\left.+2\sin\frac{u(t+s)\sin(\varphi+\frac{\pi}{k})}{2}\sin\frac{u(s-t)\sin(\varphi+\frac{\pi}{k})}{2}\right|d\varphi=\frac{1}{2}\int_{-\pi}^{\pi}|\cos(k\varphi)|\times$$
$$\times\left|\left(\sin\frac{\lambda(t+s)\sin\varphi}{2}\sin\frac{\lambda(s-t)\sin\varphi}{2}-\sin\frac{u(t+s)\sin\varphi}{2}\times\right.\right.$$

$$\times \sin\frac{u(s-t)\sin\varphi}{2}\Bigg) - \Bigg(\sin\frac{\lambda(t+s)\sin(\varphi+\frac{\pi}{k})}{2}\sin\frac{\lambda(s-t)\sin(\varphi+\frac{\pi}{k})}{2} -$$

$$-\sin\frac{u(t+s)\sin(\varphi+\frac{\pi}{k})}{2}\sin\frac{u(s-t)\sin(\varphi+\frac{\pi}{k})}{2}\Bigg)\Bigg| d\varphi = \frac{1}{2}\int\limits_{-\pi}^{\pi} |\cos(k\varphi)| \times$$

$$\times \Bigg|\sin\frac{\lambda(t+s)\sin\varphi}{2}\Bigg(\sin\frac{\lambda(s-t)\sin\varphi}{2} - \sin\frac{u(s-t)\sin\varphi}{2}\Bigg) +$$

$$+\sin\frac{u(s-t)\sin\varphi}{2}\Bigg(\sin\frac{\lambda(t+s)\sin\varphi}{2} - \sin\frac{u(t+s)\sin\varphi}{2}\Bigg) -$$

$$-\sin\frac{\lambda(t+s)\sin(\varphi+\frac{\pi}{k})}{2}\Bigg(\sin\frac{\lambda(s-t)\sin(\varphi+\frac{\pi}{k})}{2} -$$

$$-\sin\frac{u(s-t)\sin(\varphi+\frac{\pi}{k})}{2}\Bigg) - \sin\frac{u(s-t)\sin(\varphi+\frac{\pi}{k})}{2}\Bigg(\sin\frac{\lambda(t+s)\sin(\varphi+\frac{\pi}{k})}{2} -$$

$$\sin\frac{u(t+s)\sin(\varphi+\frac{\pi}{k})}{2}\Bigg)\Bigg| d\varphi = \frac{1}{2}\int\limits_{-\pi}^{\pi} |\cos(k\varphi)| \cdot \Bigg|2\sin\frac{\lambda(t+s)\sin\varphi}{2} \times$$

$$\times \cos\frac{(\lambda+u)(s-t)\sin\varphi}{4}\sin\frac{(\lambda-u)(s-t)\sin\varphi}{4} + 2\sin\frac{u(s-t)\sin\varphi}{2} \times$$

$$\times \cos\frac{(\lambda+u)(t+s)\sin\varphi}{4}\sin\frac{(\lambda-u)(t+s)\sin\varphi}{4} - 2\sin\frac{\lambda(t+s)\sin(\varphi+\frac{\pi}{k})}{2} \times$$

$$\times \cos\frac{(\lambda+u)(s-t)\sin(\varphi+\frac{\pi}{k})}{4}\sin\frac{(\lambda-u)(s-t)\sin(\varphi+\frac{\pi}{k})}{4} -$$

$$-2\sin\frac{u(s-t)\sin(\varphi+\frac{\pi}{k})}{2}\cos\frac{(\lambda+u)(t+s)\sin(\varphi+\frac{\pi}{k})}{4} \times$$

$$\times \sin\frac{(\lambda-u)(t+s)\sin(\varphi+\frac{\pi}{k})}{4}\Bigg| d\varphi = \int\limits_{-\pi}^{\pi} |\cos(k\varphi)| \cdot \Bigg|\sin\frac{\lambda(t+s)\sin\varphi}{2} \times$$

$$\times \cos\frac{(\lambda+u)(s-t)\sin\varphi}{4}\Bigg(\sin\frac{(\lambda-u)(s-t)\sin\varphi}{4} -$$

$$-\sin\frac{(\lambda-u)(s-t)\sin(\varphi+\frac{\pi}{k})}{4}\Bigg) + \sin\frac{(\lambda-u)(s-t)\sin(\varphi+\frac{\pi}{k})}{4} \times$$

$$\times \Bigg(\sin\frac{\lambda(t+s)\sin\varphi}{2}\cos\frac{(\lambda+u)(s-t)\sin\varphi}{4} - \sin\frac{\lambda(t+s)\sin(\varphi+\frac{\pi}{k})}{2} \times$$

$$\times \cos\frac{(\lambda+u)(s-t)\sin(\varphi+\frac{\pi}{k})}{4}\Bigg) + \sin\frac{u(s-t)\sin\varphi}{2}\cos\frac{(\lambda+u)(t+s)\sin\varphi}{4} \times$$

$$\times\left(\sin\frac{(\lambda-u)(t+s)\sin\varphi}{4}-\sin\frac{(\lambda-u)(t+s)\sin(\varphi+\frac{\pi}{k})}{4}\right)+$$

$$+\sin\frac{(\lambda-u)(t+s)\sin(\varphi+\frac{\pi}{k})}{4}\left(\sin\frac{u(s-t)\sin\varphi}{2}\cos\frac{(\lambda+u)(t+s)\sin\varphi}{4}-\right.$$

$$\left.-\sin\frac{u(s-t)\sin(\varphi+\frac{\pi}{k})}{2}\cos\frac{(\lambda+u)(t+s)\sin(\varphi+\frac{\pi}{k})}{4}\right)\Bigg|\,d\varphi=$$

$$=\int\limits_{-\pi}^{\pi}|\cos(k\varphi)|\cdot\Bigg|2\sin\frac{\lambda(t+s)\sin\varphi}{2}\cos\frac{(\lambda-u)(s-t)(\sin\varphi+\sin(\varphi+\frac{\pi}{k}))}{8}\times$$

$$\times\sin\frac{(\lambda-u)(s-t)(\sin\varphi-\sin(\varphi+\frac{\pi}{k}))}{8}\cos\frac{(\lambda+u)(s-t)\sin\varphi}{4}+$$

$$+\sin\frac{(\lambda-u)(s-t)\sin(\varphi+\frac{\pi}{k})}{4}\left[\sin\frac{\lambda(t+s)\sin\varphi}{2}\left(\cos\frac{(\lambda+u)(s-t)\sin\varphi}{4}-\right.\right.$$

$$\left.-\cos\frac{(\lambda+u)(s-t)\sin(\varphi+\frac{\pi}{k})}{4}\right)+\cos\frac{(\lambda+u)(s-t)\sin(\varphi+\frac{\pi}{k})}{4}\times$$

$$\left.\times\left(\sin\frac{\lambda(t+s)\sin\varphi}{2}-\sin\frac{\lambda(t+s)\sin(\varphi+\frac{\pi}{k})}{2}\right)\right]+2\sin\frac{u(s-t)\sin\varphi}{2}\times$$

$$\times\cos\frac{(\lambda+u)(t+s)\sin\varphi}{4}\cos\frac{(\lambda-u)(t+s)(\sin\varphi+\sin(\varphi+\frac{\pi}{k}))}{8}\times$$

$$\times\sin\frac{(\lambda-u)(t+s)(\sin\varphi-\sin(\varphi+\frac{\pi}{k}))}{8}+\sin\frac{(\lambda-u)(t+s)\sin(\varphi+\frac{\pi}{k})}{4}\times$$

$$\times\left[\sin\frac{u(s-t)\sin\varphi}{2}\left(\cos\frac{(\lambda+u)(t+s)\sin\varphi}{4}-\right.\right.$$

$$\left.-\cos\frac{(\lambda+u)(t+s)\sin(\varphi+\frac{\pi}{k})}{4}\right)+\cos\frac{(\lambda+u)(t+s)\sin(\varphi+\frac{\pi}{k})}{4}\times$$

$$\left.\times\left(\sin\frac{u(s-t)\sin\varphi}{2}-\sin\frac{u(s-t)\sin(\varphi+\frac{\pi}{k})}{2}\right)\right]\Bigg|\,d\varphi=\int\limits_{-\pi}^{\pi}|\cos(k\varphi)|\times$$

$$\times\Bigg|2\sin\frac{\lambda(t+s)\sin\varphi}{2}\cos\frac{(\lambda-u)(s-t)(\sin\varphi+\sin(\varphi+\frac{\pi}{k}))}{8}\times$$

$$\times\sin\frac{(\lambda-u)(s-t)\cos(\varphi+\frac{\pi}{2k})\sin(-\frac{\pi}{2k})}{4}\cos\frac{(\lambda+u)(s-t)\sin\varphi}{4}+$$

$$+2\sin\frac{(\lambda-u)(s-t)\sin(\varphi+\frac{\pi}{k})}{4}\sin\frac{(\lambda+u)(s-t)(\sin\varphi+\sin(\varphi+\frac{\pi}{k}))}{8}\times$$

$$\times \sin\frac{(\lambda+u)(s-t)(\sin(\varphi+\frac{\pi}{k})-\sin\varphi)}{8}\sin\frac{\lambda(t+s)\sin\varphi}{2}+$$

$$+2\sin\frac{(\lambda-u)(s-t)\sin(\varphi+\frac{\pi}{k})}{4}\cos\frac{(\lambda+u)(s-t)\sin(\varphi+\frac{\pi}{k})}{4}\times$$

$$\times\cos\frac{\lambda(t+s)(\sin\varphi+\sin(\varphi+\frac{\pi}{k}))}{4}\sin\frac{\lambda(t+s)(\sin\varphi-\sin(\varphi+\frac{\pi}{k}))}{4}+$$

$$+2\sin\frac{u(s-t)\sin\varphi}{2}\cos\frac{(\lambda-u)(t+s)(\sin\varphi+\sin(\varphi+\frac{\pi}{k}))}{8}\times$$

$$\times\sin\frac{(\lambda-u)(t+s)\cos(\varphi+\frac{\pi}{2k})\sin(-\frac{\pi}{2k})}{4}\cos\frac{(\lambda+u)(t+s)\sin\varphi}{4}+$$

$$+2\sin\frac{(\lambda-u)(t+s)\sin(\varphi+\frac{\pi}{k})}{4}\sin\frac{(\lambda+u)(t+s)(\sin\varphi+\sin(\varphi+\frac{\pi}{k}))}{8}\times$$

$$\times\sin\frac{u(s-t)\sin\varphi}{2}\sin\frac{(\lambda+u)(t+s)(\sin(\varphi+\frac{\pi}{k})-\sin\varphi)}{8}+$$

$$+2\sin\frac{(\lambda-u)(t+s)\sin(\varphi+\frac{\pi}{k})}{4}\cos\frac{(\lambda+u)(t+s)\sin(\varphi+\frac{\pi}{k})}{4}\times$$

$$\times\cos\frac{u(s-t)(\sin\varphi+\sin(\varphi+\frac{\pi}{k}))}{4}\sin\frac{u(s-t)(\sin\varphi-\sin(\varphi+\frac{\pi}{k}))}{4}\Bigg|\,d\varphi\leqslant$$

$$\leqslant 2\int\limits_{-\pi}^{\pi}|\cos(k\varphi)|\cdot\Bigg[\Bigg|\sin\frac{\lambda(t+s)\sin\varphi}{2}\cos\frac{(\lambda-u)(s-t)(\sin\varphi+\sin(\varphi+\frac{\pi}{k}))}{8}\times$$

$$\times\sin\frac{(\lambda-u)(s-t)\cos(\varphi+\frac{\pi}{2k})\sin(-\frac{\pi}{2k})}{4}\cos\frac{(\lambda+u)(s-t)\sin\varphi}{4}\Bigg|+$$

$$+\Bigg|\sin\frac{(\lambda-u)(s-t)\sin(\varphi+\frac{\pi}{k})}{4}\sin\frac{(\lambda+u)(s-t)(\sin\varphi+\sin(\varphi+\frac{\pi}{k}))}{8}\times$$

$$\times\sin\frac{\lambda(t+s)\sin\varphi}{2}\sin\frac{(\lambda+u)(s-t)\cos(\varphi+\frac{\pi}{2k})\sin\frac{\pi}{2k}}{4}\Bigg|+$$

$$+\Bigg|\sin\frac{(\lambda-u)(s-t)\sin(\varphi+\frac{\pi}{k})}{4}\cos\frac{(\lambda+u)(s-t)\sin(\varphi+\frac{\pi}{k})}{4}\times$$

$$\times\cos\frac{\lambda(t+s)(\sin\varphi+\sin(\varphi+\frac{\pi}{k}))}{4}\sin\frac{\lambda(t+s)\cos(\varphi+\frac{\pi}{2k})\sin(-\frac{\pi}{2k})}{2}\Bigg|+$$

$$+\Bigg|\sin\frac{u(s-t)\sin\varphi}{2}\cos\frac{(\lambda-u)(t+s)(\sin\varphi+\sin(\varphi+\frac{\pi}{k}))}{8}\times$$

$$\times\sin\frac{(\lambda-u)(t+s)\cos(\varphi+\frac{\pi}{2k})\sin(-\frac{\pi}{2k})}{4}\cos\frac{(\lambda+u)(t+s)\sin\varphi}{4}\Bigg|+$$

$$+\left|\sin\frac{(\lambda-u)(t+s)\sin(\varphi+\frac{\pi}{k})}{4}\sin\frac{u(s-t)\sin\varphi}{2}\times\right.$$

$$\left.\times\sin\frac{(\lambda+u)(t+s)(\sin\varphi+\sin(\varphi+\frac{\pi}{k}))}{8}\sin\frac{(\lambda+u)(t+s)\cos(\varphi+\frac{\pi}{2k})\sin\frac{\pi}{2k}}{4}\right|+$$

$$+\left|\sin\frac{(\lambda-u)(t+s)\sin(\varphi+\frac{\pi}{k})}{4}\cos\frac{(\lambda+u)(t+s)\sin(\varphi+\frac{\pi}{k})}{4}\times\right.$$

$$\left.\left.\times\cos\frac{u(s-t)(\sin\varphi+\sin(\varphi+\frac{\pi}{k}))}{4}\sin\frac{u(s-t)\cos(\varphi+\frac{\pi}{2k})\sin(-\frac{\pi}{2k})}{2}\right|\right]d\varphi\leqslant$$

$$\leqslant 4\pi\left(\frac{|\lambda-u|^\alpha|s-t|^\alpha}{4^\alpha}\left(\frac{\pi}{2k}\right)^\alpha+\frac{|\lambda-u|^\alpha|s-t|^{2\alpha}|\lambda+u|^\alpha}{4^{2\alpha}}\left(\frac{\pi}{2k}\right)^\alpha+\right.$$

$$+\frac{|\lambda-u|^\alpha|s-t|^\alpha|t+s|^\alpha\lambda^\alpha}{4^\alpha\cdot 2^\alpha}\left(\frac{\pi}{2k}\right)^\alpha+\frac{|\lambda-u|^\alpha|s-t|^\alpha|t+s|^\alpha u^\alpha}{4^\alpha\cdot 2^\alpha}\left(\frac{\pi}{2k}\right)^\alpha+$$

$$+\frac{|\lambda-u|^\alpha|s-t|^\alpha|t+s|^{2\alpha}u^\alpha|\lambda+u|^\alpha}{4^{2\alpha}\cdot 2^\alpha}\left(\frac{\pi}{2k}\right)^\alpha+\frac{|\lambda-u|^\alpha|t+s|^\alpha|s-t|^\alpha u^\alpha}{4^\alpha\cdot 2^\alpha}\times$$

$$\left.\times\left(\frac{\pi}{2k}\right)^\alpha\right)=4^{1-\alpha}\pi|\lambda-u|^\alpha|s-t|^\alpha\left(\frac{\pi}{2k}\right)^\alpha\left(1+\frac{|\lambda+u|^\alpha|s-t|^\alpha}{4^\alpha}+\right.$$

$$\left.+\frac{|t+s|^\alpha(\lambda^\alpha+2u^\alpha)}{2^\alpha}+\frac{|t+s|^{2\alpha}u^\alpha|\lambda+u|^\alpha}{4^\alpha\cdot 2^\alpha}\right).$$

Similarly, we obtain a bound for $|K_2|$

$$|K_2|=\frac{1}{4}\left|\int\limits_{-\pi}^{\pi}\sin(k\varphi)\left[(\sin(t\lambda\sin\varphi)-\sin(s\lambda\sin\varphi))-\left(\sin\left(t\lambda\sin\left(\varphi+\frac{\pi}{k}\right)\right)-\right.\right.\right.$$

$$\left.-\sin\left(s\lambda\sin\left(\varphi+\frac{\pi}{k}\right)\right)\right)-(\sin(tu\sin\varphi)-\sin(su\sin\varphi))+$$

$$\left.\left.+\left(\sin\left(tu\sin\left(\varphi+\frac{\pi}{k}\right)\right)-\sin\left(su\sin\left(\varphi+\frac{\pi}{k}\right)\right)\right)\right]d\varphi\right|\leqslant\frac{1}{4}\int\limits_{-\pi}^{\pi}|\sin(k\varphi)|\times$$

$$\times\left|2\cos\frac{\lambda(t+s)\sin\varphi}{2}\sin\frac{\lambda(t-s)\sin\varphi}{2}-2\cos\frac{\lambda(t+s)\sin(\varphi+\frac{\pi}{k})}{2}\times\right.$$

$$\times\sin\frac{\lambda(t-s)\sin(\varphi+\frac{\pi}{k})}{2}-2\cos\frac{u(t+s)\sin\varphi}{2}\sin\frac{u(t-s)\sin\varphi}{2}+$$

$$\left.+2\cos\frac{u(t+s)\sin(\varphi+\frac{\pi}{k})}{2}\sin\frac{u(t-s)\sin(\varphi+\frac{\pi}{k})}{2}\right|d\varphi=\frac{1}{2}\int\limits_{-\pi}^{\pi}|\sin(k\varphi)|\times$$

$$\times\left|\left(\cos\frac{\lambda(t+s)\sin\varphi}{2}\sin\frac{\lambda(t-s)\sin\varphi}{2}-\cos\frac{u(t+s)\sin\varphi}{2}\times\right.\right.$$

$$\left.\times\sin\frac{u(t-s)\sin\varphi}{2}\right)-\left(\cos\frac{\lambda(t+s)\sin(\varphi+\frac{\pi}{k})}{2}\sin\frac{\lambda(t-s)\sin(\varphi+\frac{\pi}{k})}{2}-\right.$$

$$\left.\left.-\cos\frac{u(t+s)\sin(\varphi+\frac{\pi}{k})}{2}\sin\frac{u(t-s)\sin(\varphi+\frac{\pi}{k})}{2}\right)\right|d\varphi=\frac{1}{2}\int\limits_{-\pi}^{\pi}|\sin(k\varphi)|\times$$

$$\times\left|\cos\frac{\lambda(t+s)\sin\varphi}{2}\left(\sin\frac{\lambda(t-s)\sin\varphi}{2}-\sin\frac{u(t-s)\sin\varphi}{2}\right)+\right.$$

$$+\sin\frac{u(t-s)\sin\varphi}{2}\left(\cos\frac{\lambda(t+s)\sin\varphi}{2}-\cos\frac{u(t+s)\sin\varphi}{2}\right)-$$

$$-\cos\frac{\lambda(t+s)\sin(\varphi+\frac{\pi}{k})}{2}\left(\sin\frac{\lambda(t-s)\sin(\varphi+\frac{\pi}{k})}{2}-\right.$$

$$\left.-\sin\frac{u(t-s)\sin(\varphi+\frac{\pi}{k})}{2}\right)-\sin\frac{u(t-s)\sin(\varphi+\frac{\pi}{k})}{2}\times$$

$$\left.\times\left(\cos\frac{\lambda(t+s)\sin(\varphi+\frac{\pi}{k})}{2}-\cos\frac{u(t+s)\sin(\varphi+\frac{\pi}{k})}{2}\right)\right|d\varphi=$$

$$=\frac{1}{2}\int\limits_{-\pi}^{\pi}|\sin(k\varphi)|\left|2\cos\frac{\lambda(t+s)\sin\varphi}{2}\cos\frac{(\lambda+u)(t-s)\sin\varphi}{4}\times\right.$$

$$\times\sin\frac{(\lambda-u)(t-s)\sin\varphi}{4}+2\sin\frac{u(t-s)\sin\varphi}{2}\sin\frac{(\lambda+u)(t+s)\sin\varphi}{4}\times$$

$$\times\sin\frac{(u-\lambda)(t+s)\sin\varphi}{4}-2\cos\frac{\lambda(t+s)\sin(\varphi+\frac{\pi}{k})}{2}\times$$

$$\times\cos\frac{(\lambda+u)(t-s)\sin(\varphi+\frac{\pi}{k})}{4}\sin\frac{(\lambda-u)(t-s)\sin(\varphi+\frac{\pi}{k})}{4}-$$

$$-2\sin\frac{u(t-s)\sin(\varphi+\frac{\pi}{k})}{2}\sin\frac{(\lambda+u)(t+s)\sin(\varphi+\frac{\pi}{k})}{4}\times$$

$$\left.\times\sin\frac{(u-\lambda)(t+s)\sin(\varphi+\frac{\pi}{k})}{4}\right|d\varphi=\int\limits_{-\pi}^{\pi}|\sin(k\varphi)|\left|\cos\frac{\lambda(t+s)\sin\varphi}{2}\times\right.$$

$$\times\cos\frac{(\lambda+u)(t-s)\sin\varphi}{4}\left(\sin\frac{(\lambda-u)(t-s)\sin\varphi}{4}-\right.$$

$$\left.-\sin\frac{(\lambda-u)(t-s)\sin(\varphi+\frac{\pi}{k})}{4}\right)+\sin\frac{(\lambda-u)(t-s)\sin(\varphi+\frac{\pi}{k})}{4}\times$$

$$\times\left(\cos\frac{\lambda(t+s)\sin\varphi}{2}\cos\frac{(\lambda+u)(t-s)\sin\varphi}{4}-\cos\frac{\lambda(t+s)\sin(\varphi+\frac{\pi}{k})}{2}\times\right.$$

$$\left.\times\cos\frac{(\lambda+u)(t-s)\sin(\varphi+\frac{\pi}{k})}{4}\right)+\sin\frac{u(t-s)\sin\varphi}{2}\times$$

$$\times\sin\frac{(\lambda+u)(t+s)\sin\varphi}{4}\left(\sin\frac{(u-\lambda)(t+s)\sin\varphi}{4}-\right.$$

$$\left.-\sin\frac{(u-\lambda)(t+s)\sin(\varphi+\frac{\pi}{k})}{4}\right)+\sin\frac{(u-\lambda)(t+s)\sin(\varphi+\frac{\pi}{k})}{4}\times$$

$$\times\left(\sin\frac{u(t-s)\sin\varphi}{2}\sin\frac{(\lambda+u)(t+s)\sin\varphi}{4}-\sin\frac{u(t-s)\sin(\varphi+\frac{\pi}{k})}{2}\times\right.$$

$$\left.\times\sin\frac{(\lambda+u)(t+s)\sin(\varphi+\frac{\pi}{k})}{4}\right)\Bigg|\,d\varphi=\int\limits_{-\pi}^{\pi}|\sin(k\varphi)|\,\Bigg|2\cos\frac{\lambda(t+s)\sin\varphi}{2}\times$$

$$\times\cos\frac{(\lambda+u)(t-s)\sin\varphi}{4}\cos\frac{(\lambda-u)(t-s)(\sin\varphi+\sin(\varphi+\frac{\pi}{k}))}{8}\times$$

$$\times\sin\frac{(\lambda-u)(t-s)(\sin\varphi-\sin(\varphi+\frac{\pi}{k}))}{8}+\sin\frac{(\lambda-u)(t-s)\sin(\varphi+\frac{\pi}{k})}{4}\times$$

$$\times\left[\left(\cos\frac{(\lambda+u)(t-s)\sin\varphi}{4}-\cos\frac{(\lambda+u)(t-s)\sin(\varphi+\frac{\pi}{k})}{4}\right)\times\right.$$

$$\times\cos\frac{\lambda(t+s)\sin\varphi}{2}+\cos\frac{(\lambda+u)(t-s)\sin(\varphi+\frac{\pi}{k})}{4}\left(\cos\frac{\lambda(t+s)\sin\varphi}{2}-\right.$$

$$\left.\left.-\cos\frac{\lambda(t+s)\sin(\varphi+\frac{\pi}{k})}{2}\right)\right]+2\sin\frac{u(t-s)\sin\varphi}{2}\sin\frac{(\lambda+u)(t+s)\sin\varphi}{4}\times$$

$$\times\cos\frac{(u-\lambda)(t+s)(\sin\varphi+\sin(\varphi+\frac{\pi}{k}))}{8}\times$$

$$\times\sin\frac{(u-\lambda)(t+s)(\sin\varphi-\sin(\varphi+\frac{\pi}{k}))}{8}+\sin\frac{(u-\lambda)(t+s)\sin(\varphi+\frac{\pi}{k})}{4}\times$$

$$\times\left[\left(\sin\frac{(\lambda+u)(t+s)\sin\varphi}{4}-\sin\frac{(\lambda+u)(t+s)\sin(\varphi+\frac{\pi}{k})}{4}\right)\times\right.$$

$$\times\sin\frac{u(t-s)\sin\varphi}{2}+\sin\frac{(\lambda+u)(t+s)\sin(\varphi+\frac{\pi}{k})}{4}\left(\sin\frac{u(t-s)\sin\varphi}{2}-\right.$$

$$\left.\left.-\sin\frac{u(t-s)\sin(\varphi+\frac{\pi}{k})}{2}\right)\right]\Bigg|\,d\varphi=\int\limits_{-\pi}^{\pi}|\sin(k\varphi)|\cdot\Bigg|2\cos\frac{\lambda(t+s)\sin\varphi}{2}\times$$

$$\times\cos\frac{(\lambda+u)(t-s)\sin\varphi}{4}\cos\frac{(\lambda-u)(t-s)(\sin\varphi+\sin(\varphi+\frac{\pi}{k}))}{8}\times$$

$$
\begin{aligned}
&\times \sin\frac{(\lambda-u)(t-s)\cos(\varphi+\frac{\pi}{2k})\sin(-\frac{\pi}{2k})}{4}+2\sin\frac{(\lambda-u)(t-s)\sin(\varphi+\frac{\pi}{k})}{4}\times\\
&\times \sin\frac{(\lambda+u)(t-s)(\sin\varphi+\sin(\varphi+\frac{\pi}{k}))}{8}\cos\frac{\lambda(t+s)\sin\varphi}{2}\times\\
&\times \sin\frac{(\lambda+u)(t-s)(\sin(\varphi+\frac{\pi}{k})-\sin\varphi)}{8}+2\sin\frac{(\lambda-u)(t-s)\sin(\varphi+\frac{\pi}{k})}{4}\times\\
&\times \cos\frac{(\lambda+u)(t-s)\sin(\varphi+\frac{\pi}{k})}{4}\sin\frac{\lambda(t+s)(\sin\varphi+\sin(\varphi+\frac{\pi}{k}))}{4}\times\\
&\times \sin\frac{\lambda(t+s)(\sin\varphi-\sin(\varphi+\frac{\pi}{k}))}{4}+2\sin\frac{u(t-s)\sin\varphi}{2}\times\\
&\times \sin\frac{(\lambda+u)(t+s)\sin\varphi}{4}\cos\frac{(u-\lambda)(t+s)(\sin\varphi+\sin(\varphi+\frac{\pi}{k}))}{8}\times\\
&\times \sin\frac{(u-\lambda)(t+s)\cos(\varphi+\frac{\pi}{2k})\sin(-\frac{\pi}{2k})}{4}+2\sin\frac{(u-\lambda)(t+s)\sin(\varphi+\frac{\pi}{k})}{4}\times\\
&\times \sin\frac{u(t-s)\sin\varphi}{2}\cos\frac{(\lambda+u)(t+s)(\sin\varphi+\sin(\varphi+\frac{\pi}{k}))}{8}\times\\
&\times \sin\frac{(\lambda+u)(t+s)(\sin\varphi-\sin(\varphi+\frac{\pi}{k}))}{8}+2\sin\frac{(u-\lambda)(t+s)\sin(\varphi+\frac{\pi}{k})}{4}\times\\
&\times \sin\frac{(\lambda+u)(t+s)\sin(\varphi+\frac{\pi}{k})}{4}\cos\frac{u(t-s)(\sin\varphi+\sin(\varphi+\frac{\pi}{k}))}{4}\times\\
&\times \sin\frac{u(t-s)(\sin\varphi-\sin(\varphi+\frac{\pi}{k}))}{4}\Bigg|\,d\varphi\leqslant 2\int\limits_{-\pi}^{\pi}|\sin(k\varphi)|\cdot\Bigg[\Bigg|\cos\frac{\lambda(t+s)\sin\varphi}{2}\times\\
&\times \cos\frac{(\lambda+u)(t-s)\sin\varphi}{4}\cos\frac{(\lambda-u)(t-s)(\sin\varphi+\sin(\varphi+\frac{\pi}{k}))}{8}\times\\
&\times \sin\frac{(\lambda-u)(t-s)\cos(\varphi+\frac{\pi}{2k})\sin(-\frac{\pi}{2k})}{4}\Bigg|+\Bigg|\sin\frac{(\lambda-u)(t-s)\sin(\varphi+\frac{\pi}{k})}{4}\times\\
&\times \cos\frac{\lambda(t+s)\sin\varphi}{2}\sin\frac{(\lambda+u)(t-s)(\sin\varphi+\sin(\varphi+\frac{\pi}{k}))}{8}\times\\
&\times \sin\frac{(\lambda+u)(t-s)\cos(\varphi+\frac{\pi}{2k})\sin\frac{\pi}{2k}}{4}\Bigg|+\Bigg|\sin\frac{(\lambda-u)(t-s)\sin(\varphi+\frac{\pi}{k})}{4}\times\\
&\times \cos\frac{(\lambda+u)(t-s)\sin(\varphi+\frac{\pi}{k})}{4}\sin\frac{\lambda(t+s)(\sin\varphi+\sin(\varphi+\frac{\pi}{k}))}{4}\times\\
&\times \sin\frac{\lambda(t+s)\cos(\varphi+\frac{\pi}{2k})\sin\frac{\pi}{2k}}{2}\Bigg|+\Bigg|\sin\frac{u(t-s)\sin\varphi}{2}\times\\
&\times \sin\frac{(\lambda+u)(t+s)\sin\varphi}{4}\cos\frac{(u-\lambda)(t+s)(\sin\varphi+\sin(\varphi+\frac{\pi}{k}))}{8}\times
\end{aligned}
$$

$$\times \sin \frac{(u-\lambda)(t+s)\cos(\varphi+\frac{\pi}{2k})\sin(-\frac{\pi}{2k})}{4}\Bigg| + \Bigg|\sin \frac{(u-\lambda)(t+s)\sin(\varphi+\frac{\pi}{k})}{4} \times$$

$$\times \sin \frac{u(t-s)\sin\varphi}{2} \cos \frac{(\lambda+u)(t+s)(\sin\varphi+\sin(\varphi+\frac{\pi}{k}))}{8} \times$$

$$\times \sin \frac{(\lambda+u)(t+s)\cos(\varphi+\frac{\pi}{2k})\sin(-\frac{\pi}{2k})}{4}\Bigg| + \Bigg|\sin \frac{(u-\lambda)(t+s)\sin(\varphi+\frac{\pi}{k})}{4} \times$$

$$\times \sin \frac{(\lambda+u)(t+s)\sin(\varphi+\frac{\pi}{k})}{4} \cos \frac{u(t-s)(\sin\varphi+\sin(\varphi+\frac{\pi}{k}))}{4} \times$$

$$\times \sin \frac{u(t-s)\cos(\varphi+\frac{\pi}{2k})\sin(-\frac{\pi}{2k})}{2}\Bigg|\Bigg] d\varphi \leqslant 4\pi \left(\frac{|\lambda-u|^\alpha |s-t|^\alpha}{4^\alpha} \left(\frac{\pi}{2k}\right)^\alpha + \right.$$

$$+ \frac{|\lambda-u|^\alpha |s-t|^{2\alpha} |\lambda+u|^\alpha}{4^{2\alpha}} \left(\frac{\pi}{2k}\right)^\alpha + \frac{|\lambda-u|^\alpha |s-t|^\alpha |t+s|^\alpha \lambda^\alpha}{4^\alpha \cdot 2^\alpha} \left(\frac{\pi}{2k}\right)^\alpha +$$

$$+ \frac{|\lambda-u|^\alpha |s-t|^\alpha |t+s|^\alpha u^\alpha}{4^\alpha \cdot 2^\alpha} \left(\frac{\pi}{2k}\right)^\alpha + \frac{|\lambda-u|^\alpha |s-t|^\alpha |t+s|^{2\alpha} u^\alpha |\lambda+u|^\alpha}{4^{2\alpha} \cdot 2^\alpha} \times$$

$$\left. \times \left(\frac{\pi}{2k}\right)^\alpha + \frac{|\lambda-u|^\alpha |t+s|^\alpha |s-t|^\alpha u^\alpha}{4^\alpha \cdot 2^\alpha} \left(\frac{\pi}{2k}\right)^\alpha \right) = 4^{1-\alpha}\pi |\lambda-u|^\alpha |s-t|^\alpha \times$$

$$\times \left(\frac{\pi}{2k}\right)^\alpha \left(1 + \frac{|\lambda+u|^\alpha |s-t|^\alpha}{4^\alpha} + \frac{|t+s|^\alpha(\lambda^\alpha+2u^\alpha)}{2^\alpha} + \frac{|t+s|^{2\alpha} u^\alpha |\lambda+u|^\alpha}{4^\alpha \cdot 2^\alpha}\right).$$

Then, we obtain

$$|J_k(t\lambda) - J_k(tu) - J_k(s\lambda) + J_k(su)| \leqslant 2 \cdot 4^{1-\alpha} |\lambda-u|^\alpha |s-t|^\alpha \left(\frac{\pi}{2k}\right)^\alpha \times$$

$$\times \left(1 + \frac{|\lambda+u|^\alpha |s-t|^\alpha}{4^\alpha} + \frac{|t+s|^\alpha(\lambda^\alpha+2u^\alpha)}{2^\alpha} + \frac{|t+s|^{2\alpha} u^\alpha |\lambda+u|^\alpha}{4^\alpha \cdot 2^\alpha}\right). \diamondsuit$$

3.4. Construction of the model of homogeneous and isotropic stochastic field

Definition 3.2. [141] A stochastic field $X = \{X(t), t \in \mathbb{R}^2\}$ is called homogeneous in the wide sense in $\mathbb{R}^2$ if $\mathbb{E}X(t) = const, t \in \mathbb{R}^2$ and

$$\mathbb{E}X(t)X(s) = B(t-s) = \int\limits_{\mathbb{R}^2} e^{i(\lambda, t-s)} dF(\lambda), t, s \in \mathbb{R}^2.$$

Definition 3.3. [141] Let $SO(2)$ be the group of all rotations about the origin of the space $\mathbb{R}^2$. A homogeneous stochastic field $X(t), t \in \mathbb{R}^2$ is called isotropic if

$$\mathbb{E}X(t)X(s) = \mathbb{E}X(gt)X(gs),$$

for all elements g of the group $SO(2)$ and for all $t, s \in \mathbb{R}^2$.

Let $X = \{X(t,x), t \in \mathbb{R}, x \in [0, 2\pi]\}$ be a mean square continuous real Gaussian homogeneous and isotropic stochastic field on $\mathbb{R}^2$. The following representation is obtained similarly to [141] where complex valued fields are considered:

$$X(t,x) = \sum_{k=1}^{\infty} \cos(kx) \int_0^{\infty} J_k(t\lambda) d\eta_{1,k}(\lambda) + \\ + \sum_{k=1}^{\infty} \sin(kx) \int_0^{\infty} J_k(t\lambda) d\eta_{2,k}(\lambda), \quad (3.4)$$

where $\eta_{i,k}(\lambda), i = 1,2, k = \overline{1,\infty}$ are independent Gaussian processes with independent increments, $\mathbf{E}\eta_{i,k}(\lambda) = 0$, $\mathbf{E}(\eta_{i,k}(b) - \eta_{i,k}(c))^2 = F(b) - F(c), b > c$, $F(\lambda)$ is the spectral function of the field. Let $J_k(u) = \frac{1}{\pi}\int_0^{\pi} \cos(k\varphi - u\sin\varphi)d\varphi$ be the integral representation of the Bessel functions of the first kind.

Consider a partition $L = \{\lambda_0, ..., \lambda_N\}$ of the set $[0, \infty)$ such that $\lambda_0 = 0, \lambda_l < \lambda_{l+1}, \lambda_{N-1} = \Lambda, \lambda_N = \infty$ and $C = \max_{0<l\leqslant N-2} \frac{\lambda_{l+1}}{\lambda_l} < \infty$.

The process

$$\hat{X}(t,x) = \sum_{k=1}^{M} \cos(kx) \sum_{l=0}^{N-1} \eta_{1,k,l} J_k(t\zeta_l) + \sum_{k=1}^{M} \sin(kx) \sum_{l=0}^{N-1} \eta_{2,k,l} J_k(t\zeta_l),$$

is viewed as a model of the field $X(t,x)$ where $\eta_{i,k,l}, i = 1,2$ are independent Gaussian random variables,

$$\eta_{i,k,l} = \int_{\lambda_l}^{\lambda_{l+1}} d\eta_{i,k}(\lambda)$$

are such that $\mathbf{E}\eta_{i,k,l} = 0$, $\mathbf{E}\eta_{i,k,l}^2 = F(\lambda_{l+1}) - F(\lambda_l) = b_l^2$, $\zeta_l, l = 0, ..., N-2$ are independent random variables being independent of $\eta_{i,k,l}$ and assuming values in the intervals $[\lambda_l, \lambda_{l+1}]$, $\zeta_{N-1} = \Lambda$, $b_l^2 > 0$ are such that

$$F_l(\lambda) = P\{\zeta_l < \lambda\} = \frac{F(\lambda) - F(\lambda_l)}{F(\lambda_{l+1}) - F(\lambda_l)}.$$

If $b_l^2 = 0$ then $\zeta_l = 0$ with probability 1. For the sake of simplicity assume that $b_l^2 > 0, l = 0, 1, ..., N-1$.

Thus $\hat{X}(t,x)$ is written as follows

$$\hat{X}(t,x) = \sum_{k=1}^{M} \cos(kx) \sum_{l=0}^{N-1} \int\limits_{\lambda_l}^{\lambda_{l+1}} J_k(t\zeta_l) d\eta_{1,k}(\lambda) + \\ + \sum_{k=1}^{M} \sin(kx) \sum_{l=0}^{N-1} \int\limits_{\lambda_l}^{\lambda_{l+1}} J_k(t\zeta_l) d\eta_{2,k}(\lambda). \quad (3.5)$$

Note that $X(t,x)$ admits the following representation

$$X(t,x) = \sum_{k=1}^{\infty} \cos(kx) \sum_{l=0}^{N-1} \int\limits_{\lambda_l}^{\lambda_{l+1}} J_k(t\lambda) d\eta_{1,k}(\lambda) + \\ + \sum_{k=1}^{\infty} \sin(kx) \sum_{l=0}^{N-1} \int\limits_{\lambda_l}^{\lambda_{l+1}} J_k(t\lambda) d\eta_{2,k}(\lambda). \quad (3.6)$$

Consider the deviation $X(t,x) - \hat{X}(t,x)$ and put

$$\chi_M(t,x) = X(t,x) - \hat{X}(t,x) = \\ = \left(\sum_{k=1}^{M} \cos(kx) \sum_{l=0}^{N-1} \int\limits_{\lambda_l}^{\lambda_{l+1}} (J_k(t\lambda) - J_k(t\zeta_l)) d\eta_{1,k}(\lambda) + \right.$$

$$\left. + \sum_{k=M+1}^{\infty} \cos(kx) \int\limits_{0}^{\infty} J_k(t\lambda) d\eta_{1,k}(\lambda) \right) + \\ + \left(\sum_{k=1}^{M} \sin(kx) \sum_{l=0}^{N-1} \int\limits_{\lambda_l}^{\lambda_{l+1}} (J_k(t\lambda) - J_k(t\zeta_l)) d\eta_{2,k}(\lambda) + \right. \\ \left. + \sum_{k=M+1}^{\infty} \sin(kx) \int\limits_{0}^{\infty} J_k(t\lambda) d\eta_{2,k}(\lambda) \right) =: \\ =: \chi_{M,1}(t,x) + \chi_{M,2}(t,x). \quad (3.7)$$

Denote the two terms on the right hand side of (ref313) by $\chi_{M,1}(t,x)$ and $\chi_{M,2}(t,x)$. Then

$$\tau(\chi_M(t,x)) \leqslant \tau(\chi_{M,1}(t,x)) + \tau(\chi_{M,2}(t,x)). \tag{3.8}$$

According to the Lemma 1.2 the following inequality holds

$$\tau^2(\chi_{M,1}(t,x)) \leqslant \tau^2\left(\sum_{k=1}^{M}\cos(kx)\sum_{l=0}^{N-1}\int\limits_{\lambda_l}^{\lambda_{l+1}}(J_k(t\lambda)-J_k(t\zeta_l))d\eta_{1,k}(\lambda)\right)+ \\ +\tau^2\left(\sum_{k=M+1}^{\infty}\cos(kx)\int\limits_0^{\infty}J_k(t\lambda)d\eta_{1,k}(\lambda)\right),$$

$$\tau^2(\chi_{M,2}(t,x)) \leqslant \tau^2\left(\sum_{k=1}^{M}\sin(kx)\sum_{l=0}^{N-1}\int\limits_{\lambda_l}^{\lambda_{l+1}}(J_k(t\lambda)-J_k(t\zeta_l))d\eta_{2,k}(\lambda)\right)+ \\ +\tau^2\left(\sum_{k=M+1}^{\infty}\sin(kx)\int\limits_0^{\infty}J_k(t\lambda)d\eta_{2,k}(\lambda)\right).$$

Lemma 3.7. *For all $\frac{1}{2} < \alpha \leqslant 1$*

$$\tau^2\left(\sum_{k=1}^{M}\cos(kx)\sum_{l=0}^{N-1}\int\limits_{\lambda_l}^{\lambda_{l+1}}(J_k(t\lambda)-J_k(t\zeta_l))d\eta_{1,k}(\lambda)\right)\leqslant \\ \leqslant \frac{M}{2\alpha-1}\left(2\alpha-\frac{1}{M^{2\alpha-1}}\right)\cdot 2\cdot 4^{2(1-\alpha)}\pi^{2\alpha}t^{2\alpha}\sum_{l=0}^{N-2}\mid\lambda_{l+1}-\lambda_l\mid^{2\alpha}\times \\ \times\left(b_l^2+\left(\frac{t(1+C)}{2}\right)^{2\alpha}\int\limits_{\lambda_l}^{\lambda_{l+1}}\lambda^{2\alpha}dF(\lambda)\right)+4M^2(F(+\infty)-F(\Lambda)),$$

$$\tau^2\left(\sum_{k=1}^{M}\sin(kx)\sum_{l=0}^{N-1}\int\limits_{\lambda_l}^{\lambda_{l+1}}(J_k(t\lambda)-J_k(t\zeta_l))d\eta_{2,k}(\lambda)\right)\leqslant$$

$$\leqslant \frac{M}{2\alpha-1}\left(2\alpha-\frac{1}{M^{2\alpha-1}}\right)\cdot 2\cdot 4^{2(1-\alpha)}\pi^{2\alpha}t^{2\alpha}\sum_{l=0}^{N-2}\mid \lambda_{l+1}-\lambda_l \mid^{2\alpha}\times$$

$$\times\left(b_l^2+\left(\frac{t(1+C)}{2}\right)^{2\alpha}\int\limits_{\lambda_l}^{\lambda_{l+1}}\lambda^{2\alpha}dF(\lambda)\right)+4M^2(F(+\infty)-F(\Lambda)),$$

where $C=\max\limits_{0<l\leqslant N-2}\frac{\lambda_{l+1}}{\lambda_l}$.

Proof. Since

$$(a_1+a_2+...+a_n)^2\leqslant n(a_1^2+a_2^2+...+a_n^2),$$

for all real $a_1,a_2,...,a_n$, we derive from Lemmas 1.2 and 1.3 that

$$\tau^2\left(\sum_{k=1}^{M}\cos(kx)\sum_{l=0}^{N-1}\int\limits_{\lambda_l}^{\lambda_{l+1}}(J_k(t\lambda)-J_k(t\zeta_l))d\eta_{1,k}(\lambda)\right)\leqslant$$

$$\leqslant M\sum_{k=1}^{M}\sum_{l=0}^{N-1}\tau^2\left(\int\limits_{\lambda_l}^{\lambda_{l+1}}(J_k(t\lambda)-J_k(t\zeta_l))d\eta_{1,k}(\lambda)\right)\leqslant$$

$$\leqslant M\sum_{k=1}^{M}\sum_{l=0}^{N-1}\theta^2\left(\int\limits_{\lambda_l}^{\lambda_{l+1}}(J_k(t\lambda)-J_k(t\zeta_l))d\eta_{1,k}(\lambda)\right)=$$

$$=M\sum_{k=1}^{M}\sum_{l=0}^{N-1}\sup_{m\geqslant 1}\left[\frac{2^m\cdot m!}{(2m)!}E\left(\int\limits_{\lambda_l}^{\lambda_{l+1}}(J_k(t\lambda)-J_k(t\zeta_l))d\eta_{1,k}(\lambda)\right)^{2m}\right]^{\frac{1}{m}}.$$

Since $E\xi=0$, $E\xi^{2k+1}=0$, $E\xi^{2k}=\frac{(2k)!}{2^k\cdot k!}\sigma^{2k}$ for a centered Gaussian random variable ξ and since the random variables ζ_l do not depend on $\eta_{i,k}(\lambda), i=1,2$, by Fubini's theorem, Cauchy–Bunyakovskiy inequality, and Lemma 3.4 we obtained

$$E\left(\int\limits_{\lambda_l}^{\lambda_{l+1}}(J_k(t\lambda)-J_k(t\zeta_l))d\eta_{1,k}(\lambda)\right)^{2m}=$$

$$= EE_{\zeta_l}\left(\int\limits_{\lambda_l}^{\lambda_{l+1}}(J_k(t\lambda)-J_k(t\zeta_l))d\eta_{1,k}(\lambda)\right)^{2m} \leqslant$$

$$\leqslant \frac{(2m)!}{2^m\cdot m!}E\left(\int\limits_{\lambda_l}^{\lambda_{l+1}}|J_k(t\lambda)-J_k(t\zeta_l)|^2dF(\lambda)\right)^{m} \leqslant$$

$$\leqslant \frac{(2m)!}{2^m\cdot m!}E\left(\int\limits_{\lambda_l}^{\lambda_{l+1}}\left(4^{1-\alpha}|\lambda-\zeta_l|^{\alpha}\cdot\left(\frac{t\cdot\pi}{k}\right)^{\alpha}\left(1+\frac{t^{\alpha}|\lambda+\zeta_l|^{\alpha}}{2^{\alpha}}\right)\right)^2dF(\lambda)\right)^{m} =$$

$$= \frac{(2m)!}{2^m\cdot m!}\cdot 4^{2m(1-\alpha)}t^{2m\alpha}\pi^{2m\alpha}\left(\frac{1}{k}\right)^{2m\alpha}\times$$

$$\times E\left(\int\limits_{\lambda_l}^{\lambda_{l+1}}|\lambda-\zeta_l|^{2\alpha}\left(1+\frac{t^{\alpha}|\lambda+\zeta_l|^{\alpha}}{2^{\alpha}}\right)^2dF(\lambda)\right)^{m} =$$

$$= \frac{(2m)!}{2^m\cdot m!}\cdot 4^{2m(1-\alpha)}t^{2m\alpha}\pi^{2m\alpha}\left(\frac{1}{k}\right)^{2m\alpha}\times$$

$$\times\int\limits_{\lambda_l}^{\lambda_{l+1}}\left(\int\limits_{\lambda_l}^{\lambda_{l+1}}|\lambda-u|^{2\alpha}\left(1+\frac{t^{\alpha}|\lambda+u|^{\alpha}}{2^{\alpha}}\right)^2dF(\lambda)\right)^{m}dF_l(u) \leqslant$$

$$\leqslant \frac{(2m)!}{2^m\cdot m!}\cdot 4^{2m(1-\alpha)}t^{2m\alpha}\pi^{2m\alpha}\left(\frac{1}{k}\right)^{2m\alpha}|\lambda_{l+1}-\lambda_l|^{2m\alpha}\times$$

$$\times\int\limits_{\lambda_l}^{\lambda_{l+1}}\left(\int\limits_{\lambda_l}^{\lambda_{l+1}}\left(1+\frac{t^{\alpha}\lambda^{\alpha}|1+\frac{u}{\lambda}|^{\alpha}}{2^{\alpha}}\right)^2dF(\lambda)\right)^{m}dF_l(u) \leqslant$$

$$\leqslant \frac{(2m)!}{2^m\cdot m!}\cdot 4^{2m(1-\alpha)}t^{2m\alpha}\pi^{2m\alpha}\left(\frac{1}{k}\right)^{2m\alpha}|\lambda_{l+1}-\lambda_l|^{2m\alpha}\times$$

$$\times\left(\int\limits_{\lambda_l}^{\lambda_{l+1}}\left(1+\frac{t^{\alpha}\lambda^{\alpha}|1+\frac{\lambda_{l+1}}{\lambda_l}|^{\alpha}}{2^{\alpha}}\right)^2dF(\lambda)\right)^{m} \leqslant$$

$$\leqslant \frac{(2m)!}{2^m\cdot m!}\cdot 4^{2m(1-\alpha)}t^{2m\alpha}\pi^{2m\alpha}\left(\frac{1}{k}\right)^{2m\alpha}|\lambda_{l+1}-\lambda_l|^{2m\alpha}\times$$

$$\times\left(\int\limits_{\lambda_l}^{\lambda_{l+1}}\left(2+2\cdot\frac{t^{2\alpha}\lambda^{2\alpha}(1+C)^{2\alpha}}{2^{2\alpha}}\right)dF(\lambda)\right)^{m} =$$

$$= \frac{(2m)!}{2^m \cdot m!} \cdot 4^{2m(1-\alpha)} t^{2m\alpha} \pi^{2m\alpha} \left(\frac{1}{k}\right)^{2m\alpha} |\lambda_{l+1} - \lambda_l|^{2m\alpha} \times$$

$$\times \left(2b_l^2 + 2 \cdot \left(\frac{t(1+C)}{2}\right)^{2\alpha} \int\limits_{\lambda_l}^{\lambda_{l+1}} \lambda^{2\alpha} dF(\lambda)\right)^m .$$

Here the symbol E_{ζ_l} denotes the conditional expectation with respect to ζ_l. Note that

$$\left(\int\limits_{\Lambda}^{\infty} \left(\int\limits_{\Lambda}^{\infty} |J_k(t\lambda) - J_k(tu)|^2 dF(\lambda)\right)^m dF_l(u)\right)^{\frac{1}{m}} =$$

$$= \left(\int\limits_{\Lambda}^{\infty} \left(\int\limits_{\Lambda}^{\infty} \left|\frac{1}{\pi}\left(\int\limits_{0}^{\pi} \cos(k\varphi - t\lambda \sin\varphi) d\varphi - \right.\right.\right.\right.$$

$$\left.\left.\left.\left. - \int\limits_{0}^{\pi} \cos(k\varphi - tu \sin\varphi) d\varphi\right)\right|^2 dF(\lambda)\right)^m dF_l(u)\right)^{\frac{1}{m}} \leqslant$$

$$\leqslant \left(\int\limits_{\Lambda}^{\infty} \left(\int\limits_{\Lambda}^{\infty} \left(\frac{1}{\pi}\int\limits_{0}^{\pi} |\cos(k\varphi - t\lambda \sin\varphi) - \right.\right.\right.$$

$$\left.\left.\left. - \cos(k\varphi - tu \sin\varphi)|\, d\varphi\right)^2 dF(\lambda)\right)^m dF_l(u)\right)^{\frac{1}{m}} =$$

$$= \left(\int\limits_{\Lambda}^{\infty} \left(\int\limits_{\Lambda}^{\infty} \left(\frac{1}{\pi}\int\limits_{0}^{\pi} \left| 2\sin(k\varphi - \frac{t(\lambda+u)\sin\varphi}{2}) \times \right.\right.\right.\right.$$

$$\left.\left.\left.\left. \times \sin(\frac{t(\lambda-u)\sin\varphi}{2})\right| d\varphi\right)^2 dF(\lambda)\right)^m dF_l(u)\right)^{\frac{1}{m}} \leqslant$$

$$\leqslant 4\left(\int\limits_{\Lambda}^{\infty} \left(\int\limits_{\Lambda}^{\infty} dF(\lambda)\right)^m dF_l(u)\right)^{\frac{1}{m}} = 4(F(+\infty) - F(\Lambda)).$$

Whence

$$\tau^2 \left(\sum_{k=1}^{M} \cos(kx) \sum_{l=0}^{N-1} \int\limits_{\lambda_l}^{\lambda_{l+1}} (J_k(t\lambda) - J_k(t\zeta_l)) d\eta_{1,k}(\lambda)\right) \leqslant$$

$$\leqslant 4^{2(1-\alpha)} t^{2\alpha} \pi^{2\alpha} \cdot M \sum_{k=1}^{M} \sum_{l=0}^{N-1} \frac{1}{k^{2\alpha}} \times$$

$$\times \sup_{m \geqslant 1} \left(\int_{\lambda_l}^{\lambda_{l+1}} \left(\int_{\lambda_l}^{\lambda_{l+1}} |\lambda - u|^{2\alpha} \left(1 + \frac{t^\alpha |\lambda + u|^\alpha}{2^\alpha} \right)^2 dF(\lambda) \right)^m dF_l(u) \right)^{\frac{1}{m}} \leqslant$$

$$\leqslant 2 \cdot 4^{2(1-\alpha)} t^{2\alpha} \pi^{2\alpha} \cdot M \sum_{k=1}^{M} \frac{1}{k^{2\alpha}} \cdot \sum_{l=0}^{N-2} |\lambda_{l+1} - \lambda_l|^{2\alpha} \times$$

$$\times \left(b_l^2 + \left(\frac{t(1+C)}{2} \right)^{2\alpha} \int_{\lambda_l}^{\lambda_{l+1}} \lambda^{2\alpha} dF(\lambda) \right) + 4M \sum_{k=1}^{M} (F(+\infty) - F(\Lambda)).$$

The sum $\sum\limits_{k=1}^{M} \frac{1}{k^{2\alpha}}$ with $\frac{1}{2} < \alpha \leqslant 1$ can be estimated as follows

$$\sum_{k=1}^{M} \frac{1}{k^{2\alpha}} \leqslant 1 + \sum_{k=2}^{M} \int_{k-1}^{k} \frac{1}{x^{2\alpha}} dx = 1 + \int_{1}^{M} \frac{1}{x^{2\alpha}} dx = 1 + \frac{x^{1-2\alpha}}{1-2\alpha} \bigg|_1^M =$$

$$= \frac{2\alpha}{2\alpha - 1} - \frac{1}{(2\alpha - 1) M^{2\alpha - 1}}.$$

Then

$$\tau^2 \left(\sum_{k=1}^{M} \cos(kx) \sum_{l=0}^{N-1} \int_{\lambda_l}^{\lambda_{l+1}} (J_k(t\lambda) - J_k(t\zeta_l)) d\eta_{1,k}(\lambda) \right) \leqslant$$

$$\leqslant \frac{M}{2\alpha - 1} \left(2\alpha - \frac{1}{M^{2\alpha-1}} \right) 2 \cdot 4^{2(1-\alpha)} t^{2\alpha} \pi^{2\alpha} \sum_{l=0}^{N-2} |\lambda_{l+1} - \lambda_l|^{2\alpha} \times$$

$$\times \left(b_l^2 + \left(\frac{t(1+C)}{2} \right)^{2\alpha} \int_{\lambda_l}^{\lambda_{l+1}} \lambda^{2\alpha} dF(\lambda) \right) + 4M^2 (F(+\infty) - F(\Lambda)).$$

The second inequality is proved similarly. ♢

Lemma 3.8. *Let the integral* $\int\limits_0^\infty \lambda^{2\alpha} dF(\lambda) < \infty$ *converge for* $\frac{1}{2} < \alpha \leqslant 1$. *Then*

$$\tau^2 \left(\sum_{k=M+1}^{\infty} \cos(kx) \int_0^\infty J_k(t\lambda) d\eta_{1,k}(\lambda) \right) \leqslant$$

$$\leqslant 2^{2(1-\alpha)} t^{2\alpha} \pi^{2\alpha} \frac{1}{(2\alpha-1)M^{2\alpha-1}} \left(\int_0^\infty \lambda^{2\alpha} dF(\lambda) \right),$$

$$\tau^2 \left(\sum_{k=M+1}^{\infty} \sin(kx) \int_0^\infty J_k(t\lambda) d\eta_{2,k}(\lambda) \right) \leqslant$$

$$\leqslant 2^{2(1-\alpha)} t^{2\alpha} \pi^{2\alpha} \frac{1}{(2\alpha-1)M^{2\alpha-1}} \left(\int_0^\infty \lambda^{2\alpha} dF(\lambda) \right).$$

Proof. Indeed,

$$\tau^2 \left(\sum_{k=M+1}^{\infty} \cos(kx) \int_0^\infty J_k(t\lambda) d\eta_{1,k}(\lambda) \right) \leqslant$$

$$\leqslant \sum_{k=M+1}^{\infty} \tau^2 \left(\int_0^\infty J_k(t\lambda) d\eta_{1,k}(\lambda) \right) \leqslant \sum_{k=M+1}^{\infty} \theta^2 \left(\int_0^\infty J_k(t\lambda) d\eta_{1,k}(\lambda) \right) =$$

$$= \sum_{k=M+1}^{\infty} \sup_{m\geqslant 1} \left[\frac{2^m \cdot m!}{(2m)!} E \left(\int_0^\infty J_k(t\lambda) d\eta_{1,k}(\lambda) \right)^{2m} \right]^{\frac{1}{m}}.$$

Applying Lemma 3.3,

$$E \left(\int_0^\infty J_k(t\lambda) d\eta_{1,k}(\lambda) \right)^{2m} \leqslant \frac{(2m)!}{2^m \cdot m!} \left(\int_0^\infty |\ J_k(t\lambda)\ |^2 \ dF(\lambda) \right)^m \leqslant$$

$$\leqslant \frac{(2m)!}{2^m \cdot m!} \left(\int_0^\infty \left(2^{1-\alpha} \mid t\lambda \mid^\alpha \pi^\alpha \cdot \frac{1}{k^\alpha} \right)^2 dF(\lambda) \right)^m =$$

$$= \frac{(2m)!}{2^m \cdot m!} \cdot \frac{2^{2m(1-\alpha)} t^{2m\alpha} \pi^{2m\alpha}}{k^{2m\alpha}} \left(\int_0^\infty \lambda^{2\alpha} dF(\lambda) \right)^m,$$

whence

$$\tau^2 \left(\sum_{k=M+1}^{\infty} \cos(kx) \int_0^\infty J_k(t\lambda) d\eta_{1,k}(\lambda) \right) \leqslant$$

$$\leqslant 2^{2(1-\alpha)} t^{2\alpha} \pi^{2\alpha} \sum_{k=M+1}^{\infty} \frac{1}{k^{2\alpha}} \left(\int_0^\infty \lambda^{2\alpha} dF(\lambda) \right).$$

The tail $\sum\limits_{k=M+1}^{\infty} \frac{1}{k^{2\alpha}}$ with $\frac{1}{2} < \alpha \leqslant 1$ can be estimated as follows

$$\sum_{k=M+1}^{\infty} \frac{1}{k^{2\alpha}} \leqslant \sum_{k=M+1}^{\infty} \int\limits_{k-1}^{k} \frac{1}{x^{2\alpha}} dx = \int\limits_{M}^{\infty} \frac{1}{x^{2\alpha}} dx = \left. \frac{x^{1-2\alpha}}{1-2\alpha} \right|_{M}^{\infty} = \frac{1}{(2\alpha-1)M^{2\alpha-1}}.$$

Thus

$$\tau^2 \left(\sum_{k=M+1}^{\infty} \cos(kx) \int\limits_{0}^{\infty} J_k(t\lambda) d\eta_{1,k}(\lambda) \right) \leqslant$$

$$\leqslant 2^{2(1-\alpha)} t^{2\alpha} \pi^{2\alpha} \frac{1}{(2\alpha-1)M^{2\alpha-1}} \left(\int\limits_{0}^{\infty} \lambda^{2\alpha} dF(\lambda) \right).$$

The second inequality is proved similarly. ♢

Theorem 3.4. *Let $X(t,x)$ and $\hat{X}(t,x)$ be defined by (3.4) and (3.5) respectively. Assume that the integral $\int\limits_{0}^{\infty} \lambda^{2\alpha} dF(\lambda) < \infty$ with $\frac{1}{2} < \alpha \leqslant 1$ converges. Then*

$$\tau^2(X(t,x) - \hat{X}(t,x)) \leqslant \frac{4M}{2\alpha-1} \left(2\alpha - \frac{1}{M^{2\alpha-1}} \right) 2 \cdot 4^{2(1-\alpha)} t^{2\alpha} \pi^{2\alpha} \times$$

$$\times \sum_{l=0}^{N-2} |\lambda_{l+1} - \lambda_l|^{2\alpha} \left(b_l^2 + \left(\frac{t(1+C)}{2} \right)^{2\alpha} \int\limits_{\lambda_l}^{\lambda_{l+1}} \lambda^{2\alpha} dF(\lambda) \right) +$$

$$+16M^2(F(+\infty)-F(\Lambda))+2^{2(1-\alpha)} t^{2\alpha} \pi^{2\alpha} \frac{4}{(2\alpha-1)M^{2\alpha-1}} \left(\int\limits_{0}^{\infty} \lambda^{2\alpha} dF(\lambda) \right),$$

where $C = \max\limits_{0<l\leqslant N-2} \frac{\lambda_{l+1}}{\lambda_l}$.

Proof. The proof of Theorem 3.4 follows from relations (3.7) and (3.8) in view of Lemmas 3.7 and 3.8. ♢

3.5. Accuracy and reliability of models for stochastic fields in the space $L_p(\mathbb{T}), p \geqslant 1$.

Theorem 3.5. *Let $\frac{1}{2} < \alpha \leqslant 1$ and let $\int\limits_{0}^{\infty} \lambda^{2\alpha} dF(\lambda) < \infty$. Assume that a partition L used to construct a model $\hat{X}(t,x)$, $t \in [0,T], x \in [0,2\pi]$, accordi-*

ng to (3.5) is such that

$$I \leqslant \frac{\varepsilon^p}{\max\left(\left(2\ln\frac{2}{\delta}\right)^{\frac{p}{2}}, p^{\frac{p}{2}}\right)},$$

where

$$I = \frac{T^{p\alpha+1}}{p\alpha+1}\left(\frac{2^p D_p^3 M^{\frac{p}{2}}}{(2\alpha-1)^{\frac{p}{2}}}\left(2\alpha - \frac{1}{M^{2\alpha-1}}\right)^{\frac{p}{2}} 2^{\frac{p}{2}+1}\cdot 4^{p(1-\alpha)}\pi^{p\alpha+1}\times\right.$$
$$\times\left(\sum_{l=0}^{N-2}|\lambda_{l+1}-\lambda_l|^{2\alpha}b_l^2\right)^{\frac{p}{2}} + D_p 2^{p(1-\alpha)+1}\pi^{p\alpha+1}\left(\frac{4}{(2\alpha-1)M^{2\alpha-1}}\right)^{\frac{p}{2}}\times$$
$$\left.\times\left(\int_0^\infty \lambda^{2\alpha}dF(\lambda)\right)^{\frac{p}{2}}\right) + \frac{T^{2p\alpha+1}}{2p\alpha+1}\cdot\frac{2^p D_p^3 M^{\frac{p}{2}}}{(2\alpha-1)^{\frac{p}{2}}}\left(2\alpha-\frac{1}{M^{2\alpha-1}}\right)^{\frac{p}{2}}\times$$
$$\times 2^{\frac{p}{2}+1}4^{p(1-\alpha)}\pi^{p\alpha+1}\left(\frac{1+C}{2}\right)^{p\alpha}\left(\sum_{l=0}^{N-2}|\lambda_{l+1}-\lambda_l|^{2\alpha}\int_{\lambda_l}^{\lambda_{l+1}}\lambda^{2\alpha}dF(\lambda)\right)^{\frac{p}{2}} +$$
$$+ T\cdot 2^{2p+1}\pi D_p^2 M^p(F(+\infty)-F(\Lambda))^{\frac{p}{2}},$$

and where $C = \max\limits_{0<l\leqslant N-2}\frac{\lambda_{l+1}}{\lambda_l}$, $D_p = \begin{cases} 1, & \text{if } 0<\frac{p}{2}\leqslant 1, \\ 2^{\frac{p}{2}-1}, & \text{if } \frac{p}{2}>1 \end{cases}$.

Then the model $\hat{X}(t,x)$ *approximates the Gaussian field* $X(t,x)$ *with reliability* $1-\ \delta, 0<\delta<1$, *and accuracy* $\varepsilon>0$ *in the space* $L_p(\mathbb{T}), p\geqslant 1$.

Proof. If

$$\varepsilon > \left(\int_0^T\int_0^{2\pi}\left(\tau(X(t,x)-\hat{X}(t,x))\right)^p dxdt\right)^{\frac{1}{p}}\cdot p^{\frac{1}{2}},$$

then Theorem 3.2 and Definition 3.1 imply that

$$\mathbf{P}\left\{\| X(t,x)-\hat{X}(t,x)\|_{L_p}>\varepsilon\right\}\leqslant 2\exp\left\{-\frac{\varepsilon^2}{2c_p^{\frac{2}{p}}}\right\}\leqslant\delta,$$

where $c_p = \int_0^T\int_0^{2\pi}\left(\tau(X(t,x)-\hat{X}(t,x))\right)^p dxdt.$

The latter inequality holds if

$$\int_0^T \int_0^{2\pi} \left(\tau(X(t,x) - \hat{X}(t,x))\right)^p dxdt \leqslant \frac{\varepsilon^p}{\left(2\ln\frac{2}{\delta}\right)^{\frac{p}{2}}}.$$

Since

$$(a+b)^{\frac{p}{2}} \leqslant D_p(a^{\frac{p}{2}} + b^{\frac{p}{2}})$$

with the constant

$$D_p = \begin{cases} 1, & \text{if } \frac{p}{2} \leqslant 1, \\ 2^{\frac{p}{2}-1}, & \text{if} \frac{p}{2} > 1 \end{cases},$$

Theorem 3.4 implies that

$$\left(\tau\left(X(t,x) - \hat{X}(t,x)\right)\right)^p \leqslant$$

$$\leqslant \left[\frac{8M}{2\alpha-1}\left(2\alpha - \frac{1}{M^{2\alpha-1}}\right) 4^{2(1-\alpha)} t^{2\alpha} \pi^{2\alpha} \sum_{l=0}^{N-2} |\lambda_{l+1} - \lambda_l|^{2\alpha} \times\right.$$

$$\times \left(b_l^2 + \left(\frac{t(1+C)}{2}\right)^{2\alpha} \int_{\lambda_l}^{\lambda_{l+1}} \lambda^{2\alpha} dF(\lambda)\right) + 16M^2(F(+\infty) - F(\Lambda)) +$$

$$\left.+ \frac{4 \cdot 2^{2(1-\alpha)} t^{2\alpha} \pi^{2\alpha}}{(2\alpha-1)M^{2\alpha-1}} \left(\int_0^\infty \lambda^{2\alpha} dF(\lambda)\right)\right]^{\frac{p}{2}} \leqslant$$

$$\leqslant D_p \left(\frac{8M}{2\alpha-1}\left(2\alpha - \frac{1}{M^{2\alpha-1}}\right) \cdot 4^{2(1-\alpha)} t^{2\alpha} \pi^{2\alpha} \sum_{l=0}^{N-2} |\lambda_{l+1} - \lambda_l|^{2\alpha} \times\right.$$

$$\left.\times \left(b_l^2 + \left(\frac{t(1+C)}{2}\right)^{2\alpha} \int_{\lambda_l}^{\lambda_{l+1}} \lambda^{2\alpha} dF(\lambda)\right) + 16M^2(F(+\infty) - F(\Lambda))\right)^{\frac{p}{2}} +$$

$$+ D_p 2^{p(1-\alpha)} t^{p\alpha} \pi^{p\alpha} \left(\frac{4}{(2\alpha-1)M^{2\alpha-1}}\right)^{\frac{p}{2}} \left(\int_0^\infty \lambda^{2\alpha} dF(\lambda)\right)^{\frac{p}{2}} \leqslant$$

$$\leqslant D_p^2 \left(\left(\frac{8M}{2\alpha-1}\right)^{\frac{p}{2}} \left(2\alpha - \frac{1}{M^{2\alpha-1}}\right)^{\frac{p}{2}} 4^{p(1-\alpha)} t^{p\alpha} \pi^{p\alpha} \times\right.$$

$$\left.\times \left(\sum_{l=0}^{N-2} |\lambda_{l+1} - \lambda_l|^{2\alpha} \left(b_l^2 + \left(\frac{t(1+C)}{2}\right)^{2\alpha} \int_{\lambda_l}^{\lambda_{l+1}} \lambda^{2\alpha} dF(\lambda)\right)\right)^{\frac{p}{2}} +\right.$$

$$+4^p M^p \left(F(+\infty) - F(\Lambda)\right)^{\frac{p}{2}}\Big) + D_p 2^{p(1-\alpha)} t^{p\alpha} \pi^{p\alpha} \left(\frac{4}{(2\alpha-1)M^{2\alpha-1}}\right)^{\frac{p}{2}} \times$$

$$\times \left(\int\limits_0^{\infty} \lambda^{2\alpha} dF(\lambda)\right)^{\frac{p}{2}} \leqslant D_p^2 \left(\frac{2^p D_p M^{\frac{p}{2}}}{(2\alpha-1)^{\frac{p}{2}}} \left(2\alpha - \frac{1}{M^{2\alpha-1}}\right)^{\frac{p}{2}} \times\right.$$

$$\times 2^{\frac{p}{2}} \cdot 4^{p(1-\alpha)} t^{p\alpha} \pi^{p\alpha} \left(\left(\sum_{l=0}^{N-2} |\lambda_{l+1} - \lambda_l|^{2\alpha} b_l^2\right)^{\frac{p}{2}} + \left(\frac{t(1+C)}{2}\right)^{p\alpha} \times\right.$$

$$\left.\left.\times \left(\sum_{l=0}^{N-2} |\lambda_{l+1} - \lambda_l|^{2\alpha} \int\limits_{\lambda_l}^{\lambda_{l+1}} \lambda^{2\alpha} dF(\lambda)\right)^{\frac{p}{2}}\right) + 4^p M^p (F(+\infty) - F(\Lambda))^{\frac{p}{2}}\right) +$$

$$+ D_p 2^{p(1-\alpha)} (t\pi)^{p\alpha} \left(\frac{4}{(2\alpha-1)M^{2\alpha-1}}\right)^{\frac{p}{2}} \left(\int\limits_0^{\infty} \lambda^{2\alpha} dF(\lambda)\right)^{\frac{p}{2}} =$$

$$= \left(\frac{2^p D_p^3 M^{\frac{p}{2}}}{(2\alpha-1)^{\frac{p}{2}}} \left(2\alpha - \frac{1}{M^{2\alpha-1}}\right)^{\frac{p}{2}} 2^{\frac{p}{2}} \cdot 4^{p(1-\alpha)} \pi^{p\alpha} \left(\sum_{l=0}^{N-2} |\lambda_{l+1} - \lambda_l|^{2\alpha} b_l^2\right)^{\frac{p}{2}} +\right.$$

$$\left. + D_p 2^{p(1-\alpha)} \pi^{p\alpha} \left(\frac{4}{(2\alpha-1)M^{2\alpha-1}}\right)^{\frac{p}{2}} \left(\int\limits_0^{\infty} \lambda^{2\alpha} dF(\lambda)\right)^{\frac{p}{2}}\right) t^{p\alpha} +$$

$$+ \frac{2^p D_p^3 M^{\frac{p}{2}}}{(2\alpha-1)^{\frac{p}{2}}} \left(2\alpha - \frac{1}{M^{2\alpha-1}}\right)^{\frac{p}{2}} 2^{\frac{p}{2}} \cdot 4^{p(1-\alpha)} \pi^{p\alpha} \left(\frac{1+C}{2}\right)^{p\alpha} \times$$

$$\times \left(\sum_{l=0}^{N-2} \mid \lambda_{l+1} - \lambda_l \mid^{2\alpha} \int\limits_{\lambda_l}^{\lambda_{l+1}} \lambda^{2\alpha} dF(\lambda)\right)^{\frac{p}{2}} t^{2p\alpha} + 4^p D_p^2 M^p (F(+\infty) - F(\Lambda))^{\frac{p}{2}}.$$

Therefore

$$\int\limits_0^T \int\limits_0^{2\pi} \left(\tau(X(t,x) - \hat{X}(t,x))\right)^p dx dt \leqslant$$

$$\leqslant \frac{T^{p\alpha+1}}{p\alpha+1} \left(\frac{2^p D_p^3 M^{\frac{p}{2}}}{(2\alpha-1)^{\frac{p}{2}}} \left(2\alpha - \frac{1}{M^{2\alpha-1}}\right)^{\frac{p}{2}} 2^{\frac{p}{2}+1} 4^{p(1-\alpha)} \pi^{p\alpha+1} \times\right.$$

$$\times \left(\sum_{l=0}^{N-2} \mid \lambda_{l+1} - \lambda_l \mid^{2\alpha} b_l^2\right)^{\frac{p}{2}} + D_p 2^{p(1-\alpha)+1} \pi^{p\alpha+1} \left(\frac{4}{(2\alpha-1)M^{2\alpha-1}}\right)^{\frac{p}{2}} \times$$

$$\times\left(\left(\int\limits_0^{\infty}\lambda^{2\alpha}dF(\lambda)\right)^{\frac{p}{2}}\right)+\frac{T^{2p\alpha+1}}{2p\alpha+1}\frac{2^pD_p^3M^{\frac{p}{2}}}{(2\alpha-1)^{\frac{p}{2}}}\left(2\alpha-\frac{1}{M^{2\alpha-1}}\right)^{\frac{p}{2}}2^{\frac{p}{2}+1}\times$$

$$4^{p(1-\alpha)}\pi^{p\alpha+1}\left(\frac{1+C}{2}\right)^{p\alpha}\left(\sum_{l=0}^{N-2}\mid\lambda_{l+1}-\lambda_l\mid^{2\alpha}\cdot\int\limits_{\lambda_l}^{\lambda_{l+1}}\lambda^{2\alpha}dF(\lambda)\right)^{\frac{p}{2}}+$$

$$+T\cdot 2^{2p+1}\pi D_p^2M^p(F(+\infty)-F(\Lambda))^{\frac{p}{2}}=I.\quad\diamondsuit$$

Corollary 3.1. *Let a partition* $L=\{\lambda_0,...,\lambda_N\}$ *of the set* $[0,\infty)$ *be such that* $\lambda_l<\lambda_{l+1}$ *and* $\lambda_{l+1}-\lambda_l=\frac{\Lambda}{N-1}$. *Then Theorem 3.5 holds with*

$$I=\left(\frac{\Lambda}{N-1}\right)^{p\alpha}\cdot A+\left(\frac{1}{M^{2\alpha-1}}\right)^{\frac{p}{2}}\cdot B+(F(+\infty)-F(\Lambda))^{\frac{p}{2}}\cdot H,$$

where

$$A=2^pD_p^3\left(\frac{2\alpha M}{2\alpha-1}\right)^{\frac{p}{2}}2^{\frac{p}{2}+1}\cdot 4^{p(1-\alpha)}\pi^{p\alpha+1}\times$$

$$\times\left(\frac{T^{p\alpha+1}}{p\alpha+1}+\left(\frac{3}{2}\right)^{p\alpha}\left(\int\limits_0^{\Lambda}\lambda^{2\alpha}dF(\lambda)\right)^{\frac{p}{2}}\frac{T^{2p\alpha+1}}{2p\alpha+1}\right),$$

$$B=\frac{2^pD_p}{(2\alpha-1)^{\frac{p}{2}}}2^{p(1-\alpha)+1}\pi^{p\alpha+1}\left(\int\limits_0^{\infty}\lambda^{2\alpha}dF(\lambda)\right)^{\frac{p}{2}}\cdot\frac{T^{p\alpha+1}}{p\alpha+1},$$

$$H=2^{2p+1}\cdot D_p^2\cdot\pi\cdot M^p\cdot T.$$

3.6. Accuracy and reliability of models for stochastic fields in the space $C(T)$

Let $X=\{X(t,x),t\in\mathbb{R},x\in[0,2\pi]\}$ be a mean square continuous real Gaussian homogeneous and isotropic stochastic field on $\mathbb{R}^2$. Images of the field and its model $\hat{X}(t,x)$ are provided in Section 5, by (3.4) and (3.5) respectively.

Also we consider $\chi_M(t,x)=X(t,x)-\hat{X}(t,x)$ that is defined in Section 3 by equality (3.7).

Consider the difference

$$\chi_M(t,x) - \chi_M(s,y) = (\chi_{M,1}(t,x) - \chi_{M,1}(s,y)) + (\chi_{M,2}(t,x) - \chi_{M,2}(s,y)).$$

It is clear that

$$\chi_{M,1}(t,x) - \chi_{M,1}(s,y) = \sum_{k=1}^{M} \cos(kx) \sum_{l=0}^{N-1} \int_{\lambda_l}^{\lambda_{l+1}} (J_k(t\lambda) - J_k(t\zeta_l)) d\eta_{1,k}(\lambda) +$$

$$+ \sum_{k=M+1}^{\infty} \cos(kx) \int_0^{\infty} J_k(t\lambda) d\eta_{1,k}(\lambda) - \sum_{k=1}^{M} \cos(ky) \sum_{l=0}^{N-1} \int_{\lambda_l}^{\lambda_{l+1}} (J_k(s\lambda) -$$

$$- J_k(s\zeta_l)) d\eta_{1,k}(\lambda) - \sum_{k=M+1}^{\infty} \cos(ky) \int_0^{\infty} J_k(s\lambda) d\eta_{1,k}(\lambda) =$$

$$= \sum_{k=1}^{M} \left(\cos(kx) \sum_{l=0}^{N-1} \int_{\lambda_l}^{\lambda_{l+1}} (J_k(t\lambda) - J_k(t\zeta_l)) d\eta_{1,k}(\lambda) - \cos(ky) \times \right.$$

$$\left. \times \sum_{l=0}^{N-1} \int_{\lambda_l}^{\lambda_{l+1}} (J_k(s\lambda) - J_k(s\zeta_l)) d\eta_{1,k}(\lambda) \right) + \sum_{k=M+1}^{\infty} \left(\cos(kx) \times \right.$$

$$\left. \times \int_0^{\infty} J_k(t\lambda) d\eta_{1,k}(\lambda) - \cos(ky) \int_0^{\infty} J_k(s\lambda) d\eta_{1,k}(\lambda) \right) =$$

$$= \sum_{k=1}^{M} \left(\cos(kx) \sum_{l=0}^{N-1} \int_{\lambda_l}^{\lambda_{l+1}} (J_k(t\lambda) - J_k(t\zeta_l) - J_k(s\lambda) + J_k(s\zeta_l)) d\eta_{1,k}(\lambda) + \right.$$

$$\left. + (\cos(kx) - \cos(ky)) \sum_{l=0}^{N-1} \int_{\lambda_l}^{\lambda_{l+1}} (J_k(s\lambda) - J_k(s\zeta_l)) d\eta_{1,k}(\lambda) \right) +$$

$$+ \sum_{k=M+1}^{\infty} \left(\cos(kx) \int_0^{\infty} (J_k(t\lambda) - J_k(s\lambda)) d\eta_{1,k}(\lambda) + \right.$$

$$\left. + (\cos(kx) - \cos(ky)) \int_0^{\infty} J_k(s\lambda) d\eta_{1,k}(\lambda) \right).$$

for all $t, s \in [0, T]$ and $x, y \in [0, 2\pi]$.

Similarly

$$\chi_{M,2}(t,x)-\chi_{M,2}(s,y)=\sum_{k=1}^{M}\sin(kx)\sum_{l=0}^{N-1}\int\limits_{\lambda_l}^{\lambda_{l+1}}(J_k(t\lambda)-J_k(t\zeta_l))d\eta_{2,k}(\lambda)+$$

$$+\sum_{k=M+1}^{\infty}\sin(kx)\int\limits_0^{\infty}J_k(t\lambda)d\eta_{2,k}(\lambda)-\sum_{k=1}^{M}\sin(ky)\sum_{l=0}^{N-1}\int\limits_{\lambda_l}^{\lambda_{l+1}}(J_k(s\lambda)-$$

$$-J_k(s\zeta_l))d\eta_{2,k}(\lambda)-\sum_{k=M+1}^{\infty}\sin(ky)\int\limits_0^{\infty}J_k(s\lambda)d\eta_{2,k}(\lambda)=\sum_{k=1}^{M}(\sin(kx)\times$$

$$\times\sum_{l=0}^{N-1}\int\limits_{\lambda_l}^{\lambda_{l+1}}(J_k(t\lambda)-J_k(t\zeta_l))d\eta_{2,k}(\lambda)-\sin(ky)\sum_{l=0}^{N-1}\int\limits_{\lambda_l}^{\lambda_{l+1}}(J_k(s\lambda)-$$

$$-J_k(s\zeta_l))\;d\eta_{2,k}(\lambda))+\sum_{k=M+1}^{\infty}\left(\sin(kx)\int\limits_0^{\infty}J_k(t\lambda)d\eta_{2,k}(\lambda)-\sin(ky)\times\right.$$

$$\left.\times\int\limits_0^{\infty}J_k(s\lambda)d\eta_{2,k}(\lambda)\right)=\sum_{k=1}^{M}\left(\sin(kx)\sum_{l=0}^{N-1}\int\limits_{\lambda_l}^{\lambda_{l+1}}(J_k(t\lambda)-J_k(t\zeta_l)-J_k(s\lambda)+\right.$$

$$\left.+J_k(s\zeta_l))d\eta_{2,k}(\lambda)+(\sin(kx)-\sin(ky))\sum_{l=0}^{N-1}\int\limits_{\lambda_l}^{\lambda_{l+1}}(J_k(s\lambda)-J_k(s\zeta_l))d\eta_{2,k}(\lambda)\right)+$$

$$+\sum_{k=M+1}^{\infty}\left(\sin(kx)\int\limits_0^{\infty}(J_k(t\lambda)-J_k(s\lambda))d\eta_{2,k}(\lambda)+\right.$$

$$\left.+(\sin(kx)-\sin(ky))\int\limits_0^{\infty}J_k(s\lambda)d\eta_{2,k}(\lambda)\right).$$

Then

$$\tau(\chi_M(t,x)-\chi_M(s,y))\leqslant$$
$$\leqslant\tau(\chi_{M,1}(t,x)-\chi_{M,1}(s,y))+\tau(\chi_{M,2}(t,x)-\chi_{M,2}(s,y)),$$

and

$$\tau^2(\chi_{M,1}(t,x)-\chi_{M,1}(s,y))\leqslant4\tau^2\left(\sum_{k=1}^{M}\cos(kx)\sum_{l=0}^{N-1}\int\limits_{\lambda_l}^{\lambda_{l+1}}(J_k(t\lambda)-J_k(t\zeta_l)-\right.$$

$$- J_k(s\lambda) + J_k(s\zeta_l))\, d\eta_{1,k}(\lambda)) + 4\tau^2 \left(\sum_{k=1}^{M} (\cos(kx) - \cos(ky)) \times \right.$$

$$\times \sum_{l=0}^{N-1} \int_{\lambda_l}^{\lambda_{l+1}} (J_k(s\lambda) - J_k(s\zeta_l))\, d\eta_{1,k}(\lambda)) +$$

$$+ 4\tau^2 \left(\sum_{k=M+1}^{\infty} \cos(kx) \int_0^{\infty} (J_k(t\lambda) - J_k(s\lambda)) d\eta_{1,k}(\lambda) \right) +$$

$$+ 4\tau^2 \left(\sum_{k=M+1}^{\infty} (\cos(kx) - \cos(ky)) \int_0^{\infty} J_k(s\lambda) d\eta_{1,k}(\lambda) \right),$$

whence

$$\tau^2(\chi_{M,2}(t,x) - \chi_{M,2}(s,y)) \leqslant 4\tau^2 \left(\sum_{k=1}^{M} \sin(kx) \sum_{l=0}^{N-1} \int_{\lambda_l}^{\lambda_{l+1}} (J_k(t\lambda) - J_k(t\zeta_l) - \right.$$

$$- J_k(s\lambda) + J_k(s\zeta_l)) d\eta_{2,k}(\lambda)) + 4\tau^2 \left(\sum_{k=1}^{M} (\sin(kx) - \sin(ky)) \times \right.$$

$$\left. \times \sum_{l=0}^{N-1} \int_{\lambda_l}^{\lambda_{l+1}} (J_k(s\lambda) - J_k(s\zeta_l)) d\eta_{2,k}(\lambda) \right) +$$

$$+ 4\tau^2 \left(\sum_{k=M+1}^{\infty} \sin(kx) \int_0^{\infty} (J_k(t\lambda) - J_k(s\lambda)) d\eta_{2,k}(\lambda) \right) +$$

$$+ 4\tau^2 \left(\sum_{k=M+1}^{\infty} (\sin(kx) - \sin(ky)) \int_0^{\infty} J_k(s\lambda) d\eta_{2,k}(\lambda) \right).$$

Let $\sigma_0 = \sup\limits_{0 \leqslant t \leqslant T} \tau(\chi_M(t,x))$ and $\sigma(h) = \sup\limits_{\substack{|t-s| \leqslant h \\ |x-y| \leqslant h}} \tau(\chi_M(t,x) - \chi_M(s,y))$.

Theorem 3.6. *Let $X(t,x)$ and $\hat{X}(t,x)$ be defined by (3.4) and (3.5) respectively. Assume that $\frac{1}{2} < \alpha \leqslant 1$ and $\int_0^{\infty} \lambda^{2\alpha} dF(\lambda) < \infty$. Then*

$$\sigma_0 \leqslant \left(\frac{2M}{2\alpha - 1} \left(2\alpha - \frac{1}{M^{2\alpha-1}} \right) 2 \cdot 4^{2(1-\alpha)} T^{2\alpha} \pi^{2\alpha} \sum_{l=0}^{N-2} |\lambda_{l+1} - \lambda_l|^{2\alpha} \times \right.$$

$$\times \left(b_l^2 + \left(\frac{T(1+C)}{2} \right)^{2\alpha} \int\limits_{\lambda_l}^{\lambda_{l+1}} \lambda^{2\alpha} dF(\lambda) \right) + 8M^2 (F(+\infty) - F(\Lambda)) +$$

$$\left. + 2^{2(1-\alpha)} T^{2\alpha} \pi^{2\alpha} \frac{2}{(2\alpha - 1) M^{2\alpha - 1}} \int\limits_0^{\infty} \lambda^{2\alpha} dF(\lambda) \right)^{\frac{1}{2}},$$

where $C = \max\limits_{0 < l \leqslant N-2} \frac{\lambda_{l+1}}{\lambda_l}$.

Proof. Since

$$\tau^2(\chi_M(t,x)) \leqslant [\tau(\chi_{M,1}(t,x)) + \tau(\chi_{M,2}(t,x))]^2 \leqslant$$
$$\leqslant 2[\tau^2(\chi_{M,1}(t,x)) + \tau^2(\chi_{M,2}(t,x))],$$

Lemmas 3.7 and 3.8 imply that

$$\tau^2(\chi_M(t,x)) \leqslant 2 \left[\tau^2 \left(\sum_{k=1}^{M} \cos(kx) \sum_{l=0}^{N-1} \int\limits_{\lambda_l}^{\lambda_{l+1}} (J_k(t\lambda) - J_k(t\zeta_l)) d\eta_{1,k}(\lambda) \right) + \right.$$

$$+ \tau^2 \left(\sum_{k=1}^{M} \sin(kx) \sum_{l=0}^{N-1} \int\limits_{\lambda_l}^{\lambda_{l+1}} (J_k(t\lambda) - J_k(t\zeta_l)) d\eta_{2,k}(\lambda) \right) +$$

$$+ \tau^2 \left(\sum_{k=M+1}^{\infty} \cos(kx) \int\limits_0^{\infty} J_k(t\lambda) d\eta_{1,k}(\lambda) \right) +$$

$$\left. + \tau^2 \left(\sum_{k=M+1}^{\infty} \sin(kx) \int\limits_0^{\infty} J_k(t\lambda) d\eta_{2,k}(\lambda) \right) \right] \leqslant$$

$$\leqslant 2 \left[2 \cdot 4^{2(1-\alpha)} t^{2\alpha} \pi^{2\alpha} M \sum_{k=1}^{M} \frac{1}{k^{2\alpha}} \left(\cos^2(kx) + \sin^2(kx) \right) \times \right.$$

$$\times \sum_{l=0}^{N-2} |\lambda_{l+1} - \lambda_l|^{2\alpha} \left(b_l^2 + \left(\frac{t(1+C)}{2} \right)^{2\alpha} \int\limits_{\lambda_l}^{\lambda_{l+1}} \lambda^{2\alpha} dF(\lambda) \right) +$$

$$+ 4M \sum_{k=1}^{M} \left(\cos^2(kx) + \sin^2(kx)\right) \left(F(+\infty) - F(\Lambda)\right) + 2^{2(1-\alpha)} t^{2\alpha} \pi^{2\alpha} \times$$

$$\times \sum_{k=M+1}^{\infty} \frac{1}{k^{2\alpha}} \left(\cos^2(kx) + \sin^2(kx)\right) \times \left(\int\limits_0^{\infty} \lambda^{2\alpha} dF(\lambda) \right) \Bigg] \leqslant$$

$$\leqslant \frac{2M}{2\alpha - 1} \left(2\alpha - \frac{1}{M^{2\alpha-1}} \right) 2 \cdot 4^{2(1-\alpha)} t^{2\alpha} \pi^{2\alpha} \sum_{l=0}^{N-2} |\lambda_{l+1} - \lambda_l|^{2\alpha} \times$$

$$\times \left(b_l^2 + \left(\frac{t(1+C)}{2} \right)^{2\alpha} \int\limits_{\lambda_l}^{\lambda_{l+1}} \lambda^{2\alpha} dF(\lambda) \right) + 8M^2 (F(+\infty) - F(\Lambda)) +$$

$$+ 2^{2(1-\alpha)} t^{2\alpha} \pi^{2\alpha} \frac{2}{(2\alpha - 1) M^{2\alpha-1}} \int\limits_0^{\infty} \lambda^{2\alpha} dF(\lambda).$$

This yields

$$\sigma_0 = \sup_{0 \leqslant t \leqslant T} \tau(\chi_M(t,x)) \leqslant \left[\frac{2M}{2\alpha - 1} \left(2\alpha - \frac{1}{M^{2\alpha-1}} \right) 2 \cdot 4^{2(1-\alpha)} T^{2\alpha} \pi^{2\alpha} \times \right.$$

$$\times \sum_{l=0}^{N-2} |\lambda_{l+1} - \lambda_l|^{2\alpha} \left(b_l^2 + \left(\frac{T(1+C)}{2} \right)^{2\alpha} \int\limits_{\lambda_l}^{\lambda_{l+1}} \lambda^{2\alpha} dF(\lambda) \right) +$$

$$\left. + 8M^2 (F(+\infty) - F(\Lambda)) + 2^{2(1-\alpha)} T^{2\alpha} \pi^{2\alpha} \frac{2}{(2\alpha - 1) M^{2\alpha-1}} \int\limits_0^{\infty} \lambda^{2\alpha} dF(\lambda) \right]^{\frac{1}{2}}. \quad \Diamond$$

Corollary 3.2. *Let a partition* $L = \{\lambda_0, ..., \lambda_N\}$ *of the set* $[0, \infty)$ *be such that* $\lambda_l < \lambda_{l+1}$ *and* $\lambda_{l+1} - \lambda_l = \frac{\Lambda}{N-1}$. *If all assumptions of Theorem 3.6 hold, then*

$$\sigma_0 \leqslant \left[\frac{4^{2(1-\alpha)+1} T^{2\alpha} \pi^{2\alpha} M}{2\alpha - 1} \left(2\alpha - \frac{1}{M^{2\alpha-1}} \right) \left(\frac{\Lambda}{N-1} \right)^{2\alpha} \times \right.$$

$$\times \left(F(\Lambda) + \left(\frac{3T}{2} \right)^{2\alpha} \int\limits_0^{\Lambda} \lambda^{2\alpha} dF(\lambda) \right) + 8M^2 (F(+\infty) - F(\Lambda)) +$$

$$\left. + \frac{2^{2(1-\alpha)+1} T^{2\alpha} \pi^{2\alpha}}{(2\alpha - 1) M^{2\alpha-1}} \int\limits_0^{\infty} \lambda^{2\alpha} dF(\lambda) \right]^{\frac{1}{2}}.$$

Lemma 3.9. *Let* $\frac{1}{2} < \alpha \leqslant 1$ *and* $\int\limits_0^\infty \lambda^{2\alpha} dF(\lambda) < \infty$. *Then*

$$\tau^2 \left(\sum_{k=1}^{M} \cos(kx) \sum_{l=0}^{N-1} \int\limits_{\lambda_l}^{\lambda_{l+1}} (J_k(t\lambda) - J_k(t\zeta_l) - J_k(s\lambda) + J_k(s\zeta_l)) d\eta_{1,k}(\lambda) \right) \leqslant$$

$$\leqslant 4^{2(2-\alpha)} |s-t|^{2\alpha} \left(\frac{\pi}{2}\right)^{2\alpha} M \left(\sum_{k=1}^{M} \cos^2(kx) \cdot \frac{1}{k^{2\alpha}} \right) \sum_{l=0}^{N-2} |\lambda_{l+1} - \lambda_l|^{2\alpha} \times$$

$$\times \left(b_l^2 + \left[\left(\frac{|s-t|(1+C)}{4} \right)^{2\alpha} + \left(\frac{|t+s|}{2} \right)^{2\alpha} (1 + 2C^\alpha) + \right.\right.$$

$$\left.\left. + \left(\frac{|t+s|^2 \lambda_{l+1}(1+C)}{8} \right)^{2\alpha} \right] \int\limits_{\lambda_l}^{\lambda_{l+1}} \lambda^{2\alpha} dF(\lambda) \right) +$$

$$+ 18 \cdot 4^{3-2\alpha} |s-t|^{2\alpha} M \sum_{k=1}^{M} \cos^2(kx) \left(\int\limits_{\Lambda}^{\infty} |\lambda - \Lambda|^{2\alpha} dF(\lambda) + 2^{2\alpha} \Lambda^{2\alpha} (F(+\infty) - F(\Lambda)) \right),$$

$$\tau^2 \left(\sum_{k=1}^{M} \sin(kx) \sum_{l=0}^{N-1} \int\limits_{\lambda_l}^{\lambda_{l+1}} (J_k(t\lambda) - J_k(t\zeta_l) - J_k(s\lambda) + J_k(s\zeta_l)) d\eta_{2,k}(\lambda) \right) \leqslant$$

$$\leqslant 4^{2(2-\alpha)} |s-t|^{2\alpha} \left(\frac{\pi}{2}\right)^{2\alpha} M \left(\sum_{k=1}^{M} \sin^2(kx) \cdot \frac{1}{k^{2\alpha}} \right) \sum_{l=0}^{N-2} |\lambda_{l+1} - \lambda_l|^{2\alpha} \times$$

$$\times \left(b_l^2 + \left[\left(\frac{|s-t|(1+C)}{4} \right)^{2\alpha} + \left(\frac{|t+s|}{2} \right)^{2\alpha} (1 + 2C^\alpha) + \right.\right.$$

$$\left.\left. + \left(\frac{|t+s|^2 \lambda_{l+1}(1+C)}{8} \right)^{2\alpha} \right] \int\limits_{\lambda_l}^{\lambda_{l+1}} \lambda^{2\alpha} dF(\lambda) \right) +$$

$$+ 18 \cdot 4^{3-2\alpha} |s-t|^{2\alpha} M \sum_{k=1}^{M} \sin^2(kx) \left(\int\limits_{\Lambda}^{\infty} |\lambda - \Lambda|^{2\alpha} dF(\lambda) + 2^{2\alpha} \Lambda^{2\alpha} (F(+\infty) - F(\Lambda)) \right),$$

where $C = \max\limits_{0 < l \leqslant N-2} \frac{\lambda_{l+1}}{\lambda_l}$.

Proof. Since

$$(a_1 + a_2 + ... + a_n)^2 \leqslant n(a_1^2 + a_2^2 + ... + a_n^2),$$

for all real $a_1, a_2, ..., a_n$ we derive from Lemmas 1.2 and 1.3 that

$$\tau^2\left(\sum_{k=1}^{M}\cos(kx)\sum_{l=0}^{N-1}\int_{\lambda_l}^{\lambda_{l+1}}(J_k(t\lambda)-J_k(t\zeta_l)-J_k(s\lambda)+J_k(s\zeta_l))d\eta_{1,k}(\lambda)\right)\leqslant$$

$$\leqslant M\sum_{k=1}^{M}\cos^2(kx)\sum_{l=0}^{N-1}\tau^2\left(\int_{\lambda_l}^{\lambda_{l+1}}(J_k(t\lambda)-J_k(t\zeta_l)-J_k(s\lambda)+J_k(s\zeta_l))d\eta_{1,k}(\lambda)\right)\leqslant$$

$$\leqslant M\sum_{k=1}^{M}\cos^2(kx)\sum_{l=0}^{N-1}\theta^2\left(\int_{\lambda_l}^{\lambda_{l+1}}(J_k(t\lambda)-J_k(t\zeta_l)-J_k(s\lambda)+J_k(s\zeta_l))d\eta_{1,k}(\lambda)\right)=$$

$$=M\sum_{k=1}^{M}\cos^2(kx)\sum_{l=0}^{N-1}\sup_{m\geqslant 1}\left[\frac{2^m\cdot m!}{(2m)!}\times\right.$$

$$\left.\times\mathbf{E}\left(\int_{\lambda_l}^{\lambda_{l+1}}(J_k(t\lambda)-J_k(t\zeta_l)-J_k(s\lambda)+J_k(s\zeta_l))d\eta_{1,k}(\lambda)\right)^{2m}\right]^{\frac{1}{m}}.$$

Since $\mathbf{E}\xi = 0$, $\mathbf{E}\xi^{2k+1} = 0$, $\mathbf{E}\xi^{2k} = \frac{(2k)!}{2^k\cdot k!}\sigma^{2k}$ for a centered Gaussian random variable ξ and since the random variables ζ_l do not depend on $\eta_{i,k}(\lambda), i = 1,2$, by Fubini's theorem, Cauchy–Bunyakovskiy inequality and Lemma 3.6 with $l \leqslant N-2$ imply that

$$\mathbf{E}\left(\int_{\lambda_l}^{\lambda_{l+1}}(J_k(t\lambda)-J_k(t\zeta_l)-J_k(s\lambda)+J_k(s\zeta_l))d\eta_{1,k}(\lambda)\right)^{2m}\leqslant$$

$$\leqslant\frac{(2m)!}{2^m\cdot m!}\mathbf{E}\left(\int_{\lambda_l}^{\lambda_{l+1}}|J_k(t\lambda)-J_k(t\zeta_l)-J_k(s\lambda)+J_k(s\zeta_l)|^2dF(\lambda)\right)^m\leqslant$$

$$\leqslant\frac{(2m)!}{2^m\cdot m!}\mathbf{E}\left(\int_{\lambda_l}^{\lambda_{l+1}}\left(4\cdot 4^{2(1-\alpha)}|\lambda-\zeta_l|^{2\alpha}|s-t|^{2\alpha}\left(\frac{\pi}{2k}\right)^{2\alpha}\times\right.\right.$$

$$\times\left(1+\frac{|\lambda+\zeta_l|^\alpha|s-t|^\alpha}{4^\alpha}+\frac{|t+s|^\alpha(\lambda^\alpha+2\zeta_l^\alpha)}{2^\alpha}+\right.$$

$$+\frac{|t+s|^{2\alpha}\zeta_l^{\alpha}|\lambda+\zeta_l|^{\alpha}}{4^{\alpha}\cdot 2^{\alpha}}\Bigg)^{2}\Bigg)\, dF(\lambda))^{m} = \frac{(2m)!}{2^m\cdot m!}4^m\times$$

$$\times\, 4^{2m(1-\alpha)}\left(\frac{\pi}{2k}\right)^{2m\alpha}|s-t|^{2m\alpha}\mathbf{E}\left(\int\limits_{\lambda_l}^{\lambda_{l+1}}|\lambda-\zeta_l|^{2\alpha}\left(1+\frac{|\lambda+\zeta_l|^{\alpha}|s-t|^{\alpha}}{4^{\alpha}}+\right.\right.$$

$$\left.\left.+\frac{|t+s|^{\alpha}(\lambda^{\alpha}+2\zeta_l^{\alpha})}{2^{\alpha}}+\frac{|t+s|^{2\alpha}\zeta_l^{\alpha}|\lambda+\zeta_l|^{\alpha}}{4^{\alpha}\cdot 2^{\alpha}}\right)^{2} dF(\lambda)\right)^{m} = \frac{(2m)!}{2^m\cdot m!}4^m\times$$

$$\times\, 4^{2m(1-\alpha)}\left(\frac{\pi}{2k}\right)^{2m\alpha}|s-t|^{2m\alpha}\int\limits_{\lambda_l}^{\lambda_{l+1}}\left(\int\limits_{\lambda_l}^{\lambda_{l+1}}|\lambda-u|^{2\alpha}\left(1+\frac{|\lambda+u|^{\alpha}|s-t|^{\alpha}}{4^{\alpha}}+\right.\right.$$

$$\left.\left.+\frac{|t+s|^{\alpha}(\lambda^{\alpha}+2u^{\alpha})}{2^{\alpha}}+\frac{|t+s|^{2\alpha}u^{\alpha}|\lambda+u|^{\alpha}}{4^{\alpha}\cdot 2^{\alpha}}\right)^{2} dF(\lambda)\right)^{m} dF_l(u)\leqslant$$

$$\leqslant \frac{(2m)!}{2^m\cdot m!}4^m\cdot 4^{2m(1-\alpha)}\left(\frac{\pi}{2k}\right)^{2m\alpha}|s-t|^{2m\alpha}|\lambda_{l+1}-\lambda_l|^{2m\alpha}\times$$

$$\times\int\limits_{\lambda_l}^{\lambda_{l+1}}\left(\int\limits_{\lambda_l}^{\lambda_{l+1}}\left(1+\frac{\lambda^{\alpha}\left(1+\frac{u}{\lambda}\right)^{\alpha}|s-t|^{\alpha}}{4^{\alpha}}+\frac{|t+s|^{\alpha}\lambda^{\alpha}(1+2\left(\frac{u}{\lambda}\right)^{\alpha})}{2^{\alpha}}+\right.\right.$$

$$\left.\left.+\frac{|t+s|^{2\alpha}u^{\alpha}\lambda^{\alpha}\left(1+\frac{u}{\lambda}\right)^{\alpha}}{4^{\alpha}\cdot 2^{\alpha}}\right)^{2} dF(\lambda)\right)^{m} dF_l(u)\leqslant\frac{(2m)!}{2^m\cdot m!}4^m\cdot 4^{2m(1-\alpha)}\times$$

$$\times\left(\frac{\pi}{2k}\right)^{2m\alpha}|s-t|^{2m\alpha}|\lambda_{l+1}-\lambda_l|^{2m\alpha}\left(\int\limits_{\lambda_l}^{\lambda_{l+1}}\left(1+\frac{\lambda^{\alpha}\left(1+\frac{\lambda_{l+1}}{\lambda_l}\right)^{\alpha}|s-t|^{\alpha}}{4^{\alpha}}+\right.\right.$$

$$\left.\left.+\frac{|t+s|^{\alpha}\lambda^{\alpha}\left(1+2\left(\frac{\lambda_{l+1}}{\lambda_l}\right)^{\alpha}\right)}{2^{\alpha}}+\frac{|t+s|^{2\alpha}\lambda_{l+1}^{\alpha}\lambda^{\alpha}\left(1+\frac{\lambda_{l+1}}{\lambda_l}\right)^{\alpha}}{4^{\alpha}\cdot 2^{\alpha}}\right)^{2} dF(\lambda)\right)^{m}\leqslant$$

$$\leqslant \frac{(2m)!}{2^m\cdot m!}4^m\cdot 4^{2m(1-\alpha)}\left(\frac{\pi}{2k}\right)^{2m\alpha}|s-t|^{2m\alpha}|\lambda_{l+1}-\lambda_l|^{2m\alpha}\times$$

$$\times\left(4\int\limits_{\lambda_l}^{\lambda_{l+1}}\left(1+\frac{\lambda^{2\alpha}(1+C)^{2\alpha}|s-t|^{2\alpha}}{4^{2\alpha}}+\frac{|t+s|^{2\alpha}\lambda^{2\alpha}(1+2C^{\alpha})^2}{2^{2\alpha}}+\right.\right.$$

$$\left.\left.+\frac{|t+s|^{4\alpha}\lambda_{l+1}^{2\alpha}\lambda^{2\alpha}(1+C)^{2\alpha}}{4^{2\alpha}\cdot 2^{2\alpha}}\right) dF(\lambda)\right)^{m} = \frac{(2m)!}{2^m\cdot m!}4^m\cdot 4^{2m(1-\alpha)}\left(\frac{\pi}{2k}\right)^{2m\alpha}\times$$

$$\times |s-t|^{2m\alpha}|\lambda_{l+1}-\lambda_l|^{2m\alpha}4^m \left[b_l^2 + \left(\frac{|s-t|(1+C)}{4}\right)^{2\alpha} \int\limits_{\lambda_l}^{\lambda_{l+1}} \lambda^{2\alpha} dF(\lambda) + \right.$$

$$+ \left(\frac{|t+s|}{2}\right)^{2\alpha} (1+2C^\alpha)^2 \int\limits_{\lambda_l}^{\lambda_{l+1}} \lambda^{2\alpha} dF(\lambda) +$$

$$\left. + \left(\frac{|t+s|^2\lambda_{l+1}(1+C)}{8}\right)^{2\alpha} \int\limits_{\lambda_l}^{\lambda_{l+1}} \lambda^{2\alpha} dF(\lambda) \right]^m .$$

Consider the case of $l = N-1$. Applying the inequality $|\sin x| \leqslant x^\alpha$ $\frac{1}{2} < \alpha \leqslant 1$ to those terms in $|K_1|$ and $|K_2|$ that do not contain $|\lambda + u|$ and $|t+s|$ and bounding the rest ones with sin and cos by 1 we obtain

$$\mathbf{E}\left(\int\limits_{\Lambda}^{\infty} (J_k(t\lambda) - J_k(t\zeta_l) - J_k(s\lambda) + J_k(s\zeta_l)) d\eta_{1,k}(\lambda)\right)^{2m} \leqslant$$

$$\leqslant \frac{(2m)!}{2^m \cdot m!} \left(\int\limits_{\Lambda}^{\infty} |J_k(t\lambda) - J_k(t\Lambda) - J_k(s\lambda) + J_k(s\Lambda)|^2 dF(\lambda)\right)^m \leqslant$$

$$\leqslant \frac{(2m)!}{2^m \cdot m!} \left(64 \int\limits_{\Lambda}^{\infty} \left(3\frac{|\lambda-\Lambda|^\alpha |s-t|^\alpha}{4^\alpha} + 3\frac{\Lambda^\alpha |s-t|^\alpha}{2^\alpha}\right)^2 dF(\lambda)\right)^m =$$

$$= \frac{(2m)!}{2^m \cdot m!} 4^{(3-2\alpha)m} 9^m |s-t|^{2m\alpha} \left(\int\limits_{\Lambda}^{\infty} \left(|\lambda-\Lambda|^\alpha + 2^\alpha \Lambda^\alpha\right)^2 dF(\lambda)\right)^m \leqslant$$

$$\leqslant \frac{(2m)!}{2^m \cdot m!} 4^{(3-2\alpha)m} 18^m |s-t|^{2m\alpha} \left(\int\limits_{\Lambda}^{\infty} \left(|\lambda-\Lambda|^{2\alpha} + 2^{2\alpha}\Lambda^{2\alpha}\right) dF(\lambda)\right)^m =$$

$$= \frac{(2m)!}{2^m \cdot m!} 4^{(3-2\alpha)m} 18^m |s-t|^{2m\alpha} \times$$

$$\times \left(\int\limits_{\Lambda}^{\infty} |\lambda-\Lambda|^{2\alpha} dF(\lambda) + \int\limits_{\Lambda}^{\infty} 2^{2\alpha}\Lambda^{2\alpha} dF(\lambda)\right)^m =$$

$$= \frac{(2m)!}{2^m \cdot m!} 4^{(3-2\alpha)m} 18^m |s-t|^{2m\alpha} \times$$

$$\times \left(\int\limits_{\Lambda}^{\infty} |\lambda-\Lambda|^{2\alpha} dF(\lambda) + 2^{2\alpha}\Lambda^{2\alpha}(F(+\infty) - F(\Lambda))\right)^m .$$

Thus

$$\tau^2\left(\sum_{k=1}^{M}\cos(kx)\sum_{l=0}^{N-1}\int\limits_{\lambda_l}^{\lambda_{l+1}}(J_k(t\lambda)-J_k(t\zeta_l)-J_k(s\lambda)+J_k(s\zeta_l))d\eta_{1,k}(\lambda)\right)\leqslant$$

$$\leqslant M\sum_{k=1}^{M}\cos^2(kx)\left[4^{2(2-\alpha)}|s-t|^{2\alpha}\left(\frac{\pi}{2k}\right)^{2\alpha}\sum_{l=0}^{N-2}|\lambda_{l+1}-\lambda_l|^{2\alpha}\times\right.$$

$$\times\left(b_l^2+\left(\left(\left(\frac{|s-t|(1+C)}{4}\right)^{2\alpha}+\left(\frac{|t+s|}{2}\right)^{2\alpha}(1+2C^{\alpha})+\right.\right.\right.$$

$$\left.\left.+\left(\frac{|t+s|^2\lambda_{l+1}(1+C)}{8}\right)^{2\alpha}\right)\int\limits_{\lambda_l}^{\lambda_{l+1}}\lambda^{2\alpha}dF(\lambda)\right)+18\cdot4^{3-2\alpha}|s-t|^{2\alpha}\times$$

$$\left.\times\left(\int\limits_{\Lambda}^{\infty}|\lambda-\Lambda|^{2\alpha}dF(\lambda)+2^{2\alpha}\Lambda^{2\alpha}(F(+\infty)-F(\Lambda))\right)\right]=4^{2(2-\alpha)}|s-t|^{2\alpha}M\times$$

$$\times\left(\frac{\pi}{2}\right)^{2\alpha}\sum_{k=1}^{M}\left(\frac{\cos^2(kx)}{k^{2\alpha}}\right)\sum_{l=0}^{N-2}|\ \lambda_{l+1}-\lambda_l\ |^{2\alpha}\left(b_l^2+\left[\left(\frac{|s-t|(1+C)}{4}\right)^{2\alpha}+\right.\right.$$

$$\left.\left.+\left(\frac{|t+s|}{2}\right)^{2\alpha}(1+2C^{\alpha})+\left(\frac{|t+s|^2\lambda_{l+1}(1+C)}{8}\right)^{2\alpha}\right]\int\limits_{\lambda_l}^{\lambda_{l+1}}\lambda^{2\alpha}dF(\lambda)\right)+$$

$$+18\cdot4^{3-2\alpha}|s-t|^{2\alpha}M\sum_{k=1}^{M}\cos^2(kx)\times$$

$$\times\left(\int\limits_{\Lambda}^{\infty}|\lambda-\Lambda|^{2\alpha}dF(\lambda)+2^{2\alpha}\Lambda^{2\alpha}(F(+\infty)-F(\Lambda))\right).\quad\diamondsuit$$

The proof of the second inequality is the same.

Lemma 3.10. *Let the integral* $\int\limits_0^\infty\lambda^{2\alpha}dF(\lambda)$ *converge for all* $\frac{1}{2}<\alpha\leqslant 1$. *Then*

$$\tau^2\left(\sum_{k=1}^{M}(\cos(kx)-\cos(ky))\sum_{l=0}^{N-1}\int\limits_{\lambda_l}^{\lambda_{l+1}}(J_k(s\lambda)-J_k(s\zeta_l))d\eta_{1,k}(\lambda)\right)\leqslant$$

$$\leqslant 2\cdot 4^{2(1-\alpha)} s^{2\alpha}\pi^{2\alpha} M\sum_{k=1}^{M}(\cos(kx)-\cos(ky))^2\frac{1}{k^{2\alpha}}\sum_{l=0}^{N-2}|\lambda_{l+1}-\lambda_l|^{2\alpha}\times$$

$$\times\left(b_l^2+\left(\frac{s(1+C)}{2}\right)^{2\alpha}\int\limits_{\lambda_l}^{\lambda_{l+1}}\lambda^{2\alpha}dF(\lambda)\right)+$$

$$+4M\sum_{k=1}^{M}(\cos(kx)-\cos(ky))^2(F(+\infty)-F(\Lambda)),$$

$$\tau^2\left(\sum_{k=1}^{M}(\sin(kx)-\sin(ky))\sum_{l=0}^{N-1}\int\limits_{\lambda_l}^{\lambda_{l+1}}(J_k(s\lambda)-J_k(s\zeta_l))d\eta_{2,k}(\lambda)\right)\leqslant$$

$$\leqslant 2\cdot 4^{2(1-\alpha)} s^{2\alpha}\pi^{2\alpha} M\sum_{k=1}^{M}(\sin(kx)-\sin(ky))^2\frac{1}{k^{2\alpha}}\sum_{l=0}^{N-2}|\lambda_{l+1}-\lambda_l|^{2\alpha}\times$$

$$\times\left(b_l^2+\left(\frac{s(1+C)}{2}\right)^{2\alpha}\int\limits_{\lambda_l}^{\lambda_{l+1}}\lambda^{2\alpha}dF(\lambda)\right)+$$

$$+4M\sum_{k=1}^{M}(\sin(kx)-\sin(ky))^2(F(+\infty)-F(\Lambda)).$$

Proof. Lemmas 1.2 and 1.3 imply

$$\tau^2\left(\sum_{k=1}^{M}(\cos(kx)-\cos(ky))\sum_{l=0}^{N-1}\int\limits_{\lambda_l}^{\lambda_{l+1}}(J_k(s\lambda)-J_k(s\zeta_l))d\eta_{1,k}(\lambda)\right)\leqslant$$

$$\leqslant M\sum_{k=1}^{M}(\cos(kx)-\cos(ky))^2\sum_{l=0}^{N-1}\tau^2\left(\int\limits_{\lambda_l}^{\lambda_{l+1}}(J_k(s\lambda)-J_k(s\zeta_l))d\eta_{1,k}(\lambda)\right)\leqslant$$

$$\leqslant M\sum_{k=1}^{M}(\cos(kx)-\cos(ky))^2\sum_{l=0}^{N-1}\theta^2\left(\int\limits_{\lambda_l}^{\lambda_{l+1}}(J_k(s\lambda)-J_k(s\zeta_l))d\eta_{1,k}(\lambda)\right)=$$

$$=M\sum_{k=1}^{M}(\cos(kx)-\cos(ky))^2\times$$

$$\times \sum_{l=0}^{N-1} \sup_{m\geqslant 1} \left[\frac{2^m \cdot m!}{(2m)!} \mathbf{E} \left(\int\limits_{\lambda_l}^{\lambda_{l+1}} (J_k(s\lambda) - J_k(s\zeta_l)) d\eta_{1,k}(\lambda) \right)^{2m} \right]^{\frac{1}{m}}.$$

We use Lemma 3.4 and a reasoning similar to that in the proof of Lemma 3.7 we estimate the expression

$$\left[\frac{2^m \cdot m!}{(2m)!} \mathbf{E} \left(\int\limits_{\lambda_l}^{\lambda_{l+1}} (J_k(s\lambda) - J_k(s\zeta_l)) d\eta_{1,k}(\lambda) \right)^{2m} \right]^{\frac{1}{m}}.$$

With this estimate, we obtain

$$\tau^2 \left(\sum_{k=1}^{M} (\cos(kx) - \cos(ky)) \sum_{l=0}^{N-1} \int\limits_{\lambda_l}^{\lambda_{l+1}} (J_k(s\lambda) - J_k(s\zeta_l)) d\eta_{1,k}(\lambda) \right) \leqslant$$

$$\leqslant 2 \cdot 4^{2(1-\alpha)} s^{2\alpha} \pi^{2\alpha} M \sum_{k=1}^{M} (\cos(kx) - \cos(ky))^2 \frac{1}{k^{2\alpha}} \sum_{l=0}^{N-2} |\lambda_{l+1} - \lambda_l|^{2\alpha} \times$$

$$\times \left(b_l^2 + \left(\frac{s(1+C)}{2} \right)^{2\alpha} \int\limits_{\lambda_l}^{\lambda_{l+1}} \lambda^{2\alpha} dF(\lambda) \right) +$$

$$+4M \sum_{k=1}^{M} (\cos(kx) - \cos(ky))^2 (F(+\infty) - F(\Lambda)).$$

The second inequality is proved similarly. ♢

Lemma 3.11. *Let the integral* $\int\limits_0^\infty \lambda^{2\nu} dF(\lambda)$ *converge for* $\nu > \frac{1}{2}$. *Then*

$$\tau^2 \left(\sum_{k=M+1}^{\infty} \cos(kx) \int\limits_0^\infty (J_k(t\lambda) - J_k(s\lambda)) d\eta_{1,k}(\lambda) \right) \leqslant$$

$$\leqslant \frac{2 \cdot 4^{2(1-\alpha)} \pi^{2\alpha}}{\left(\ln \left(1 + \frac{1}{|s-t|} \right) \right)^{2\delta}} \sum_{k=M+1}^{\infty} \left(\cos^2(kx) \cdot \frac{1}{k^{2\alpha}} \right) \times$$

$$\times \left(\left(\frac{\delta}{\alpha} \right)^{2\delta} \int\limits_0^\infty \lambda^{2\alpha} dF(\lambda) + \left(\frac{\delta}{\beta} \right)^{2\delta} |s+t|^{2\alpha} \int\limits_0^\infty \lambda^{2\nu} dF(\lambda) \right),$$

$$\tau^2\left(\sum_{k=M+1}^{\infty}\sin(kx)\int_0^{\infty}(J_k(t\lambda)-J_k(s\lambda))d\eta_{2,k}(\lambda)\right)\leqslant$$

$$\leqslant\frac{2\cdot 4^{2(1-\alpha)}\pi^{2\alpha}}{\left(\ln\left(1+\frac{1}{|s-t|}\right)\right)^{2\delta}}\sum_{k=M+1}^{\infty}\left(\sin^2(kx)\cdot\frac{1}{k^{2\alpha}}\right)\times$$

$$\times\left(\left(\frac{\delta}{\alpha}\right)^{2\delta}\int_0^{\infty}\lambda^{2\alpha}dF(\lambda)+\left(\frac{\delta}{\beta}\right)^{2\delta}|s+t|^{2\alpha}\int_0^{\infty}\lambda^{2\nu}dF(\lambda)\right),$$

where $\frac{\alpha}{\delta}\leqslant 1$, $\frac{1}{2}<\alpha\leqslant 1$, $\delta>0$ *and* $0<\beta\leqslant 1$.

Proof. Lemmas 1.2 and 1.3 imply that

$$\tau^2\left(\sum_{k=M+1}^{\infty}\cos(kx)\int_0^{\infty}(J_k(t\lambda)-J_k(s\lambda))d\eta_{1,k}(\lambda)\right)\leqslant$$

$$\leqslant\sum_{k=M+1}^{\infty}\cos^2(kx)\tau^2\left(\int_0^{\infty}(J_k(t\lambda)-J_k(s\lambda))d\eta_{1,k}(\lambda)\right)\leqslant$$

$$\leqslant\sum_{k=M+1}^{\infty}\cos^2(kx)\theta^2\left(\int_0^{\infty}(J_k(t\lambda)-J_k(s\lambda))d\eta_{1,k}(\lambda)\right)=$$

$$=\sum_{k=M+1}^{\infty}\cos^2(kx)\sup_{m\geqslant 1}\left[\frac{2^m\cdot m!}{(2m)!}\mathbf{E}\left(\int_0^{\infty}(J_k(t\lambda)-J_k(s\lambda))d\eta_{1,k}(\lambda)\right)^{2m}\right]^{\frac{1}{m}}.$$

Given $h>0$ and $0<\gamma\leqslant 1$ we obtain

$$\ln\left(1+\frac{1}{h}\right)=\frac{1}{\gamma}\ln\left(1+\frac{1}{h}\right)^{\gamma}\leqslant\frac{1}{\gamma}\ln\left(1+\left(\frac{1}{h}\right)^{\gamma}\right)\leqslant\frac{1}{h^{\gamma}\cdot\gamma},$$

whence $h^{\gamma}\leqslant\frac{1}{\gamma\ln\left(1+\frac{1}{h}\right)}$. Thus $h^{\gamma\delta}\leqslant\frac{1}{\gamma^{\delta}\left(\ln\left(1+\frac{1}{h}\right)\right)^{\delta}}$, $\delta>0$. If $\gamma\cdot\delta=\alpha$, then

$$h^{\alpha}\leqslant\frac{1}{\left(\frac{\alpha}{\delta}\right)^{\delta}\left(\ln\left(1+\frac{1}{h}\right)\right)^{\delta}}.\tag{3.9}$$

We conclude from Lemma 3.5 and inequality (3.9) that

$$\mathbf{E}\left(\int_0^\infty (J_k(t\lambda)-J_k(s\lambda))d\eta_{1,k}(\lambda)\right)^{2m}\leqslant$$

$$\leqslant\frac{(2m)!}{2^m\cdot m!}\left(\int_0^\infty (J_k(t\lambda)-J_k(s\lambda))dF(\lambda)\right)^m\leqslant$$

$$\leqslant\frac{(2m)!}{2^m\cdot m!}\left(\int_0^\infty\left(4^{1-\alpha}\pi^\alpha\frac{1}{k^\alpha}(\lambda^\alpha|s-t|^\alpha+\lambda^{\alpha+\beta}|s-t|^\beta\cdot|s+t|^\alpha)\right)^2 dF(\lambda)\right)^m\leqslant$$

$$\leqslant\frac{(2m)!}{2^m\cdot m!}\frac{2\cdot 4^{2m(1-\alpha)}\pi^{2m\alpha}}{k^{2m\alpha}}\times$$

$$\times\left(|s-t|^{2\alpha}\int_0^\infty\lambda^{2\alpha}dF(\lambda)+|s-t|^{2\beta}\cdot|s+t|^{2\alpha}\int_0^\infty\lambda^{2(\alpha+\beta)}dF(\lambda)\right)^m\leqslant$$

$$\leqslant\frac{(2m)!}{2^m\cdot m!}\frac{2\cdot 4^{2m(1-\alpha)}\pi^{2m\alpha}}{k^{2m\alpha}}\frac{1}{\left(\ln\left(1+\frac{1}{|s-t|}\right)\right)^{2m\delta}}\times$$

$$\left(\left(\frac{\delta}{\alpha}\right)^{2\delta}\int_0^\infty\lambda^{2\alpha}dF(\lambda)+\left(\frac{\delta}{\beta}\right)^{2\delta}\cdot|s+t|^{2\alpha}\int_0^\infty\lambda^{2(\alpha+\beta)}dF(\lambda)\right)^m.$$

We introduce the numbers α and β as follows $\alpha=\frac{1}{2}+\beta$, $\beta=\frac{\nu-\frac{1}{2}}{2}$, where $\frac{1}{2}<\alpha\leqslant 1$, $0<\beta\leqslant 1$ and $\nu>\frac{1}{2}$. Then

$$\tau^2\left(\sum_{k=M+1}^\infty\cos(kx)\int_0^\infty(J_k(t\lambda)-J_k(s\lambda))d\eta_{1,k}(\lambda)\right)\leqslant$$

$$\leqslant\frac{2\cdot 4^{2(1-\alpha)}\pi^{2\alpha}}{\left(\ln\left(1+\frac{1}{|s-t|}\right)\right)^{2\delta}}\sum_{k=M+1}^\infty\left(\cos^2(kx)\cdot\frac{1}{k^{2\alpha}}\right)\times$$

$$\times\left(\left(\frac{\delta}{\alpha}\right)^{2\delta}\int_0^\infty\lambda^{2\alpha}dF(\lambda)+\left(\frac{\delta}{\beta}\right)^{2\delta}|s+t|^{2\alpha}\int_0^\infty\lambda^{2\nu}dF(\lambda)\right).$$

The proof of the second inequality is the same. ♢

Lemma 3.12. *Let the integral* $\int_0^\infty\lambda^{2\alpha}dF(\lambda)<\infty$ *converge for* $\frac{1}{2}<\alpha\leqslant 1$. *Then*

$$\tau^2\left(\sum_{k=M+1}^{\infty}(\cos(kx)-\cos(ky))\int_0^{\infty}J_k(s\lambda)d\eta_{1,k}(\lambda)\right)\leqslant$$

$$\leqslant 2^{2(1-\alpha)}s^{2\alpha}\pi^{2\alpha}\sum_{k=M+1}^{\infty}(\cos(kx)-\cos(ky))^2\cdot\frac{1}{k^{2\alpha}}\int_0^{\infty}\lambda^{2\alpha}dF(\lambda),$$

$$\tau^2\left(\sum_{k=M+1}^{\infty}(\sin(kx)-\sin(ky))\int_0^{\infty}J_k(s\lambda)d\eta_{2,k}(\lambda)\right)\leqslant$$

$$\leqslant 2^{2(1-\alpha)}s^{2\alpha}\pi^{2\alpha}\sum_{k=M+1}^{\infty}(\sin(kx)-\sin(ky))^2\cdot\frac{1}{k^{2\alpha}}\int_0^{\infty}\lambda^{2\alpha}dF(\lambda).$$

Proof. Lemma 3.12 follows from Lemma 3.8. ♢

Theorem 3.7. *Let $X(t,x)$ and $\hat{X}(t,x)$ be defined by (3.4) and (3.5), respectively, and*

$$\sigma(h)=\sup_{\substack{|t-s|\leqslant h\\|x-y|\leqslant h}}\tau(\chi_M(t,x)-\chi_M(s,y)),$$

where $\chi_M(t,x)$ is defined by (3.7). Assume that $\int_0^{\infty}\lambda^{2\nu}dF(\lambda)<\infty$ for $\nu>\frac{1}{2}$. Then

$$\sigma(h)\leqslant\frac{1}{\left(\ln\left(\frac{1}{h}+1\right)\right)^{\delta}}\left[2\cdot 4^{2(2-\alpha)}\left(\frac{\delta}{\alpha}\right)^{2\delta}\left(\frac{\pi}{2}\right)^{2\alpha}\frac{M}{2\alpha-1}\left(2\alpha-\frac{1}{M^{2\alpha-1}}\right)\times\right.$$

$$\times\sum_{l=0}^{N-2}|\lambda_{l+1}-\lambda_l|^{2\alpha}\left(b_l^2+\left(\left(\frac{T(1+C)}{4}\right)^{2\alpha}+T^{2\alpha}(1+2C^{\alpha})+\right.\right.$$

$$\left.\left.+\left(\frac{T^2\Lambda(1+C)}{2}\right)^{2\alpha}\right)\int_{\lambda_l}^{\lambda_{l+1}}\lambda^{2\alpha}dF(\lambda)\right)+9\cdot 4^{4-2\alpha}\cdot M^2\left(\frac{\delta}{\alpha}\right)^{2\delta}\times$$

$$\times\left(\int\limits_{\Lambda}^{\infty}|\lambda-\Lambda|^{2\alpha}dF(\lambda)+2^{2\alpha}\Lambda^{2\alpha}(F(+\infty)-F(\Lambda))\right)+4^{4-2\alpha}T^{2\alpha}\pi^{2\alpha}M\times$$

$$\times\sum_{k=1}^{M}\frac{\left(\ln\left(k^2+e^{\delta}\right)\right)^{2\delta}}{k^{2\alpha}}\sum_{l=0}^{N-2}|\lambda_{l+1}-\lambda_l|^{2\alpha}\left(b_l^2+\left(\frac{T(1+C)}{2}\right)^{2\alpha}\int\limits_{\lambda_l}^{\lambda_{l+1}}\lambda^{2\alpha}dF(\lambda)\right)+$$

$$+16M(F(+\infty)-F(\Lambda))\sum_{k=1}^{M}\left(\ln\left(k^2+e^{\delta}\right)\right)^{2\delta}+\frac{4^{3-2\alpha}\pi^{2\alpha}}{(2\alpha-1)M^{2\alpha-1}}\times$$

$$\times\left(\left(\frac{\delta}{\alpha}\right)^{2\delta}\int\limits_{0}^{\infty}\lambda^{2\alpha}dF(\lambda)+(2T)^{2\alpha}\left(\frac{\delta}{\beta}\right)^{2\delta}\int\limits_{0}^{\infty}\lambda^{2\nu}dF(\lambda)\right)+$$

$$\left.+2^{2(2-\alpha)}T^{2\alpha}\pi^{2\alpha}\int\limits_{0}^{\infty}\lambda^{2\alpha}dF(\lambda)\sum_{k=M+1}^{\infty}\frac{\left(\ln\left(k^2+e^{\delta}\right)\right)^{2\delta}}{k^{2\alpha}}\right]^{\frac{1}{2}},$$

where $C=\max\limits_{0<l\leqslant N-2}\frac{\lambda_{l+1}}{\lambda_l}$, $\frac{1}{2}<\alpha\leqslant 1$, $\frac{\alpha}{\delta}\leqslant 1$, $\delta>0$ *and* $0<\beta\leqslant 1$.

Proof. Lemmas 3.9 - 3.12 imply

$$\tau^2(\chi_M(t,x)-\chi_M(s,y))\leqslant[\tau(\chi_{M,1}(t,x)-\chi_{M,1}(s,y))+$$
$$+\tau(\chi_{M,2}(t,x)-\chi_{M,2}(s,y))]^2\leqslant 2\tau^2(\chi_{M,1}(t,x)-\chi_{M,1}(s,y))+$$
$$+2\tau^2(\chi_{M,2}(t,x)-\chi_{M,2}(s,y))\leqslant 2\cdot 4^{2(2-\alpha)}|s-t|^{2\alpha}\left(\frac{\pi}{2}\right)^{2\alpha}M\times$$
$$\times\sum_{k=1}^{M}\frac{1}{k^{2\alpha}}\sum_{l=0}^{N-2}|\lambda_{l+1}-\lambda_l|^{2\alpha}\left(b_l^2+\left[\left(\frac{|s-t|(1+C)}{4}\right)^{2\alpha}+\right.\right.$$
$$\left.\left.+\left(\frac{|t+s|}{2}\right)^{2\alpha}(1+2C^{\alpha})+\left(\frac{|t+s|^2\lambda_{l+1}(1+C)}{8}\right)^{2\alpha}\right]\int\limits_{\lambda_l}^{\lambda_{l+1}}\lambda^{2\alpha}dF(\lambda)\right)+$$
$$+9\cdot 4^{4-2\alpha}|s-t|^{2\alpha}M^2\left(\int\limits_{\Lambda}^{\infty}|\lambda-\Lambda|^{2\alpha}dF(\lambda)+2^{2\alpha}\Lambda^{2\alpha}(F(+\infty)-F(\Lambda))\right)+$$
$$+4^{3-2\alpha}s^{2\alpha}\pi^{2\alpha}M\left(\sum_{k=1}^{M}\left((\cos(kx)-\cos(ky))^2+\right.\right.$$
$$\left.\left.+(\sin(kx)-\sin(ky))^2\right)\frac{1}{k^{2\alpha}}\right)\sum_{l=0}^{N-2}|\lambda_{l+1}-\lambda_l|^{2\alpha}\times$$

$$\times\left(b_l^2+\left(\frac{s(1+C)}{2}\right)^{2\alpha}\int\limits_{\lambda_l}^{\lambda_{l+1}}\lambda^{2\alpha}dF(\lambda)\right)+8M(F(+\infty)-F(\Lambda))\times$$

$$\times\sum_{k=1}^{M}\left((\cos(kx)-\cos(ky))^2+(\sin(kx)-\sin(ky))^2\right)+$$

$$+\frac{4^{3-2\alpha}\pi^{2\alpha}}{\left(\ln\left(1+\frac{1}{|s-t|}\right)\right)^{2\delta}}\sum_{k=M+1}^{\infty}\left(\frac{1}{k^{2\alpha}}\right)\times$$

$$\times\left(\left(\frac{\delta}{\alpha}\right)^{2\delta}\int\limits_0^{\infty}\lambda^{2\alpha}dF(\lambda)+\left(\frac{\delta}{\beta}\right)^{2\delta}|s+t|^{2\alpha}\int\limits_0^{\infty}\lambda^{2\nu}dF(\lambda)\right)+$$

$$+2^{3-2\alpha}s^{2\alpha}\pi^{2\alpha}\int\limits_0^{\infty}\lambda^{2\alpha}dF(\lambda)\sum_{k=M+1}^{\infty}\frac{1}{k^{2\alpha}}\left((\cos(kx)-\cos(ky))^2+\right.$$

$$\left.+(\sin(kx)-\sin(ky))^2\right).$$

Now we apply the inequality

$$|\cos(kx)-\cos(ky)|\leqslant\frac{\left(\ln\left(k^2+e^{\delta}\right)\right)^{\delta}}{\left(\ln\left(\frac{1}{|x-y|}+e^{\delta}\right)\right)^{\delta}},$$

for some $\delta>0$ (this is inequality (10) in [79]). Since

$$\sum_{k=1}^{M}\frac{1}{k^{2\alpha}}\leqslant 1+\sum_{k=2}^{M}\int\limits_{k-1}^{k}\frac{1}{x^{2\alpha}}dx=1+\int\limits_1^M\frac{1}{x^{2\alpha}}dx=$$

$$=1+\left.\frac{x^{1-2\alpha}}{1-2\alpha}\right|_1^M=\frac{2\alpha}{2\alpha-1}-\frac{1}{(2\alpha-1)M^{2\alpha-1}},$$

for all $\frac{1}{2}<\alpha\leqslant 1$ and

$$\sum_{k=M+1}^{\infty}\frac{1}{k^{2\alpha}}\leqslant\sum_{k=M+1}^{\infty}\int\limits_{k-1}^{k}\frac{1}{x^{2\alpha}}dx=\int\limits_M^{\infty}\frac{1}{x^{2\alpha}}dx=\left.\frac{x^{1-2\alpha}}{1-2\alpha}\right|_M^{\infty}=\frac{1}{(2\alpha-1)M^{2\alpha-1}},$$

we have

$$\tau^2(\chi_M(t,x)-\chi_M(s,y)) \leqslant 2\cdot 4^{2(2-\alpha)}|s-t|^{2\alpha}\left(\frac{\pi}{2}\right)^{2\alpha}\frac{M}{2\alpha-1}\left(2\alpha-\frac{1}{M^{2\alpha-1}}\right)\times$$

$$\times\sum_{l=0}^{N-2}|\lambda_{l+1}-\lambda_l|^{2\alpha}\left(b_l^2+\left[\left(\frac{|s-t|(1+C)}{4}\right)^{2\alpha}+\left(\frac{|t+s|}{2}\right)^{2\alpha}(1+2C^\alpha)+\right.\right.$$

$$\left.\left.+\left(\frac{|t+s|^2\lambda_{l+1}(1+C)}{8}\right)^{2\alpha}\right]\int\limits_{\lambda_l}^{\lambda_{l+1}}\lambda^{2\alpha}dF(\lambda)\right)+9\cdot 4^{4-2\alpha}|s-t|^{2\alpha}\cdot M^2\times$$

$$\times\left(\int\limits_{\Lambda}^{\infty}|\lambda-\Lambda|^{2\alpha}dF(\lambda)+2^{2\alpha}\Lambda^{2\alpha}(F(+\infty)-F(\Lambda))\right)+\frac{4^{4-2\alpha}s^{2\alpha}\pi^{2\alpha}M}{\left(\ln\left(\frac{1}{|x-y|}+e^\delta\right)\right)^{2\delta}}\times$$

$$\times\sum_{k=1}^{M}\frac{\left(\ln\left(k^2+e^\delta\right)\right)^{2\delta}}{k^{2\alpha}}\sum_{l=0}^{N-2}|\lambda_{l+1}-\lambda_l|^{2\alpha}\left(b_l^2+\left(\frac{s(1+C)}{2}\right)^{2\alpha}\int\limits_{\lambda_l}^{\lambda_{l+1}}\lambda^{2\alpha}dF(\lambda)\right)+$$

$$+\frac{16M(F(+\infty)-F(\Lambda))}{\left(\ln\left(\frac{1}{|x-y|}+e^\delta\right)\right)^{2\delta}}\sum_{k=1}^{M}\left(\ln\left(k^2+e^\delta\right)\right)^{2\delta}+\frac{4^{3-2\alpha}\pi^{2\alpha}}{(2\alpha-1)M^{2\alpha-1}}\times$$

$$\times\frac{1}{\left(\ln\left(1+\frac{1}{|s-t|}\right)\right)^{2\delta}}\left(\left(\frac{\delta}{\alpha}\right)^{2\delta}\int\limits_0^\infty\lambda^{2\alpha}dF(\lambda)+\left(\frac{\delta}{\beta}\right)^{2\delta}|s+t|^{2\alpha}\int\limits_0^\infty\lambda^{2\text{æ}}dF(\lambda)\right)+$$

$$+\frac{2^{2(2-\alpha)}s^{2\alpha}\pi^{2\alpha}}{\left(\ln\left(\frac{1}{|x-y|}+e^\delta\right)\right)^{2\delta}}\int\limits_0^\infty\lambda^{2\alpha}dF(\lambda)\sum_{k=M+1}^{\infty}\frac{\left(\ln\left(k^2+e^\delta\right)\right)^{2\delta}}{k^{2\alpha}}.$$

Therefore

$$\sup_{\substack{|t-s|\leqslant h\\|x-y|\leqslant h}}\tau(\chi_M(t,x)-\chi_M(s,y))\leqslant\left[2\cdot 4^{2(2-\alpha)}h^{2\alpha}\left(\frac{\pi}{2}\right)^{2\alpha}\frac{M}{2\alpha-1}\left(2\alpha-\frac{1}{M^{2\alpha-1}}\right)\times\right.$$

$$\times\sum_{l=0}^{N-2}\mid\lambda_{l+1}-\lambda_l\mid^{2\alpha}\left(b_l^2+\left[\left(\frac{T(1+C)}{4}\right)^{2\alpha}+T^{2\alpha}(1+2C^\alpha)+\right.\right.$$

$$\left.\left.+\left(\frac{T^2\Lambda(1+C)}{2}\right)^{2\alpha}\right]\int\limits_{\lambda_l}^{\lambda_{l+1}}\lambda^{2\alpha}dF(\lambda)\right)+9\cdot 4^{4-2\alpha}h^{2\alpha}\cdot M^2\times$$

$$\times\left(\int_{\Lambda}^{\infty}|\lambda-\Lambda|^{2\alpha}dF(\lambda)+2^{2\alpha}\Lambda^{2\alpha}(F(+\infty)-F(\Lambda))\right)+\frac{4^{4-2\alpha}T^{2\alpha}\pi^{2\alpha}M}{\left(\ln\left(\frac{1}{h}+1\right)\right)^{2\delta}}\times$$

$$\times\left(\sum_{k=1}^{M}\frac{\left(\ln\left(k^2+e^{\delta}\right)\right)^{2\delta}}{k^{2\alpha}}\right)\sum_{l=0}^{N-2}\mid\lambda_{l+1}-\lambda_l\mid^{2\alpha}\left(b_l^2+\left(\frac{T(1+C)}{2}\right)^{2\alpha}\int_{\lambda_l}^{\lambda_{l+1}}\lambda^{2\alpha}dF(\lambda)\right)+$$

$$+\frac{16M(F(+\infty)-F(\Lambda))}{\left(\ln\left(\frac{1}{h}+1\right)\right)^{2\delta}}\sum_{k=1}^{M}\left(\ln\left(k^2+e^{\delta}\right)\right)^{2\delta}+\frac{4^{3-2\alpha}\pi^{2\alpha}}{(2\alpha-1)M^{2\alpha-1}}\times$$

$$\times\frac{1}{\left(\ln\left(\frac{1}{h}+1\right)\right)^{2\delta}}\left(\left(\frac{\delta}{\alpha}\right)^{2\delta}\int_{0}^{\infty}\lambda^{2\alpha}dF(\lambda)+(2T)^{2\alpha}\left(\frac{\delta}{\beta}\right)^{2\delta}\int_{0}^{\infty}\lambda^{2\nu}dF(\lambda)\right)+$$

$$\left.+\frac{2^{2(2-\alpha)}T^{2\alpha}\pi^{2\alpha}}{\left(\ln\left(\frac{1}{h}+1\right)\right)^{2\delta}}\int_{0}^{\infty}\lambda^{2\alpha}dF(\lambda)\sum_{k=M+1}^{\infty}\frac{\left(\ln\left(k^2+e^{\delta}\right)\right)^{2\delta}}{k^{2\alpha}}\right]^{\frac{1}{2}}.$$

Now inequality (3.9) implies

$$\sup_{\substack{|t-s|\leqslant h\\|x-y|\leqslant h}}\tau(\chi_M(t,x)-\chi_M(s,y))\leqslant\left[2\cdot4^{2(2-\alpha)}\frac{1}{\left(\frac{\alpha}{\delta}\right)^{2\delta}\left(\ln\left(\frac{1}{h}+1\right)\right)^{2\delta}}\left(\frac{\pi}{2}\right)^{2\alpha}\times\right.$$

$$\times\frac{M}{2\alpha-1}\left(2\alpha-\frac{1}{M^{2\alpha-1}}\right)\sum_{l=0}^{N-2}|\lambda_{l+1}-\lambda_l|^{2\alpha}\left(b_l^2+\left[\left(\frac{T(1+C)}{4}\right)^{2\alpha}+\right.\right.$$

$$\left.\left.+T^{2\alpha}(1+2C^{\alpha})+\left(\frac{T^2\Lambda(1+C)}{2}\right)^{2\alpha}\right]\int_{\lambda_l}^{\lambda_{l+1}}\lambda^{2\alpha}dF(\lambda)\right)+\frac{9\cdot4^{4-2\alpha}\cdot M^2}{\left(\frac{\alpha}{\delta}\right)^{2\delta}\left(\ln\left(\frac{1}{h}+1\right)\right)^{2\delta}}\times$$

$$\times\left(\int_{\Lambda}^{\infty}|\lambda-\Lambda|^{2\alpha}dF(\lambda)+2^{2\alpha}\Lambda^{2\alpha}(F(+\infty)-F(\Lambda))\right)+\frac{4^{4-2\alpha}T^{2\alpha}\pi^{2\alpha}M}{\left(\ln\left(\frac{1}{h}+1\right)\right)^{2\delta}}\times$$

$$\times\left(\sum_{k=1}^{M}\frac{\left(\ln\left(k^2+e^{\delta}\right)\right)^{2\delta}}{k^{2\alpha}}\right)\sum_{l=0}^{N-2}|\lambda_{l+1}-\lambda_l|^{2\alpha}\left(b_l^2+\left(\frac{T(1+C)}{2}\right)^{2\alpha}\times\right.$$

$$\left.\times\int_{\lambda_l}^{\lambda_{l+1}}\lambda^{2\alpha}dF(\lambda)\right)+\frac{16M(F(+\infty)-F(\Lambda))}{\left(\ln\left(\frac{1}{h}+1\right)\right)^{2\delta}}\sum_{k=1}^{M}\left(\ln\left(k^2+e^{\delta}\right)\right)^{2\delta}+$$

$$+\frac{4^{3-2\alpha}\pi^{2\alpha}}{(2\alpha-1)M^{2\alpha-1}}\cdot\frac{1}{\left(\ln\left(\frac{1}{h}+1\right)\right)^{2\delta}}\left(\left(\frac{\delta}{\alpha}\right)^{2\delta}\int_{0}^{\infty}\lambda^{2\alpha}dF(\lambda)+\right.$$

$$+(2T)^{2\alpha}\left(\frac{\delta}{\beta}\right)^{2\delta}\int\limits_0^\infty \lambda^{2\nu}dF(\lambda)\Bigg)+\frac{2^{2(2-\alpha)}T^{2\alpha}\pi^{2\alpha}}{\left(\ln\left(\frac{1}{h}+1\right)\right)^{2\delta}}\int\limits_0^\infty \lambda^{2\alpha}dF(\lambda)\times$$

$$\times\sum_{k=M+1}^{\infty}\frac{\left(\ln\left(k^2+e^{\delta}\right)\right)^{2\delta}}{k^{2\alpha}}\Bigg]^{\frac{1}{2}},$$

whence

$$\sigma(h)\leqslant\frac{1}{\left(\ln\left(\frac{1}{h}+1\right)\right)^{\delta}}\Bigg[2\cdot 4^{2(2-\alpha)}\left(\frac{\delta}{\alpha}\right)^{2\delta}\left(\frac{\pi}{2}\right)^{2\alpha}\frac{M}{2\alpha-1}\left(2\alpha-\frac{1}{M^{2\alpha-1}}\right)\times$$

$$\times\sum_{l=0}^{N-2}|\lambda_{l+1}-\lambda_l|^{2\alpha}\Bigg(b_l^2+\Bigg[\left(\frac{T(1+C)}{4}\right)^{2\alpha}+T^{2\alpha}(1+2C^{\alpha})+$$

$$+\left(\frac{T^2\Lambda(1+C)}{2}\right)^{2\alpha}\Bigg]\int\limits_{\lambda_l}^{\lambda_{l+1}}\lambda^{2\alpha}dF(\lambda)\Bigg)+9\cdot 4^{4-2\alpha}\cdot M^2\left(\frac{\delta}{\alpha}\right)^{2\delta}\times$$

$$\times\left(\int\limits_{\Lambda}^{\infty}|\lambda-\Lambda|^{2\alpha}dF(\lambda)+2^{2\alpha}\Lambda^{2\alpha}(F(+\infty)-F(\Lambda))\right)+4^{4-2\alpha}T^{2\alpha}\pi^{2\alpha}M\times$$

$$\times\left(\sum_{k=1}^{M}\frac{\left(\ln\left(k^2+e^{\delta}\right)\right)^{2\delta}}{k^{2\alpha}}\right)\sum_{l=0}^{N-2}|\lambda_{l+1}-\lambda_l|^{2\alpha}\left(b_l^2+\left(\frac{T(1+C)}{2}\right)^{2\alpha}\int\limits_{\lambda_l}^{\lambda_{l+1}}\lambda^{2\alpha}dF(\lambda)\right)+$$

$$+16M(F(+\infty)-F(\Lambda))\sum_{k=1}^{M}\left(\ln\left(k^2+e^{\delta}\right)\right)^{2\delta}+\frac{4^{3-2\alpha)}\pi^{2\alpha}}{(2\alpha-1)M^{2\alpha-1}}\times$$

$$\times\left(\left(\frac{\delta}{\alpha}\right)^{2\delta}\int\limits_0^\infty\lambda^{2\alpha}dF(\lambda)+(2T)^{2\alpha}\left(\frac{\delta}{\beta}\right)^{2\delta}\int\limits_0^\infty\lambda^{2\nu}dF(\lambda)\right)+$$

$$+2^{2(2-\alpha)}T^{2\alpha}\pi^{2\alpha}\int\limits_0^\infty\lambda^{2\alpha}dF(\lambda)\sum_{k=M+1}^{\infty}\frac{\left(\ln\left(k^2+e^{\delta}\right)\right)^{2\delta}}{k^{2\alpha}}\Bigg]^{\frac{1}{2}}.\quad\diamondsuit$$

Corollary 3.3. *Let a partition $L=\{\lambda_0,...,\lambda_N\}$ of the set $[0,\infty)$ be such that $\lambda_l<\lambda_{l+1}$ and $\lambda_{l+1}-\lambda_l=\frac{\Lambda}{N-1}$. Let all the assumptions of Theorem 3.7 hold. Then*

$$\sigma(h)\leqslant\frac{C_1}{\left(\ln\left(\frac{1}{h}+1\right)\right)^{\delta}},$$

where

$$C_1 = \left[2 \cdot 4^{2(2-\alpha)} \left(\frac{\delta}{\alpha}\right)^{2\delta} \left(\frac{\pi}{2}\right)^{2\alpha} \frac{M}{2\alpha - 1} \left(2\alpha - \frac{1}{M^{2\alpha-1}}\right) \left(\frac{\Lambda}{N-1}\right)^{2\alpha} \times\right.$$

$$\times \left(F(\Lambda) + \left[\left(\frac{3T}{4}\right)^{2\alpha} + (1 + 2^{\alpha+1})T^{2\alpha} + \left(\frac{3T^2\Lambda}{2}\right)^{2\alpha}\right] \int_0^\Lambda \lambda^{2\alpha} dF(\lambda)\right) + 9 \cdot 4^{4-2\alpha} \times$$

$$\times M^2 \left(\frac{\delta}{\alpha}\right)^{2\delta} \left(\int_\Lambda^\infty |\lambda - \Lambda|^{2\alpha} dF(\lambda) + 2^{2\alpha}\Lambda^{2\alpha}(F(+\infty) - F(\Lambda))\right) + 4^{4-2\alpha}T^{2\alpha}\pi^{2\alpha}M \times$$

$$\times \left(\sum_{k=1}^{M} \frac{\left(\ln\left(k^2 + e^\delta\right)\right)^{2\delta}}{k^{2\alpha}}\right) \left(\frac{\Lambda}{N-1}\right)^{2\alpha} \left(F(\Lambda) + \left(\frac{3T}{2}\right)^{2\alpha} \int_0^\Lambda \lambda^{2\alpha} dF(\lambda)\right) +$$

$$+ 16M(F(+\infty) - F(\Lambda)) \sum_{k=1}^{M} \left(\ln\left(k^2 + e^\delta\right)\right)^{2\delta} + \frac{4^{3-2\alpha}\pi^{2\alpha}}{(2\alpha - 1)M^{2\alpha-1}} \times$$

$$\times \left(\left(\frac{\delta}{\alpha}\right)^{2\delta} \int_0^\infty \lambda^{2\alpha} dF(\lambda) + (2T)^{2\alpha} \left(\frac{\delta}{\beta}\right)^{2\delta} \int_0^\infty \lambda^{2\nu} dF(\lambda)\right) + 2^{4-\alpha}T^{2\alpha}\pi^{2\alpha} \times$$

$$\left.\times \int_0^\infty \lambda^{2\alpha} dF(\lambda) \sum_{k=M+1}^{\infty} \frac{\left(\ln\left(k^2 + e^\delta\right)\right)^{2\delta}}{k^{2\alpha}}\right]^{\frac{1}{2}}, \quad (3.10)$$

$\frac{1}{2} < \alpha \leqslant 1$, $\frac{\alpha}{\delta} \leqslant 1$, $\delta > 0$ *and* $0 < \beta \leqslant 1$.

Definition 3.4. A stochastic field $\hat{X}(t,x)$ approximated Gaussian field $X(t,x)$ with the reliability of $1 - \gamma$, $0 < \gamma < 1$ and accuracy $q > 0$ in the space $C(\mathbb{T})$, if there exists such partition of L, that inequality

$$\mathbf{P}\left\{\sup_{t \in \mathbb{T}} \mid X(t,x) - \hat{X}(t,x) \mid > q\right\} \leqslant \gamma$$

holds.

Theorem 3.8. *Consider* $\mathbb{R}^2$, $d(t,s) = \max_{1 \leqslant i \leqslant 2} \mid t_i - s_i \mid$, $\mathbb{T} = \{0 \leqslant t_i \leqslant T, i = 1,2\}, T > 0$ *and let* $X = \{X(t), t \in \mathbb{T}\}$ *be sub-Gaussian stochastic field. If* $\sup_{d(t,s) \leqslant h} \tau(X(t) - X(s)) \leqslant \sigma(h)$, *where* $\sigma(h)$ *is continuous, monotonically decreasing function, such that* $\sigma(h) \to 0$ *as* $h \to 0$ *and* $\int_{0+} \psi\left(\ln \frac{1}{\sigma^{(-1)}(\varepsilon)}\right) d\varepsilon <$

∞ *where* $\psi(u) = \left(\frac{u}{2}\right)^{\frac{1}{2}}$ *and* $\sigma^{(-1)}(\varepsilon)$ *is an inverse function to* $\sigma(\varepsilon)$.

Then $\mathbf{P}\left\{\sup\limits_{t\in\mathbb{T}} \mid X(t) \mid > u\right\} \leqslant 2\tilde{A}(u,\theta)$ *for all* $0 < \theta < 1$ *and* $u > \frac{2\tilde{I}(\theta\varepsilon_0)}{\theta(1-\theta)}$, *where*

$$\tilde{A}(u,\theta) = \exp\left\{-\frac{1}{2\varepsilon_0^2}\left(u(1-\theta) - \frac{2}{\theta}\tilde{I}(\theta\varepsilon_0)\right)^2\right\},$$

$$\varepsilon_0 = \sup_{t\in\mathbb{T}}\left(\mathbb{E} \mid X(t) \mid^2\right)^{\frac{1}{2}},$$

$$\tilde{I}(v) = \int\limits_0^v \left(\ln\left(\frac{T}{2\sigma^{(-1)}(\varepsilon)} + 1\right)\right)^{\frac{1}{2}} d\varepsilon.$$

The Theorem 3.8 is a particular case of the Theorem 1.8 from the [29].

Theorem 3.9. *Let in model* $\hat{X}(t,x)$ *the split* L *be such that when* $q > \frac{2\tilde{I}(\theta\varepsilon_0)}{\theta(1-\theta)}$, $0 < \theta < 1$ *the following relationship takes place*

$$2\exp\left\{-\frac{1}{2\varepsilon_0^2}\left(q(1-\theta) - \frac{2}{\theta}\tilde{I}(\theta\varepsilon_0)\right)^2\right\} \leqslant \gamma,$$

where $\varepsilon_0 = \sup\limits_{0\leqslant t\leqslant T} \tau(\chi_M(t,x)) = \sigma_0$, $\chi_M(t,x)$ *is defined in (3.7) and let* $\tilde{I}(\theta\varepsilon_0) \leqslant \hat{I}(\theta\varepsilon_0)$, *where*

$$\hat{I}(\theta\varepsilon_0) = \int\limits_0^{\theta\varepsilon_0} \sqrt{\ln\left(\frac{T}{2}\left(\exp\left\{\left(\frac{C_1}{\varepsilon}\right)^{\frac{1}{\delta}}\right\} - 1\right) + 1\right)} d\varepsilon,$$

C_1 *represented by the formula (3.10),* $\frac{1}{2} < \alpha \leqslant 1$, $\frac{\alpha}{\delta} \leqslant 1$, $\delta > 0$, $0 < \beta \leqslant 1$ *and* $\nu > \frac{1}{2}$.

Then the model $\hat{X}(t,x)$ *approximates Gaussian stochastic field* $X(t,x)$ *with a given reliability* $1-\gamma$, $0 < \gamma < 1$ *and accuracy* $q > 0$ *in the space* $C(\mathbb{T})$.

Proof. According to the Theorem 3.8 if $q > \frac{2\tilde{I}(\theta\varepsilon_0)}{\theta(1-\theta)}$, $0 < \theta < 1$ then for

$\chi_M(t,x)$ the following inequality holds

$$\mathbf{P}\left\{\sup_{t\in\mathbb{T}} \mid \chi_M(t,x) \mid > q\right\} \leqslant 2\exp\left\{-\frac{1}{2\varepsilon_0^2}\left(q(1-\theta)-\frac{2}{\theta}\tilde{I}(\theta\varepsilon_0)\right)^2\right\},$$

де $\tilde{I}(\theta\varepsilon_0) = \int\limits_0^{\theta\varepsilon_0}\left(\ln\left(\frac{T}{2\sigma^{(-1)}(\varepsilon)}+1\right)\right)^{\frac{1}{2}} d\varepsilon$, $\sigma(h) = \sup\limits_{\substack{|t-s|\leqslant h\\ |x-y|\leqslant h}} \tau(\chi_M(t,x)-\chi_M(s,y))$.

From Theorem 3.7 for $\sigma(h)$ we have

$$\sigma^{(-1)}(h) = \frac{1}{\exp\left\{\left(\frac{C_1}{h}\right)^{\frac{1}{\delta}}\right\}-1},$$

where $\frac{1}{2} < \alpha \leqslant 1$, $\frac{\alpha}{\delta} \leqslant 1$, $\delta > 0$, $0 < \beta \leqslant 1$, $\nu > \frac{1}{2}$ and C_1 is defined as in (3.10), then

$$\tilde{I}(\theta\varepsilon_0) \leqslant \int\limits_0^{\theta\varepsilon_0}\sqrt{\ln\left(\frac{T}{2}\left(\exp\left\{\left(\frac{C_1}{\varepsilon}\right)^{\frac{1}{\delta}}\right\}-1\right)+1\right)}d\varepsilon = \hat{I}(\theta\varepsilon_0),$$

that can be made an arbitrarily small at a certain selection M, Λ and N. Specifically, for a given accuracy and reliability we choose M so that the fifth and sixth terms in the (3.10) were arbitrarily small. Further, considering the resulting value of M, we choose Λ so that the second and the fourth terms were small in the ratio (3.10). And finally, considering the value of the M and Λ, we choose N so that one and three terms in the (3.10) were arbitrarily small. It should be noted that with this choice of M, Λ and N is an arbitrarily small not only the C_1, but also ε_0, which is defined in the Theorem 3.6. That there exists a partition L, for which holds

$$2\exp\left\{-\frac{1}{2\varepsilon_0^2}\left(q(1-\theta)-\frac{2}{\theta}\tilde{I}(\theta\varepsilon_0)\right)^2\right\} \leqslant \gamma.$$

This, together with the Definition 3.4 means that the constructed model $\hat{X}(t,x)$ approximates $X(t,x)$ with a given reliability $1-\gamma$, $0<\gamma<1$ and accuracy $q>0$ in the space $C(\mathbb{T})$. $\diamondsuit$

Example 3.3. Consider the model $\hat{X}(t,x)$ of Gaussian homogeneous and isotropic stochastic field, representation of which is given in (3.5). For this model we put

$$F(\lambda)=\begin{cases} 1-\frac{1}{\lambda^4}, & \text{if } \lambda\geqslant 1,\\ 0, & \text{if } \lambda<1\end{cases}.$$

We estimate the value C_1 and ε_0. For this, we presents them in the following forms

$$C_1=\left(C_I+C_{II}+C_{III}\right)^{\frac{1}{2}},$$

where

$$C_I=\frac{4^{3-2\alpha}\pi^{2\alpha}}{(2\alpha-1)M^{2\alpha-1}}\left(\left(\frac{\delta}{\alpha}\right)^{2\delta}\int\limits_0^\infty\lambda^{2\alpha}dF(\lambda)+(2T)^{2\alpha}\left(\frac{\delta}{\beta}\right)^{2\delta}\int\limits_0^\infty\lambda^{2\nu}dF(\lambda)\right)+$$

$$+2^{4-\alpha}T^{2\alpha}\pi^{2\alpha}\int\limits_0^\infty\lambda^{2\alpha}dF(\lambda)\sum_{k=M+1}^{\infty}\frac{\left(\ln\left(k^2+e^\delta\right)\right)^{2\delta}}{k^{2\alpha}},$$

$$C_{II}=9{\cdot}4^{4-2\alpha}M^2\left(\frac{\delta}{\alpha}\right)^{2\delta}\left(\int\limits_\Lambda^\infty|\lambda-\Lambda|^{2\alpha}dF(\lambda)+2^{2\alpha}\Lambda^{2\alpha}(F(+\infty)-F(\Lambda))\right)+$$

$$+16M(F(+\infty)-F(\Lambda))\sum_{k=1}^{M}\left(\ln\left(k^2+e^\delta\right)\right)^{2\delta},$$

$$C_{III}=2\cdot4^{2(2-\alpha)}\left(\frac{\delta}{\alpha}\right)^{2\delta}\left(\frac{\pi}{2}\right)^{2\alpha}\frac{M}{2\alpha-1}\left(2\alpha-\frac{1}{M^{2\alpha-1}}\right)\left(\frac{\Lambda}{N-1}\right)^{2\alpha}\times$$

$$\times\left(F(\Lambda)+\left[\left(\frac{3T}{4}\right)^{2\alpha}+(1+2^{\alpha+1})T^{2\alpha}+\left(\frac{3T^2\Lambda}{2}\right)^{2\alpha}\right]\int\limits_0^\Lambda\lambda^{2\alpha}dF(\lambda)\right)+4^{4-2\alpha}T^{2\alpha}\times$$

$$\pi^{2\alpha}M\left(\sum_{k=1}^{M}\frac{\left(\ln\left(k^2+e^\delta\right)\right)^{2\delta}}{k^{2\alpha}}\right)\left(\frac{\Lambda}{N-1}\right)^{2\alpha}\left(F(\Lambda)+\left(\frac{3T}{2}\right)^{2\alpha}\int\limits_0^\Lambda\lambda^{2\alpha}dF(\lambda)\right).$$

and

$$\varepsilon_0=\left(\varepsilon_I+\varepsilon_{II}+\varepsilon_{III}\right)^{\frac{1}{2}},$$

where

$$\varepsilon_I=\frac{2^{2(1-\alpha)+1}T^{2\alpha}\pi^{2\alpha}}{(2\alpha-1)M^{2\alpha-1}}\int\limits_0^\infty\lambda^{2\alpha}dF(\lambda),$$

$$\varepsilon_{II} = 8M^2(F(+\infty) - F(\Lambda)),$$

$$\varepsilon_{III} = \frac{4^{2(1-\alpha)+1}T^{2\alpha}\pi^{2\alpha}M}{2\alpha - 1}\left(2\alpha - \frac{1}{M^{2\alpha-1}}\right)\left(\frac{\Lambda}{N-1}\right)^{2\alpha} \times$$
$$\times \left(F(\Lambda) + \left(\frac{3T}{2}\right)^{2\alpha}\int\limits_0^\Lambda \lambda^{2\alpha} dF(\lambda)\right).$$

We choose $\alpha = 1$, $\beta = \frac{1}{2}$, $\delta = 1$, $\nu = \frac{3}{2}$, $T = 1$, after transformations we obtain

$$C_I = \frac{784\pi^2}{3M} + 16\pi^2 \sum_{k=M+1}^{\infty} \frac{\left(\ln(k^2+e)\right)^2}{k^2},$$

$$C_{II} = \frac{336M^2}{\Lambda^2} + \frac{16M}{\Lambda^4}\sum_{k=1}^{M}\left(\ln\left(k^2+e\right)\right)^2,$$

$$C_{III} = 8\pi^2(2M-1)\left(\frac{\Lambda}{N-1}\right)^2\left(\frac{9}{2}\Lambda^2 - \frac{89}{8\Lambda^2} - \frac{1}{\Lambda^4} + \frac{61}{8}\right) +$$
$$+ 16\pi^2 M\left(\frac{\Lambda}{N-1}\right)\left(\frac{11}{2} - \frac{9}{2\Lambda^2} - \frac{1}{\Lambda^4}\right)\sum_{k=1}^{M}\frac{\left(\ln\left(k^2+e\right)\right)^2}{k^2}.$$

Then we choose accuracy and reliability with which our model approximates the stochastic field, namely $q = 0,06$, $1-\gamma = 0,99$. In addition, let $\theta = \frac{1}{2}$. Then by the Theorem 3.9 we obtain

$$2\exp\left\{-\frac{1}{2\varepsilon_0^2}\left(0,06\cdot\frac{1}{2} - 4\hat{I}\left(\frac{\varepsilon_0}{2}\right)\right)^2\right\} \leqslant 0,01,$$

where

$$\hat{I}\left(\frac{\varepsilon_0}{2}\right) = \int\limits_0^{\frac{\varepsilon_0}{2}}\sqrt{\ln\left(\frac{1}{2}\left(\exp\left\{\left(\frac{C_1}{\varepsilon}\right)\right\} - 1\right) + 1\right)}d\varepsilon =$$
$$= \int\limits_0^{\frac{\varepsilon_0}{2}}\sqrt{\ln\left(\frac{1}{2}\exp\left\{\frac{C_1}{\varepsilon}\right\} + \frac{1}{2}\right)}d\varepsilon,$$

therefore

$$2\exp\left\{-\frac{1}{2\varepsilon_0^2}\left(0,03-4\int\limits_0^{\frac{\varepsilon_0}{2}}\sqrt{\ln\left(\frac{1}{2}\exp\left\{\frac{C_1}{\varepsilon}\right\}+\frac{1}{2}\right)}d\varepsilon\right)^2\right\}\leqslant 0,01.$$

By the help of approximate numerical methods, we can obtain, that for $\widehat{C_1}=1,99$ and $\widehat{\varepsilon_0}=3,91$ this inequality holds, so we obtained

$$(C_I+C_{II}+C_{III})^{\frac{1}{2}}\leqslant\widehat{C_1}$$

and

$$(\varepsilon_I+\varepsilon_{II}+\varepsilon_{III})^{\frac{1}{2}}\leqslant\widehat{\varepsilon_0}.$$

Without decreasing of the generality, we put $C_I\leqslant\frac{\widehat{C_1}^2}{3}$, $C_{II}\leqslant\frac{\widehat{C_1}^2}{3}$, $C_{III}\leqslant\frac{\widehat{C_1}^2}{3}$ and $\varepsilon_I\leqslant\frac{\widehat{\varepsilon_0}^2}{3}$, $\varepsilon_{II}\leqslant\frac{\widehat{\varepsilon_0}^2}{3}$, $\varepsilon_{III}\leqslant\frac{\widehat{\varepsilon_0}^2}{3}$.

Solving the inequality for C_I and ε_I by M, we obtain two values for M, from these values we select the maximum. Taking into account the found value of M we solve the inequalities for C_{II} and ε_{II} by Λ and we select the maximum of them. Substituting the found values for M and Λ to the inequalities for C_{III} and ε_{III}, similarly we find value of N.

By using the software package Mathematica, we found that $M=32$, $\Lambda=65$, $N=69442$. Using these values we can construct a model $\hat{X}(t,x)$ that that approximates Gaussian homogeneous and isotropic field $X(t,x)$ with reliability 0,99 and accuracy 0,06 in the space $C(\mathbb{T})$.

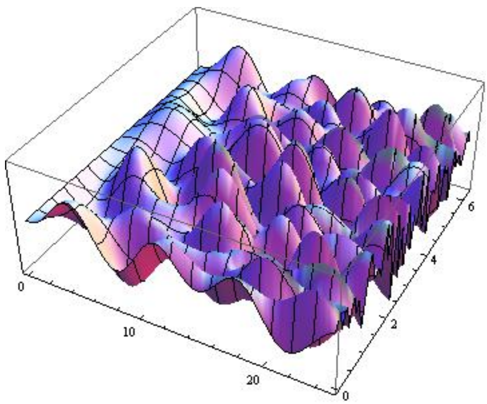

Figure 3.2. The sample path of the model of Gaussian homogeneous and isotropic stochastic field with reliability 0,99 and accuracy 0,06 in $C(\mathbb{T})$.

Chapter 4
The estimation of the correlation function of stationary Gaussian process in L_2 metric.

The estimation of correlation functions of stochastic processes and construction of the criteria for identification of these functions stay an important task in the theory of stochastic processes and fields. Intensive study of these problems is associated with the active use of obtained results in the theory of stochastic processes and in areas where it is used. Criteria for testing of hypothesis about correlation function are based on the estimations for distribution of correlogram deviation from correlation function. Many books are devoted to correlogram-type estimates of the correlation function of a stationary Gaussian process([7], [28], [90], [144], [141].) Among them the book [19] should be specially mentioned. In this book correlograms of stationary stochastic processes and their main properties are investigated. Correlogram-type estimates are considered also in the works [80], [57], [61], [78].

In this chapter a separable real-valued stationary Gaussian process $\xi(t)$ is considered. The estimates for distribution of correlogram deviation from correlation function of this process in L_2-metric are obtained. The estimation is carried out by observing one sample path of the process. Sample correlation function or correlogram is used as an estimate.

4.1. The estimation of the correlation function of stationary Gaussian process by using correlograms

Assume that $\xi = (\xi(t), t \in [0, T+B], 0 < B < \infty)$ is a separable real-valued stationary Gaussian process defined on a probability space $\{\Omega, \mathcal{B}, P\}$, with mean zero and a continuous correlation function

$$\rho(\tau) = E\xi(t+\tau)\xi(t), \quad 0 \leqslant \tau \leqslant B.$$

(this means that process is continuous in mean square)

By the well-known Belyaev alternative, sample paths of separable stationary continuous in mean square Gaussian process are continuous with probability one on bounded interval or are such that with probability one on any interval I

$$\sup_{t \in I} X(t) = +\infty, \quad \inf_{t \in I} X(t) = -\infty.$$

So, we can estimate correlation function only in the case, when sample paths of stationary Gaussian process are sample continuous with probability one.

Assume, that sample paths of process $\xi(t)$ are continuous with

probability one on any interval $[\mathbf{0}, \mathbf{T}], \mathbf{T} \geqslant \mathbf{0}$. Necessary and sufficient conditions of this fact are Dudley-Fernique's conditions:

for some $\varepsilon > 0$

$$\int_0^{\varepsilon} (H_T(u))^{1/2}\, du < \infty,$$

$H_T(\varepsilon)$ - metrical entropy of the space $([0, T], \rho)$, where ρ - pseudometric, $\rho(t, s) = \left(E(\xi(t) - \xi(s))^2\right)^{1/2}$, a $H_T(\varepsilon) = \ln N_T(\varepsilon)$, where $N_T(\varepsilon)$ the minimum number of closed balls of radius ε, which cover $([0, T], \rho)$.

Thus, will assume, that for process $\xi(t)$ Dudley-Fernique's condition holds. We note, that Dudley-Fernique's condition holds if for some $\varepsilon > 0$ at sufficiently small τ one of following condition holds:

$$E|\xi(t + \tau) - \xi(t)|^2 \leqslant \frac{1}{|\ln|\tau||^{1+\varepsilon}}$$

or

$$\int_0^{\infty} \ln^{1+\varepsilon}(1 + \lambda) dF(\lambda) < \infty, \quad \varepsilon > 0,$$

where $F(\lambda)$ - spectral function of stochastic process ξ . Latest two inequalities are close to necessary conditions.

Let $\xi(t)$ be a single sample path of the stationary process. Consider sample correlation function or correlogram

$$\widehat{\rho}_T(\tau) = \frac{1}{T} \int_0^T \xi(t + \tau)\xi(t)dt, \quad T \geqslant 0 \tag{4.1}$$

as an estimate of correlation function $\rho(\tau)$. Since $\rho(\tau)$ is an even function, then only positive τ $(\tau \geqslant 0)$ will be considered. Under our conditions integral in (4.1) becomes a usual Riemann integral constructed after a single sample path of the process $\xi(t)$, and, as a matter of fact, this integral represents an almost surely continuous process with respect to τ. Therefore the correlogram can be viewed as a continuous in probability stochastic process. This argument enables us to assume that the process $\widehat{\rho}_T(\tau)$ is separable.

It is easy to calculate mean for $\widehat{\rho}_T(\tau)$:

$$E\widehat{\rho}_T(\tau) = E\left(\frac{1}{T} \int_0^T \xi(t + \tau)\xi(t)dt\right) = \frac{1}{T} \int_0^T E\xi(t + \tau)\xi(t)dt = \rho(\tau)$$

for each $T > 0$ та $\tau \geqslant 0$.

Hence, $\widehat{\rho}_T(\tau)$ is an unbiased estimate of $\rho(\tau)$. The accuracy of estimation is given by the difference $\widehat{\rho}_T(\tau)-\rho(\tau)$. Using the Isserlis formula for jointly Gaussian random variables $\xi_i, i=1,\ldots,4$, $E\xi_i=0$:

$$E\xi_1\xi_2\xi_3\xi_4 = E\xi_1\xi_2 E\xi_3\xi_4 + E\xi_1\xi_3 E\xi_2\xi_4 + E\xi_1\xi_4 E\xi_2\xi_3 \qquad (4.2)$$

we can calculate $D\widehat{\rho}_T(\tau)$:

$$D\widehat{\rho}_T(\tau) = E\left(\widehat{\rho}_T(\tau)-\rho(\tau)\right)^2 = E\left(\widehat{\rho}_T(\tau)\right)^2 - \rho^2(\tau) =$$

$$= E\left(\frac{1}{T^2}\int_0^T\int_0^T \xi(t+\tau)\xi(t)\xi(s+\tau)\xi(s)dtds\right) - \rho^2(\tau) =$$

$$= \frac{1}{T^2}\int_0^T\int_0^T [E\xi(t+\tau)\xi(t)E\xi(s+\tau)\xi(s) + E\xi(t+\tau)\xi(s+\tau)E\xi(t)\xi(s)+$$

$$+ E\xi(t+\tau)\xi(s)E\xi(t)\xi(s+\tau)]\, dtds - \rho^2(\tau) =$$

$$= \frac{1}{T^2}\int_0^T\int_0^T \left[\rho^2(\tau)+\rho^2(t-s)+\rho(t-s+\tau)\rho(t-s-\tau)\right] dtds - \rho^2(\tau) =$$

$$= \frac{1}{T^2}\int_0^T\int_0^T \left[\rho^2(t-s)+\rho(t-s+\tau)\rho(t-s-\tau)\right] dtds.$$

Consider the difference $\zeta(\tau)=\widehat{\rho}_T(\tau)-\rho(\tau)$.

Lemma 4.1. *For any $\tau \geqslant 0$ $\zeta(\tau)$ is square Gaussian random variable.*

Proof. Since $\widehat{\rho}_T(\tau)$ is a mean square limit of integral sums of the type $\frac{1}{T}\sum\limits_k \xi(t_k+\tau)\xi(t_k)\Delta t_k$, and each integral sum is quadratic form of Gaussian random vectors, then $\zeta(\tau)$ is square Gaussian random variable for any $\tau \geqslant 0$. Therefore, $\zeta(\tau)$ is square Gaussian stochastic process. ◊

Consider random variable $\eta = \int_0^B \left(\widehat{\rho}_T(\tau)-\rho(\tau)\right)^2 d\tau$, $0<B<\infty$. We can calculate $E\eta$:

$$E\eta = E\int_0^B \left(\widehat{\rho}_T(\tau)-\rho(\tau)\right)^2 d\tau =$$

$$= \frac{1}{T^2}\int_0^B\int_0^T\int_0^T \left[\rho^2(t-s)+\rho(t-s+\tau)\rho(t-s-\tau)\right] dtdsd\tau =$$

$$= \frac{2}{T^2}\int_0^B\int_0^T (T-u)\left[\rho^2(u)+\rho(u+\tau)\rho(u-\tau)\right] dud\tau.$$

Since η is a mean square limit of integral sums $\sum\limits_k \zeta^2(\tau_k)\Delta\tau_k$, where $\zeta(\tau_k)$ is square Gaussian random variable, then the next theorem holds.

Theorem 4.1. *For the estimate $\widehat{\rho}_T(\tau)$ of correlation function $\rho(\tau)$ stationary Gaussian process $\xi = \{\xi(t), t \in [0, B+T]\}$ the following inequalities hold*

$$P\left\{\int\limits_0^B \left(\widehat{\rho}_T(\tau) - \rho(\tau)\right)^2 d\tau > x\int\limits_0^B D\widehat{\rho}_T(\tau)d\tau\right\} \geqslant 1 - g(u)\exp\left\{\frac{u^2x}{2}\right\} \quad (4.3)$$

for $u > 0$, $0 < x < -\frac{2\ln g(u)}{u^2}$,
where $g(u) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\exp\left\{-\frac{s^2}{2}\right\}\frac{ds}{(1+s^2u^2)^{\frac{1}{4}}}$ *and*

$$P\left\{\int\limits_0^B \left(\widehat{\rho}_T(\tau) - \rho(\tau)\right)^2 d\tau > y\int\limits_0^B D\widehat{\rho}_T(\tau)d\tau\right\} \leqslant \frac{2^{\frac{1}{4}}y^{\frac{1}{4}}}{\operatorname{ch}\left(\sqrt{\frac{y}{2}} - \frac{1}{2}\right)} \quad (4.4)$$

for $y > \frac{1}{2}$.

Proof. The proof is immediate by theorem 1.3, corollary 1.5 and previous lemma. ♢

Remark 4.1. Theorem 4.1 enable us to construct confidence sets for correlation function of stationary Gaussian process $\xi(t)$.

Let H be the hypothesis that for $0 \leqslant \tau \leqslant B$ the correlation function of separable real-valued stationary Gaussian process ξ equals $\rho(\tau)$. As an estimator for $\rho(\tau)$ we choose $\widehat{\rho}_T(\tau)$. To test the hypothesis H one can use the following criterion.

Criterion 4.1. For given level of confidence α, $0 < \alpha < 1$, we can find such positive x_α and y_α, that

$$s(x_\alpha, u) + f(y_\alpha) = \alpha,$$

where
$s(x, u) = g(u)\exp\left\{\frac{u^2x}{2}\right\}$, $u > 0$, $f(x) = \frac{2^{\frac{1}{4}}x^{\frac{1}{4}}}{\operatorname{ch}\left(\sqrt{\frac{x}{2}} - \frac{1}{2}\right)}$.

The hypothesis H is accepted if

$$x_\alpha < \frac{\int_0^B \left(\widehat{\rho}_T(\tau) - \rho(\tau)\right)^2 d\tau}{E\int_0^B \left(\widehat{\rho}_T(\tau) - \rho(\tau)\right)^2 d\tau} < y_\alpha$$

and hypothesis is rejected otherwise.

Remark 4.2. The probability of the first type's error does not exceed α when we use this criterion.

Remark 4.3. For given α, we can choose x_α and y_α in the following way.

Since

$$P\left\{x_\alpha \leqslant \frac{\eta}{E\eta} \leqslant y_\alpha\right\} \geqslant 1-\alpha,$$

then

$$P\left\{\frac{\eta}{E\eta} \overline{\in} [x_\alpha, y_\alpha]\right\} \leqslant \alpha.$$

The latter inequality holds if

$$P\left\{\frac{\eta}{E\eta} \leqslant tx_\alpha\right\} \leqslant \alpha\gamma \qquad \text{and} \qquad P\left\{\frac{\eta}{E\eta} \geqslant y_\alpha\right\} \leqslant \alpha(1-\gamma),$$

where $0 < \gamma < 1$. This means that x_α and y_α we can find from equations

$$g(u)\exp\left\{\frac{u^2 x_\alpha}{2}\right\} = \alpha\gamma,$$

$$\frac{2^{\frac{1}{4}} y_\alpha^{\frac{1}{4}}}{\operatorname{ch}\left(\sqrt{\frac{y_\alpha}{2}} - \frac{1}{2}\right)} = \alpha(1-\gamma).$$

It should be noted, that in this case $x_\alpha(\gamma)$ and $y_\alpha(\gamma)$ depend on γ. So, choice of γ $(0 < \gamma < 1)$ will enable minimize the difference $y_\alpha(\gamma) - x_\alpha(\gamma)$.

Remark 4.4. Consider the equation

$$g(u)\exp\left\{\frac{u^2 x_\alpha}{2}\right\} = \alpha.$$

For existence the positive solution

$$x_\alpha = \frac{2\ln\frac{\alpha}{g(u)}}{u^2}$$

the condition $\ln\frac{\alpha}{g(u)} > 0$ should holds, namely $g(u) < \alpha$, then
$u < g^{(-1)}(\alpha)$. Such u exist, because $g(u) \longrightarrow 0$ for $u \longrightarrow \infty$. If we denote

$$u_\alpha = g^{(-1)}(\alpha),$$

then

$$x_\alpha = \sup_{u > u_\alpha} \frac{2\ln\frac{\alpha}{g(u)}}{u^2}.$$

Example 4.1. Let the hipotesis consists in the fact that

$$\rho(\tau) = Ae^{-a|\tau|^2}, \quad A, a > 0$$

is the correlation function of stationary Gaussian stochastic process with mean zero. To test the hipotesis we can use criterion 4.1, where estimate $\widehat{\rho}_T(\tau)$ is defined in (4.1).

We would like to calculate $E\int_0^B (\widehat{\rho}_T(\tau) - \rho(\tau))^2 \, d\tau$, where $0 < B < \infty$ for this case.

$$E\int_0^B (\widehat{\rho}_T(\tau) - \rho(\tau))^2 \, d\tau = \frac{2B}{T^2}\int_0^T (T-u)\rho^2(u)du+$$

$$\frac{2}{T^2}\int_0^B\int_0^T (T-u)\rho(u+\tau)\rho(u-\tau)dud\tau = I_1 + I_2.$$

Taking into account that

$$\int_0^T e^{-2au^2}du = \frac{1}{2\sqrt{a}}\int_0^{2T\sqrt{a}} e^{-\frac{t^2}{2}}dt = \sqrt{\frac{\pi}{2a}}\Phi(2T\sqrt{a}),$$

$$\int_0^T ue^{-2au^2}du = -\frac{1}{4a}\int_0^{-2aT^2} e^t dt = \frac{1}{4a}\left(1 - e^{-2aT^2}\right),$$

where $\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_0^x e^{-\frac{t^2}{2}}dt$, we obtain

$$I_1 = \frac{2BA^2}{T^2}\left[T\sqrt{\frac{\pi}{2a}}\Phi(2T\sqrt{a}) + \frac{1}{4a}\left(e^{-2aT^2} - 1\right)\right],$$

$$I_2 = \frac{2A^2}{T^2}\sqrt{\frac{\pi}{2a}}\Phi(2B\sqrt{a})\left[T\sqrt{\frac{\pi}{2a}}\Phi(2T\sqrt{a}) + \frac{1}{4a}\left(e^{-2aT^2} - 1\right)\right]$$

Thus,

$$I_1+I_2 = \frac{2A^2}{T^2}\left(B + \sqrt{\frac{\pi}{2a}}\Phi(2B\sqrt{a})\right)\left(T\sqrt{\frac{\pi}{2a}}\Phi(2T\sqrt{a}) + \frac{1}{4a}\left(e^{-2T^2} - 1\right)\right).$$

Example 4.2. Let the hipotesis consists in the fact that

$$\rho(\tau) = Ae^{-a|\tau|}, \quad A, a > 0.$$

is the correlation function of stationary Gaussian stochastic process with mean zero. To test the hipotesis we can use criterion 4.1, where estimate $\widehat{\rho}_T(\tau)$ is defined in (4.1).

Let us calculate
$E\int_0^B\left(\widehat{\rho}_T(\tau)-\rho(\tau)\right)^2 d\tau$, where $0<B<\infty$.

$$E\int_0^B\left(\widehat{\rho}_T(\tau)-\rho(\tau)\right)^2 d\tau=\frac{2B}{T^2}\int_0^T(T-u)\rho^2(u)du+$$

$$\frac{2}{T^2}\int_0^B\int_0^T(T-u)\rho(u+\tau)\rho(u-\tau)dud\tau=I_1+I_2.$$

Since

$$\int_0^T e^{-2au}du=\frac{1}{2a}\left(1-e^{-2aT}\right),$$

$$\int_0^T ue^{-2au}du=-\frac{T}{2a}e^{-2aT}+\frac{1}{4a^2}\left(1-e^{-2aT}\right),$$

$$\int_0^B\int_0^T A^2e^{-a|u+\tau|}e^{-a|u-\tau|}dud\tau=A^2\int_0^B\left(\int_0^\tau e^{-2a\tau}du+\int_\tau^T e^{-2au}du\right)d\tau,$$

we will have

$$I_1=\frac{2BA^2}{T^2}\left[\frac{T}{2a}\left(1-e^{-2aT}\right)-\frac{1}{4a^2}\left(1-e^{-2aT}\right)-\frac{T}{2a}e^{-2aT}\right],$$

$$I_2=\frac{2A^2}{T^2}\left[\left(1-e^{-2aB}\right)\left(\frac{T}{2a^2}-\frac{3}{8a^3}\right)+\right.$$

$$\left.+e^{-2aB}\left(\frac{BT}{2a}+\frac{B^2}{4a}+\frac{B}{4a^2}\right)+\frac{B}{4a^2}e^{-2aT}\right].$$

So

$$I_1+I_2=\frac{2A^2}{T^2}\left[\frac{B}{a}\left(\frac{1}{2a}-T\right)e^{-2aT}+\frac{B^2}{4a}e^{-2aB}-\frac{1}{4a^3}\left(1-e^{-2aB}\right)\times\right.$$

$$\left.(32-2Ta+2a^2BT+Ba)\right].$$

4.2. The estimation of the correlation function of stationary Gaussian process when its value is known only in a finite set of points

During statistical processing of the results the estimate $\widehat{\rho}_T(\tau)$ can be obtained only approximately.

Assume that $\xi = (\xi(t), t \in [0, T+B], 0 < B < \infty)$ is a separable real-valued stationary Gaussian process defined on a probability space $\{\Omega, \mathcal{B}, P\}$, with mean zero and correlation function

$$\rho(\tau) = E\xi(t+\tau)\xi(t), \quad 0 \leqslant \tau \leqslant B.$$

Assume also that we know the value of this process in points $t_i = \frac{iT}{n}$, $i = 0, 1, \ldots, n$, $n \in \mathbb{N}$, $\Delta t_i = \frac{T}{n}$.

Consider

$$\widehat{\rho}_{T,n}(\tau) = \frac{1}{T}\sum_{i=0}^{n-1} \xi(t_i+\tau)\xi(t_i)\Delta t_i = \frac{1}{n}\sum_{i=0}^{n-1} \xi\left(\frac{iT}{n}+\tau\right)\xi\left(\frac{iT}{n}\right), \qquad (4.5)$$

as an estimate of correlation function $\rho(\tau)$. where $\xi(t_i)$ and $\xi(t_i+\tau)$ are known values of this process, $t_i = \frac{iT}{n}$, $i = 0, 1, \ldots, n$, $n \in \mathbb{N}$, $\Delta t_i = \frac{T}{n}$.

It is easy to calculate that $\widehat{\rho}_{T,n}(\tau)$ is unbiased estimate of $\rho(\tau)$:

$$E\widehat{\rho}_{T,n}(\tau) = E\left(\frac{1}{n}\sum_{i=0}^{n-1} E\xi\left(\frac{iT}{n}+\tau\right)\xi\left(\frac{iT}{n}\right)\right) = \frac{1}{n}\sum_{i=0}^{n-1}\rho(\tau) = \rho(\tau).$$

Using the Isserlis formula (4.2), we obtain

$$D\widehat{\rho}_{T,n}(\tau) = E\left(\widehat{\rho}_{T,n}(\tau) - \rho(\tau)\right)^2 = E\widehat{\rho}^2_{T,n}(\tau) - \rho^2(\tau) =$$

$$= E\left(\frac{1}{n^2}\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} \xi\left(\frac{iT}{n}+\tau\right)\xi\left(\frac{iT}{n}\right)\xi\left(\frac{jT}{n}+\tau\right)\xi\left(\frac{jT}{n}\right)\right) - \rho^2(\tau) =$$

$$= \frac{1}{n^2}\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\left[E\xi\left(\frac{iT}{n}+\tau\right)\xi\left(\frac{iT}{n}\right)E\xi\left(\frac{jT}{n}+\tau\right)\xi\left(\frac{jT}{n}\right) + \right.$$

$$+ E\xi\left(\frac{iT}{n}+\tau\right)\xi\left(\frac{jT}{n}+\tau\right)E\xi\left(\frac{iT}{n}\right)\xi\left(\frac{jT}{n}\right) +$$

$$\left. + E\xi\left(\frac{iT}{n}+\tau\right)\xi\left(\frac{jT}{n}\right)E\xi\left(\frac{iT}{n}\right)\xi\left(\frac{jT}{n}+\tau\right)\right] - \rho^2(\tau) =$$

$$= \frac{1}{n^2}\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\left[\rho^2(\tau)+\rho^2\left(\frac{(i-j)T}{n}\right)+\rho\left(\frac{(i-j)T}{n}+\tau\right)\rho\left(\frac{(i-j)T}{n}-\tau\right)\right]-$$

$$-\rho^2(\tau) = \frac{1}{n^2}\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\left[\rho^2\left(\frac{(i-j)T}{n}\right)+\rho\left(\frac{(i-j)T}{n}+\tau\right)\rho\left(\frac{(i-j)T}{n}-\tau\right)\right].$$

It is easy to see, that $\widehat{\rho}_{T,n}(\tau)$ is is quadratic form of Gaussian random vectors, therefore $\zeta(\tau) = \widehat{\rho}_{T,n}(\tau) - \rho(\tau)$, $\tau \geqslant 0$ is square Gaussian stochastic process.

Consider $\eta = \int_0^B (\widehat{\rho}_{T,n}(\tau) - \rho(\tau))^2\, d\tau$, $B > 0$. $E\eta$ can be calculated on the following way:

$$E\eta = E\int_0^B (\widehat{\rho}_{T,n}(\tau) - \rho(\tau))^2\, d\tau =$$

$$= \frac{1}{n^2}\int_0^B\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\left[\rho^2\left(\frac{(i-j)T}{n}\right)+\rho\left(\frac{(i-j)T}{n}+\tau\right)\rho\left(\frac{(i-j)T}{n}-\tau\right)\right]d\tau =$$

$$= \frac{B}{n^2}\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\rho^2\left(\frac{(i-j)T}{n}\right)+\frac{1}{n^2}\int_0^B\left[\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\rho\left(\frac{(i-j)T}{n}+\tau\right)\times\right.$$

$$\left.\times\rho\left(\frac{(i-j)T}{n}-\tau\right)\right]d\tau.$$

Theorem 4.2. *For the estimate $\widehat{\rho}_{T,n}(\tau)$ of correlation function $\rho(\tau)$ stationary Gaussian process ξ the following inequalities hold*

$$P\left\{\int_0^B (\widehat{\rho}_{T,n}(\tau) - \rho(\tau))^2\, d\tau > x\int_0^B D\widehat{\rho}_{T,n}(\tau)d\tau\right\} \geqslant 1 - g(u)\exp\left\{\frac{u^2x}{2}\right\} \tag{4.6}$$

for $u > 0$, $0 < x < -\frac{2\ln g(u)}{u^2}$,
where $g(u) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\exp\left\{-\frac{s^2}{2}\right\}\frac{ds}{(1+s^2u^2)^{\frac{1}{4}}}$ and

$$P\left\{\int_0^B (\widehat{\rho}_{T,n}(\tau) - \rho(\tau))^2\, d\tau > y\int_0^B D\widehat{\rho}_{T,n}(\tau)d\tau\right\} \leqslant \frac{2^{\frac{1}{4}}y^{\frac{1}{4}}}{\operatorname{ch}\left(\sqrt{\frac{y}{2}}-\frac{1}{2}\right)} \tag{4.7}$$

for $y > \frac{1}{2}$.

Proof. The proof is immediate by theorem 1.3, corollary 1.5 and previous calculations. ◊

Remark 4.5. Theorem 4.2 enable us to construct confidence sets for correlation function of stationary Gaussian process $\xi(t)$.

Let H be the hypothesis that for $0 \leqslant \tau \leqslant B$ the correlation function of separable real-valued stationary Gaussian process ξ equals $\rho(\tau)$. As an estimator for $\rho(\tau)$ we choose $\widehat{\rho}_{T,n}(\tau)$. To test the hypothesis H one can use the following criterion.

Criterion 4.2. For given level of confidence α, $0 < \alpha < 1$, we can find such positive x_α and y_α, that

$$s(x_\alpha, u) + f(y_\alpha) = \alpha,$$

where
$s(x,u) = g(u) \exp\left\{\frac{u^2 x}{2}\right\}$, $u > 0$, $f(x) = \frac{2^{\frac{1}{4}} x^{\frac{1}{4}}}{\mathrm{ch}\left(\sqrt{\frac{x}{2}} - \frac{1}{2}\right)}$.

The hypothesis H is accepted if

$$x_\alpha < \frac{\int_0^B \left(\widehat{\rho}_{T,n}(\tau) - \rho(\tau)\right)^2 d\tau}{E \int_0^B \left(\widehat{\rho}_{T,n}(\tau) - \rho(\tau)\right)^2 d\tau} < y_\alpha$$

and hypothesis is rejected otherwise.

4.3. The estimation of the correlation function of stationary noncentered Gaussian process by using correlograms

Assume that $\xi = (\xi(t), t \in [0, T+B], 0 < B < \infty)$ is a separable real-valued stationary Gaussian process defined on a probability space $\{\Omega, \mathcal{B}, P\}$, with $E\xi(t) = m$ and correlation function

$$r(\tau) = E(\xi(t+\tau) - m)(\xi(t) - m), \quad \tau \geqslant 0.$$

Suppose, that we know observation of one sample path of the process.

As an estimate of correlation function $\rho(\tau)$ we consider

$$\widehat{r}_T(\tau) = \frac{1}{T} \int_0^T (\xi(t+\tau) - \widehat{m}_\tau)(\xi(t) - \widehat{m}) dt, \quad 0 \leqslant \tau \leqslant B, \qquad (4.8)$$

where $\widehat{m}$ and $\widehat{m}_\tau$ are the estimates for process's mean that are defined as following

$$\widehat{m} = \frac{1}{T} \int_0^T \xi(t) dt,$$

$$\widehat{m}_\tau = \frac{1}{T}\int_0^T \xi(t+\tau)dt.$$

Since $r(\tau) = E(\xi(t+\tau)-m)(\xi(t)-m) = E\xi(t+\tau)\xi(t) - m^2$, then $E\xi(t+\tau)\xi(t) = r(\tau) + m^2$ and the following equalities are correct:

$$E\widehat{m}_\tau\xi(t) = \frac{1}{T}\int_0^T E\xi(s+\tau)\xi(t)ds = \frac{1}{T}\int_0^T r(s-t+\tau)ds + m^2,$$

$$E\widehat{m}\xi(t) = \frac{1}{T}\int_0^T E\xi(s)\xi(t)ds = \frac{1}{T}\int_0^T r(s-t)ds + m^2,$$

$$E\widehat{m}_\tau\xi(t+\tau) = \frac{1}{T}\int_0^T E\xi(s+\tau)\xi(t+\tau)ds = \frac{1}{T}\int_0^T r(s-t)ds + m^2,$$

$$E\widehat{m}\xi(t+\tau) = \frac{1}{T}\int_0^T E\xi(s)\xi(t+\tau)ds = \frac{1}{T}\int_0^T r(s-t-\tau)ds + m^2,$$

$$E\widehat{m}_\tau\widehat{m} = \frac{1}{T^2}\int_0^T\int_0^T E\xi(s+\tau)\xi(y)dsdy = \frac{1}{T^2}\int_0^T\int_0^T r(s-y+\tau)dsdy + m^2,$$

$$E\widehat{m}_\tau^2 = \frac{1}{T^2}\int_0^T\int_0^T E\xi(s+\tau)\xi(y+\tau)dsdy = \frac{1}{T^2}\int_0^T\int_0^T r(s-y)dsdy + m^2,$$

$$E\widehat{m}^2 = \frac{1}{T^2}\int_0^T\int_0^T E\xi(s)\xi(y)dsdy = \frac{1}{T^2}\int_0^T\int_0^T r(s-y)dsdy + m^2.$$

We will use these results further. Let us calculate $E\widehat{r}_T(\tau)$.

$$E\widehat{r}_T(\tau) = \frac{1}{T}\int_0^T E\left(\xi(t+\tau)\xi(t) - \widehat{m}\xi(t+\tau) - \widehat{m}_\tau\xi(t) + \widehat{m}_\tau\widehat{m}\right)dt =$$

$$= \frac{1}{T}\int_0^T \left[E\xi(t+\tau)\xi(t) - E\widehat{m}\xi(t+\tau) - E\widehat{m}_\tau\xi(t) + E\widehat{m}_\tau\widehat{m}\right]dt =$$

$$= r(\tau) + m^2 - \frac{1}{T^2}\int_0^T\int_0^T r(s-t-\tau)dsdt - m^2 - \frac{1}{T^2}\int_0^T\int_0^T r(s-t+\tau)dsdt - m^2 +$$

$$+ \frac{1}{T^2}\int_0^T\int_0^T r(s-y+\tau)dsdy + m^2 = r(\tau) - \frac{1}{T^2}\int_0^T\int_0^T r(s-t-\tau)dsdt.$$

Hence, $\widehat{r}_T(\tau)$ is biased estimate for function $r(\tau)$.

Consider

$$\widetilde{r}_T(\tau) = \widehat{r}_T(\tau) + \frac{1}{T^2}\int\limits_0^T\int\limits_0^T r(s-t-\tau)dsdt.$$

$\widetilde{r}_T(\tau)$ is unbiased estimate for $r(\tau)$, because $E\widetilde{r}_T(\tau) = r(\tau)$.

$$D\widetilde{r}_T(\tau) = D\widehat{r}_T(\tau) = E\widehat{r}_T^2(\tau) - \left(r(\tau) - \frac{1}{T^2}\int\limits_0^T\int\limits_0^T r(s-t-\tau)dsdt\right)^2.$$

We denote $a(\tau) = \frac{1}{T^2}\int_0^T\int_0^T r(t-s+\tau)dtds$. Since $r(\tau)$ is an even function, then $r(\tau) = r(-\tau)$. Therefore

$$D\widehat{r}_T(\tau) = E\widehat{r}_T^2(\tau) - (r(\tau) - a(\tau))^2.$$

Let us calculate $E\widehat{r}_T^2(\tau)$.

$$E\widehat{r}_T^2(\tau) =$$

$$= E\left(\frac{1}{T^2}\int\limits_0^T\int\limits_0^T (\xi(t+\tau)-\widehat{m}_\tau)(\xi(t)-\widehat{m})(\xi(u+\tau)-\widehat{m}_\tau)(\xi(u)-\widehat{m})dtdu\right) =$$

$$= \frac{1}{T^2}\int\limits_0^T\int\limits_0^T [E(\xi(t+\tau)-\widehat{m}_\tau)(\xi(t)-\widehat{m})E(\xi(u+\tau)-\widehat{m}_\tau)(\xi(u)-\widehat{m})+$$

$$+E(\xi(t+\tau)-\widehat{m}_\tau)(\xi(u+\tau)-\widehat{m}_\tau)E(\xi(t)-\widehat{m})(\xi(u)-\widehat{m})+$$

$$+\,E(\xi(t+\tau)-\widehat{m}_\tau)(\xi(u)-\widehat{m})E(\xi(t)-\widehat{m})(\xi(u+\tau)-\widehat{m}_\tau)]\,dtdu =$$

$$= \frac{1}{T^2}\int\limits_0^T\int\limits_0^T [I_1 + I_2 + I_3]\,dtdu,$$

where

$$I_1 = E(\xi(t+\tau)-\widehat{m}_\tau)(\xi(t)-\widehat{m})E(\xi(u+\tau)-\widehat{m}_\tau)(\xi(u)-\widehat{m}) =$$

$$= \left[(r(\tau)+a(\tau)) - \left(\frac{1}{T}\int\limits_0^T r(s-t-\tau)ds + \frac{1}{T}\int\limits_0^T r(s-t+\tau)ds\right)\right]\times$$

$$\times\left[(r(\tau)+a(\tau)) - \left(\frac{1}{T}\int\limits_0^T r(s-u-\tau)ds + \frac{1}{T}\int\limits_0^T r(s-u+\tau)ds\right)\right] =$$

$$= (r(\tau)+a(\tau))^2 - (r(\tau)+a(\tau))\left(\frac{1}{T}\int\limits_0^T r(s-t-\tau)ds+\right.$$

$$+\frac{1}{T}\int\limits_0^T r(s-t+\tau)ds+\frac{1}{T}\int\limits_0^T r(s-u-\tau)ds+\frac{1}{T}\int\limits_0^T r(s-u+\tau)ds\Bigg)+$$

$$+\Bigg(\frac{1}{T}\int\limits_0^T r(s-t-\tau)ds\frac{1}{T}\int\limits_0^T r(s-u-\tau)ds+$$

$$+\frac{1}{T}\int\limits_0^T r(s-t-\tau)ds\frac{1}{T}\int\limits_0^T r(s-u+\tau)ds+$$

$$+\frac{1}{T}\int\limits_0^T r(s-t+\tau)ds\frac{1}{T}\int\limits_0^T r(s-u-\tau)ds+$$

$$+\frac{1}{T}\int\limits_0^T r(s-t+\tau)ds\frac{1}{T}\int\limits_0^T r(s-u+\tau)ds\Bigg);$$

$$I_2=E(\xi(t+\tau)-\widehat{m}_\tau)(\xi(u+\tau)-\widehat{m}_\tau)E(\xi(t)-\widehat{m})(\xi(u)-\widehat{m})=$$

$$=\left[(r(t-u)+a(0))-\left(\frac{1}{T}\int\limits_0^T r(s-t)ds+\frac{1}{T}\int\limits_0^T r(s-u)ds\right)\right]\times$$

$$\times\left[(r(t-u)+a(0))-\left(\frac{1}{T}\int\limits_0^T r(s-t)ds+\frac{1}{T}\int\limits_0^T r(s-u)ds\right)\right]=$$

$$=(r(t-u)+a(0))^2-2r(t-u)\left(\frac{1}{T}\int\limits_0^T r(s-t)ds+\frac{1}{T}\int\limits_0^T r(s-u)ds\right)-$$

$$-2a(0)\left(\frac{1}{T}\int\limits_0^T r(s-t)ds+\frac{1}{T}\int\limits_0^T r(s-u)ds\right)+$$

$$+\left[\left(\frac{1}{T}\int\limits_0^T r(s-t)ds\right)^2+2\frac{1}{T}\int\limits_0^T r(s-t)ds\frac{1}{T}\int\limits_0^T r(s-u)ds+\right.$$

$$\left.+\left(\frac{1}{T}\int\limits_0^T r(s-u)ds\right)^2\right];$$

$$I_3=E(\xi(t+\tau)-\widehat{m}_\tau)(\xi(u)-\widehat{m})E(\xi(t)-\widehat{m})(\xi(u+\tau)-\widehat{m}_\tau)=$$

$$=\left[r(t-u+\tau)-\frac{1}{T}\int\limits_0^T r(s-t-\tau)ds-\frac{1}{T}\int\limits_0^T r(s-u+\tau)ds+a(\tau)\right]\times$$

$$\times \left[r(t-u-\tau) - \frac{1}{T}\int\limits_0^T r(s-t+\tau)ds - \frac{1}{T}\int\limits_0^T r(s-u-\tau)ds + a(\tau) \right] =$$

$$= r(t-u+\tau)r(t-u-\tau) - r(t-u+\tau)\frac{1}{T}\int\limits_0^T r(s-t+\tau)ds -$$

$$-r(t-u+\tau)\frac{1}{T}\int\limits_0^T r(s-u-\tau)ds + r(t-u+\tau)a(\tau) -$$

$$-r(t-u-\tau)\frac{1}{T}\int\limits_0^T r(s-t-\tau)ds + \frac{1}{T}\int\limits_0^T r(s-t-\tau)ds\frac{1}{T}\int\limits_0^T r(s-t+\tau)ds +$$

$$+\frac{1}{T}\int\limits_0^T r(s-t-\tau)ds\frac{1}{T}\int\limits_0^T r(s-u-\tau)ds - a(\tau)\frac{1}{T}\int\limits_0^T r(s-t-\tau)ds -$$

$$-r(t-u-\tau)\frac{1}{T}\int\limits_0^T r(s-u+\tau)ds + \frac{1}{T}\int\limits_0^T r(s-u+\tau)ds\frac{1}{T}\int\limits_0^T r(s-t+\tau)ds +$$

$$+\frac{1}{T}\int\limits_0^T r(s-u+\tau)ds\frac{1}{T}\int\limits_0^T r(s-u-\tau)ds - a(\tau)\frac{1}{T}\int\limits_0^T r(s-u+\tau)ds +$$

$$+a(\tau)r(t-u-\tau) - a(\tau)\frac{1}{T}\int\limits_0^T r(s-t+\tau)ds - a(\tau)\frac{1}{T}\int\limits_0^T r(s-u-\tau)ds + a^2(\tau).$$

Then

$$\frac{1}{T^2}\int\limits_0^T\int\limits_0^T [I_1 + I_2 + I_3]\, dtdu = (r(\tau) - a(\tau))^2 +$$

$$+a^2(0) + a^2(\tau) + \frac{1}{T^2}\int\limits_0^T\int\limits_0^T \left[r^2(t-u)dtdu + r(t-u+\tau)r(t-u-\tau)dtdu \right] -$$

$$-\frac{1}{T^3}\int\limits_0^T\int\limits_0^T\int\limits_0^T [2r(s-t)r(t-u) + r(t-s-\tau)r(t-s+\tau) +$$

$$+r(t-s+\tau)r(t-u-\tau) - r(t-s+\tau)r(t-u+\tau) + r(t-u+\tau)r(s-u-\tau) +$$
$$+ r(t-u-\tau)r(s-u+\tau) - r(t-u-\tau)r(s-u+\tau)]\, dsdtdu =$$

$$= \frac{1}{T^4}\left[\int\limits_0^T\int\limits_0^T r(t-s)dtds \right]^2 + \frac{1}{T^4}\left[\int\limits_0^T\int\limits_0^T r(t-s+\tau)dtds \right]^2 +$$

$$+\frac{2}{T^2}\int\limits_0^T (T-s)[r^2(s) + r(s+\tau)r(s-\tau)]ds -$$

$$-\frac{1}{T^3}\int_0^T\int_0^T\int_0^T [r(t-s+\tau)r(s-u+\tau)+2r(t-s)r(s-u)+$$
$$+r(t-s-\tau)r(s-u-\tau)]\,dtdsdu+$$
$$+\left(r(\tau)-a(\tau)\right)^2.$$

Hence,

$$D\widehat{r}_T(\tau)=\frac{1}{T^4}\left[\int_0^T\int_0^T r(t-s)dtds\right]^2+\frac{1}{T^4}\left[\int_0^T\int_0^T r(t-s+\tau)dtds\right]^2+$$
$$+\frac{2}{T^2}\int_0^T (T-s)[r^2(s)+r(s+\tau)r(s-\tau)]ds-$$
$$-\frac{1}{T^3}\int_0^T\int_0^T\int_0^T [r(t-s+\tau)r(s-u+\tau)+2r(t-s)r(s-u)+$$
$$+r(t-s-\tau)r(s-u-\tau)]\,dtdsdu.$$

Let us show, that $\widehat{r}_T(\tau)$ is square Gaussian stochastic process.

Consider a partition $\lambda=\left\{t_0=0,t_1=\frac{T}{n},\ldots,t_k=\frac{kT}{n},\ldots,t_n=T\right\}$ of the segment $[0,T]$ and replace integrals in $\widehat{r}_T(\tau)$ by corresponding integral sums

$$\widehat{r}_{T,n}(\tau)=\frac{1}{n}\sum_{i=0}^{n-1}\left(\xi(t_i+\tau)-\widehat{m}_\tau\right)\left(\xi(t_i)-\widehat{m}\right),$$

and $\widehat{m}_\tau$ with $\widehat{m}$ by integral sums

$$\widehat{m}_{\tau,n}=\frac{1}{n}\sum_{i=0}^{n-1}\xi\left(\frac{iT}{n}+\tau\right)$$

and

$$\widehat{m}_n=\frac{1}{n}\sum_{i=0}^{n-1}\xi\left(\frac{iT}{n}\right).$$

Then

$$\widehat{r}_{T,n}(\tau)=\frac{1}{n^3}\sum_{k=0}^{n-1}\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\left[\xi\left(\frac{kT}{n}+\tau\right)-\xi\left(\frac{iT}{n}+\tau\right)\right]\times$$
$$\times\left[\xi\left(\frac{kT}{n}\right)-\xi\left(\frac{jT}{n}\right)\right]=$$

$$= \sum_{k=0}^{n-1} \left(\frac{1}{n\sqrt{n}} \sum_{i=0}^{n-1} \left[\xi\left(\frac{kT}{n} + \tau\right) - \xi\left(\frac{iT}{n} + \tau\right) \right] \right) \times$$

$$\times \left(\frac{1}{n\sqrt{n}} \sum_{j=0}^{n-1} \left[\xi\left(\frac{kT}{n}\right) - \xi\left(\frac{jT}{n}\right) \right] \right) = \sum_{k=0}^{n-1} \alpha_k(\tau)\alpha_k(0),$$

where

$$\alpha_k(\tau) = \frac{1}{n\sqrt{n}} \sum_{i=0}^{n-1} \left[\xi\left(\frac{kT}{n} + \tau\right) - \xi\left(\frac{iT}{n} + \tau\right) \right], \quad k = \overline{0, n-1}$$

Hence,

$$\widehat{r}_{T,n}(\tau) = \overline{\alpha}^T(\tau)\overline{\alpha}(0),$$

where $\overline{\alpha}(\tau)$ is the centered random vector with components $\alpha_k(\tau)$, $k = \overline{0, n-1}$.

Obviously, that $\widehat{r}_T(\tau) = l.i.m._{n \longrightarrow \infty} \widehat{r}_{T,n}(\tau)$. Therefore $\zeta(\tau) = \widehat{r}_T(\tau) - E\widehat{r}_T(\tau)$ is square Gaussian stochastic process.

Consider $\eta = \int_0^B \left(\widehat{r}_T(\tau) - E\widehat{r}_T(\tau)\right)^2 d\tau$, $0 < B < \infty$.

$$E\eta = \int_0^B E\left(\widehat{r}_T(\tau) - E\widehat{r}_T(\tau)\right)^2 d\tau =$$

$$= \int_0^B \left(\frac{1}{T^4} \left[\int_0^T \int_0^T r(t-s)dtds \right]^2 + \frac{1}{T^4} \left[\int_0^T \int_0^T r(t-s+\tau)dtds \right]^2 + \right.$$

$$+ \frac{2}{T^2} \int_0^T (T-s)[r^2(s) + r(s+\tau)r(s-\tau)]ds -$$

$$- \frac{1}{T^3} \int_0^T \int_0^T \int_0^T [r(t-s+\tau)r(s-u+\tau) + 2r(t-s)r(s-u) +$$

$$\left. + \; r(t-s-\tau)r(s-u-\tau)\right] dtdsdu \Big) \, d\tau$$

Theorem 4.3. *For the estimate $\widehat{r}_T(\tau)$ of correlation function $r(\tau)$ stationary Gaussian process the following inequalities hold*

$$P\left\{ \int_0^B \left(\widehat{r}_T(\tau) - r(\tau)\right)^2 d\tau > x \int_0^B D\widehat{r}_T(\tau)d\tau \right\} \geqslant 1 - g(u)\exp\left\{ \frac{u^2 x}{2} \right\} \quad (4.9)$$

for $u > 0,\ 0 < x < -\frac{2\ln g(u)}{u^2}$*,*
where $g(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp\left\{-\frac{s^2}{2}\right\} \frac{ds}{(1+s^2u^2)^{\frac{1}{4}}}$ *and*

$$P\left\{\int\limits_0^B \left(\widehat{r}_T(\tau) - r(\tau)\right)^2 d\tau > y \int\limits_0^B D\widehat{r}_T(\tau) d\tau\right\} \leqslant \frac{2^{\frac{1}{4}} y^{\frac{1}{4}}}{\operatorname{ch}\left(\sqrt{\frac{y}{2}} - \frac{1}{2}\right)} \qquad (4.10)$$

for $y > \frac{1}{2}$.

Proof. The proof is immediate by theorem 1.3, corollary 1.5 and previous calculations. ♢

Remark 4.6. Theorem 4.3 enable us to construct confidence sets in the space $L_2(0, B)$ for correlation function of stationary Gaussian process $\xi(t)$.

Let H be the hypothesis that for $0 \leqslant \tau \leqslant B$ the correlation function of separable real-valued stationary Gaussian process ξ with $E\xi(t) = m$ equals $r(\tau)$. As an estimator $r(\tau)$ we choose $\widehat{r}_T(\tau)$. To test the hypothesis H one can use the following criterion.

Criterion 4.3. For given level of confidence α, $0 < \alpha < 1$, we can find such positive x_α and y_α, that

$$s(x_\alpha, u) + f(y_\alpha) = \alpha,$$

where
$s(x, u) = g(u) \exp\left\{\frac{u^2 x}{2}\right\}$, $u > 0$, $f(x) = \frac{2^{\frac{1}{4}} x^{\frac{1}{4}}}{\operatorname{ch}\left(\sqrt{\frac{x}{2}} - \frac{1}{2}\right)}$.

The hypothesis H is accepted if

$$x_\alpha < \frac{\int_0^B \left(\widehat{r}_T(\tau) - r(\tau)\right)^2 d\tau}{E\int_0^B \left(\widehat{r}_T(\tau) - r(\tau)\right)^2 d\tau} < y_\alpha$$

and hypothesis is rejected otherwise.

Remark 4.7. Since $\widehat{r}_{T,n}(\tau)$ - is the quadratic form of Gaussian centered random vectors, then $\zeta(\tau) = \widehat{r}_{T,n}(\tau) - E\widehat{r}_{T,n}(\tau)$ is square Gaussian stochastic process for $\tau \geqslant 0$. Therefore, in the case when values of process $\xi(t)$ are known in the points t_i and $t_i + \tau$, where $t_i = \frac{iT}{n}$, $i = \overline{0, n-1}$, the criterion 4.3 can be used for testing the hypothesis about the correlation function. As an estimate of correlation function in this case $\widehat{r}_{T,n}(\tau)$ must be considered.

Chapter 5
Estimation of the covariation function of Gaussian stochastic process in the space $L_p(T), p \geqslant 1$.

Estimation of spectral and covariance functions of stochastic processes and criteria construction to identify these characteristics are the matter of active research and topical direction in the theory of stochastic processes. The interest to study of these problems is caused by wide application of the obtained results, in particular for solving different problems in geology and meteorology.

There are several methods to obtain these estimates and to construct criteria for testing hypotheses about the covariance functions. One of these methods is based on Bartlett's asymptotic limit formula (see [9], Brockwell and Davis,1991, Chap. 7). In the papers by Coates and Diggle (1986) [24], Shumway (2006) [122], Choi, Ombao, Ray, (2008) [23], Taheriyoun (2012) [123] the criteria for comparison covariance functions of two stochastic sets were constructed. The criteria in the case of the separable, symmetric or stationary covariance function are obtained in the papers by Scaccia and Martin (2005) [120], Park and Fuentes (2008) [108], Fuentes (2005, 2006) [37], [38] and Lund, Bassily, and Vidakovic (2009) [95]. In the papers by Fan, Zhang, Zhang (2001) [33] and Fan, Zhang (2004) [32] generalised likelihood ratio test for a stationary time series was constructed. Li, Genton and Sherman in the paper [92] proposed a methodology to evaluate the appropriateness of several types of common assumptions on multivariate covariance functions in the spatio-temporal context.

In this chapter another approach is used to construct criteria for testing hypotheses about the covariance function of Gaussian stationary stochastic process. Namely, this criterion is based on the fact, that we can evaluate the deviation of covariance function from its estimators with a given accuracy and reliability in L_p metric. In the above-mentioned papers the limit distribution of the estimates was found. Instead, we find such T for which with a given accuracy and reliability the norm of deviation the covariance function and its estimator will be the smallest. Therefore, it is difficult to say which approach is the best. Probabily, we can say that the simultaneous use of the different approaches will be the best.

Similar approaches have been used, for examples, in the papers [11], [20], [47], [61], [80] and in the book [90], where some estimates of covariance functions with given accuracy in uniform metrics were obtained. In the papers [86] and [34] Yu. Kozachenko and T. Fedoryanych constructed criteria for testing hypotheses about covariance function of a stationary

Gaussian process with given reliability and accuracy in the space $L_2[0, A]$. To construct the criteria in this paper the estimates are used for the norm of square Gaussian stochastic processes in the space $L_p[0, A]$, $p \geqslant 1$, which were obtained in the paper [83] by Yu. Kozachenko and V. Troshki. More detailed information on the theory of square Gaussian random variables can be found in the book [19] and in the paper [81]. In particular, in these manuscripts the properties of the space of square Gaussian random variables were studied and its connection with other spaces of random variables was identified.

In this chapter we have obtained estimates of probability of deviations $\hat{\rho}(\tau)$ from $\rho(\tau)$ in the norm of the space $L_p[0, A]$, $p \geqslant 1$. In addition, this chapter deals with the construction of a criterion for testing hypothesis about covariance function of a stationary Gaussian process in the case of unknown mean of the process (see [84]), criteria for testing hypotheses about the covariance functions of Gaussian stationary random process when the values of this process are known only in a finite set of points, criterion for testing hypotheses about the covariance functions of Gaussian random process when available alternative hypothesis and criterion for testing hypotheses about the covariance functions of Gaussian non-stationary stochastic process. In fact, we continue studies initiated in the paper [83].

5.1. Estimation of the norm of deviation the covariation function from correlogram.

Consider a measurable stationary Gaussian stochastic process X which is define for any $t \in \mathbb{R}$. Without any loss of generality, we can assume that $X = \{X(t), t \in \mathbb{T} = [0, T + A], 0 < T < \infty, 0 < A < \infty\}$ and $\mathbf{E}X(t) = 0$. The covariance function of this process $\rho(\tau) = \mathbf{E}X(t + \tau)X(t)$ is defined for any $\tau \in \mathbb{R}$, $\rho(\tau)$ is an even function. Let $\rho(\tau)$ be a function that is continuous on $\mathbb{T}$.

Theorem 5.1. *Let correlogram*

$$\hat{\rho}(\tau) = \frac{1}{T}\int_0^T X(t + \tau)X(t)dt, 0 \leqslant \tau \leqslant A \tag{5.1}$$

be an estimator of the covariance function $\rho(\tau)$. Then the following inequali-

ty holds for all $\varepsilon \geqslant \left(\frac{p}{\sqrt{2}} + \sqrt{(\frac{p}{2}+1)p}\right)^p C_p$

$$P\left\{\int\limits_0^A (\hat\rho(\tau)-\rho(\tau))^p d\tau > \varepsilon\right\} \leqslant 2\sqrt{1+\frac{\varepsilon^{1/p}\sqrt{2}}{C_p^{\frac{1}{p}}}}\exp\left\{-\frac{\varepsilon^{\frac{1}{p}}}{\sqrt{2}C_p^{\frac{1}{p}}}\right\},$$

where $C_p = \int\limits_0^A \left(\frac{2}{T^2}\int\limits_0^T (T-u)(\rho^2(u)+\rho(u+\tau)\rho(u-\tau))du\right)^{\frac{p}{2}} d\tau$ *and* $0 < A < \infty$.

Remark 5.1. Since the sample paths of the process $X(t)$ are continuous with probability one on the set $\mathbb{T}$, $\hat\rho(\tau)$ is a Riemann integral.

Proof. Consider

$$\mathbf{E}(\hat\rho(\tau)-\rho(\tau))^2 = \mathbf{E}(\hat\rho(\tau))^2 - \rho^2(\tau).$$

From the Isserlis equality for jointly Gaussian random variables it follows that

$$\begin{aligned}
\mathbf{E}(\hat\rho(\tau))^2 - \rho^2(\tau) &= \mathbf{E}\left(\frac{1}{T^2}\int\limits_0^T\int\limits_0^T X(t+\tau)X(t)X(s+\tau)X(s)dtds\right) - \rho^2(\tau)\\
&= \frac{1}{T^2}\int\limits_0^T\int\limits_0^T (\mathbf{E}X(t+\tau)X(t)\mathbf{E}X(s+\tau)X(s) + \mathbf{E}X(t+\tau)X(s+\tau)\\
&\times \mathbf{E}X(t)X(s) + \mathbf{E}X(t+\tau)X(s)\mathbf{E}X(s+\tau)X(t))dtds - \rho^2(\tau)\\
&= \frac{1}{T^2}\int\limits_0^T\int\limits_0^T (\rho^2(\tau)+\rho^2(t-s)+\rho(t-s+\tau)\rho(t-s-\tau))dtds - \rho^2(\tau)\\
&= \frac{1}{T^2}\int\limits_0^T\int\limits_0^T (\rho^2(t-s)+\rho(t-s+\tau)\rho(t-s-\tau))dtds\\
&= \frac{2}{T^2}\int\limits_0^T (T-u)(\rho^2(u)+\rho(u+\tau)\rho(u-\tau))du.
\end{aligned}$$

We obtained that

$$\mathbf{E}(\hat\rho(\tau)-\rho(\tau))^2 = \frac{2}{T^2}\int\limits_0^T (T-u)(\rho^2(u)+\rho(u+\tau)\rho(u-\tau))du. \qquad (5.2)$$

Since $\hat{\rho}(\tau) - \rho(\tau)$ is a square Gaussian stochastic process (see Lemma 3.1, Chapter 6 in book [19]), then it follows from the Theorem 3.4 that

$$P\left\{\int_0^A (\hat{\rho}(\tau) - \rho(\tau))^p d\tau > \varepsilon\right\} \leqslant 2\sqrt{1 + \frac{\varepsilon^{1/p}\sqrt{2}}{C_p^{\frac{1}{p}}}} \exp\left\{-\frac{\varepsilon^{\frac{1}{p}}}{\sqrt{2}C_p^{\frac{1}{p}}}\right\},$$

where $C_p = \mathbf{E}(\hat{\rho}(\tau))^2 - \rho^2(\tau)$. Applying equality (5.2) we get

$$C_p = \int_0^A \left(\frac{2}{T^2}\int_0^T (T-u)(\rho^2(u) + \rho(u+\tau)\rho(u-\tau))du\right)^{\frac{p}{2}} d\tau.$$

The theorem is proved. ◊

Denote

$$g(\varepsilon) = 2\sqrt{1 + \frac{\varepsilon^{1/p}\sqrt{2}}{C_p^{\frac{1}{p}}}} \exp\left\{-\frac{\varepsilon^{\frac{1}{p}}}{\sqrt{2}C_p^{\frac{1}{p}}}\right\}.$$

From the Theorem 5.1 it follows that if $\varepsilon \geqslant z_p = C_p\left(\frac{p}{\sqrt{2}} + \sqrt{(\frac{p}{2}+1)p}\right)^p$ then

$$P\left\{\int_0^A (\hat{\rho}(\tau) - \rho(\tau))^p d\tau > \varepsilon\right\} \leqslant g(\varepsilon).$$

Let ε_δ be a solution of the equation $g(\varepsilon) = \delta$, $0 < \delta < 1$. Put $S_\delta = \max\{\varepsilon_\delta, z_p\}$. It is obviously that $g(S_\delta) \leqslant \delta$ and

$$P\left\{\int_0^A (\hat{\rho}(\tau) - \rho(\tau))^p d\tau > S_\delta\right\} \leqslant \delta. \tag{5.3}$$

Let $\mathbb{H}$ be the hypothesis that the covariance function of a measurable real-valued stationary Gaussian stochastic process $X(t)$ equals $\rho(\tau)$ for $0 \leqslant \tau \leqslant A$. From the Theorem 5.1 and (5.3) it follows that to test the hypothesis $\mathbb{H}$ one can use the following criterion.

Criterion 5.1. For a given level of confidence δ the hypothesis $\mathbb{H}$ is accepted if

$$\int_0^A (\hat{\rho}(\tau) - \rho(\tau))^p d\mu(\tau) < S_\delta$$

otherwise hypothesis is rejected.

Remark 5.2. The equation $g(\varepsilon) = \delta$ has a solution for any $\delta > 0$, since $g(\varepsilon)$ is a monotonically decreasing function. We can find the solution of equation using numerical methods.

Remark 5.3. One can easily see that Criterion 5.1 can be used if $C_p \to 0$ as $T \to \infty$.

Next theorem contain assumptions under which C_p tend to zero as $T \to 0$.

Theorem 5.2. *Let $\rho(\tau)$ be covariance function of centered stationary stochastic process. Let $\rho(\tau)$ be continuous function. If $\rho(T) \to 0$ as $T \to \infty$ then $C_p \to 0$ as $T \to \infty$, where $C_p = \int\limits_0^A (\psi(T,\tau))^{p/2} dt$ and $\psi(T,\tau) = \frac{2}{T^2}\int\limits_0^T (T - u)(\rho^2(u) + \rho(u+\tau)\rho(u-\tau))du$, $A > 0$, $T > 0$.*

Proof. $\psi(T,\tau) \leqslant \frac{2}{T}\int\limits_0^T (\rho^2(u) + \rho(u+\tau)\rho(u-\tau))du \leqslant 4\rho^2(0)$. Now it is sufficiently to prove that $\psi(T,\tau) \to 0$ as $T \to \infty$. From the L'Hopital's rule it follows that

$$\lim_{T\to\infty} \psi(T,\tau) = \lim_{T\to\infty} \frac{2}{T}\int\limits_0^T (\rho^2(u) + \rho(u+\tau)\rho(u-\tau))du =$$
$$= \lim_{T\to\infty} (\rho^2(T) + \rho(T+\tau)\rho(T-\tau)) = 0.$$

The application of Lebesgue's dominated convergence theorem completes the proof. ♢

Here are examples in which we find the estimates for C_p.

Example 5.1. Let $\mathbb{H}$ be the hypothesis that the covariance function of a centered measurable stationary Gaussian stochastic process equals $\rho(\tau) = B\exp\{-a \mid \tau \mid\}$, where $B > 0$ and $a > 0$.

To test the hypothesis $\mathbb{H}$ one can use the Criterion 5.1 by selecting $\hat{\rho}_T(\tau)$ which is defined in (5.1) as the estimator of the function $\rho(\tau)$. Let $0 < A < \infty$. We shall find the value of the following expression

$$I = \int\limits_0^T (T-u)\left(e^{-2au} + e^{-a|u+\tau|}e^{-a|u-\tau|}\right)du = \int\limits_0^T Te^{-2au}du$$
$$+ T\int\limits_0^T e^{-a|u+\tau|}e^{-a|u-\tau|}du - \int\limits_0^T ue^{-2au}du - \int\limits_0^T ue^{-a|u+\tau|}e^{-a|u-\tau|}du$$
$$= I_1 + I_2 + I_3 + I_4.$$

For the similar calculations as in the example, we obtain that

$$C_p \leqslant \left(\frac{2B}{T^2}\right)^{\frac{p}{2}} \int\limits_0^A \left(\left(T\tau + \frac{T}{2a}\right)e^{-2a\tau} + \frac{T}{2a} + \frac{1}{2a^2}e^{-2aT}\right)^{p/2} d\tau =$$

$$= (2B)^{\frac{p}{2}} \frac{T^{p/2}}{T^p} I_5 = (2B)^{\frac{p}{2}} \frac{1}{T^{p/2}} I_5,$$

where $I_5 = \int\limits_0^A \left(\left(\tau + \frac{1}{2a}\right)e^{-2a\tau} + \frac{1}{2a} + \frac{1}{2a^2}e^{-2aT}\right)^{p/2} d\tau.$

Example 5.2. Let $\mathbb{H}$ be the hypothesis that the covariance function of a a centered measurable stationary Gaussian stochastic process equals $\rho(\tau) = B\exp\{-a \mid \tau \mid^2\}$, where $B > 0$ and $a > 0$.

Similarly as in the previous example to test the hypothesis $\mathbb{H}$ one can use the Criterion 5.1 by selecting $\hat{\rho}_T(\tau)$, which is defined in (5.1) as the estimator of the function $\rho(\tau)$. Let $0 < A < \infty$. We shall find the value of the following expression

$$I = \int\limits_0^T (T-u)\left(e^{-2au^2} + e^{-a|u+\tau|^2}e^{-a|u-\tau|^2}\right) du = \int\limits_0^T Te^{-2au^2} du$$

$$+ T\int\limits_0^T e^{-a|u+\tau|^2}e^{-a|u-\tau|^2} du - \int\limits_0^T ue^{-2au^2} du - \int\limits_0^T ue^{-a|u+\tau|^2}e^{-a|u-\tau|^2} du$$

$$= I_1 + I_2 + I_3 + I_4.$$

For the similar calculations as in the example, we get that

$$C_p \leqslant \left(\frac{2B}{T^2}\right)^{\frac{p}{2}} \int\limits_0^A \left(T\left(\frac{\sqrt{\pi}}{2\sqrt{2a}} + \frac{\sqrt{\pi}}{2\sqrt{2a}}e^{-2a\tau^2}\right)\right)^{p/2} d\tau = (2B)^{\frac{p}{2}} \frac{1}{T^{p/2}} I_6,$$

where $I_6 = \int\limits_0^A \left(\frac{\sqrt{\pi}}{2\sqrt{2a}} + \frac{\sqrt{\pi}}{2\sqrt{2a}}e^{-2a\tau^2}\right)^{p/2} d\tau.$

Lemma 5.1. *Let X be a stationary Gaussian stochastic process with the spectral density $f(\lambda)$ and covariance function*

$$\rho(\tau) = \int\limits_{-\infty}^{\infty} \cos\lambda\tau f(\lambda) d\lambda.$$

Let $f(\lambda)$ be a differentiated function and

$$\lambda f(\lambda) \to 0, as \lambda \to \infty,$$

$$\int_0^\infty |f'(\lambda)|\, d\lambda < \infty,$$

$$\int_0^\infty |\lambda f'(\lambda)|\, d\lambda < \infty.$$

Then, to test the hypothesis about the covariance function can be used the Criterion 5.1 and

$$C_p \leqslant \left(\frac{4}{T}\right)^{p/2} d_p \left(\frac{1}{p}\left(2(A+2)^{p/2+1} - 2^{p/2+2}\right) I_1^{p/2} + A\left(\left(2-\frac{1}{T}\right) I_2\right)^{p/2}\right),$$

where

$$I_1 = \int_0^\infty \int_0^\infty |\lambda\gamma f^{'}(\lambda) f^{'}(\gamma)| d\lambda d\gamma;$$

$$I_2 = \int_0^\infty \int_0^\infty |f^{'}(\lambda) f^{'}(\gamma)| d\lambda d\gamma;$$

$$d_p = \begin{cases} 1, & as\ 0 < p \leqslant 1, \\ 2^p, & as\ p > 1. \end{cases}$$

Proof. Since the function $\rho(\tau)$ is an even function, then

$$\rho(\tau) = \int_{-\infty}^{\infty} \cos\lambda\tau f(\lambda) d\lambda = 2\int_0^\infty \cos\lambda\tau f(\lambda) d\lambda.$$

Using Theorem 5.1 we obtain

$$C_p \leqslant \int_0^A \left(\frac{1}{T}\int_0^T (\rho^2(u) + \rho(u+\tau)\rho(u-\tau)) du\right)^{\frac{p}{2}} d\tau =$$

$$= \int_0^A \left(\frac{4}{T}\int_0^T \left(\int_0^\infty \cos\lambda u f(\lambda) d\lambda \int_0^\infty \cos\gamma u f(\gamma) d\gamma + \right.\right.$$

$$\left.\left. + \int_0^\infty \cos\lambda(u+\tau) f(\lambda) d\lambda \int_0^\infty \cos\gamma(u-\tau) f(\gamma) d\gamma\right) du\right)^{\frac{p}{2}} d\tau.$$

Consider the next integral

$$\int_0^\theta \cos\lambda u f(\lambda)d\lambda = f(\lambda)\ \frac{sin\lambda u}{u}\bigg|_0^\theta - \int_0^\theta \frac{\sin\lambda u}{u} f^{'}(\lambda)d\lambda =$$

$$= f(\theta)\frac{sin\theta u}{u} - \int_0^\theta \frac{\sin\lambda u}{u} f^{'}(\lambda)d\lambda \quad (5.4)$$

From the properties of definite integral it follows that

$$\int_0^T \rho^2(u)du = \int_0^1 \rho^2(u)du + \int_0^T \rho^2(u)du.$$

Consider $\left|\int_0^1 \rho^2(u)du\right|$. It is known that for any $\theta > 0$ and $u > 0$

$$\left|\frac{\sin\theta u}{u}\right| \leqslant \theta.$$

From (5.4) and from properties of definite integral, we obtain the following inequalities

$$\left|\int_0^1 \rho^2(u)du\right| = \left|\int_0^1 \left(f(\theta)\frac{sin\theta u}{u} - \int_0^\theta \frac{\sin\lambda u}{u} f^{'}(\lambda)d\lambda\right) \times \right.$$

$$\left. \times \left(f(\theta)\frac{sin\theta u}{u} - \int_0^\theta \frac{\sin\gamma u}{u} f^{'}(\gamma)d\gamma\right) du\right| \leqslant$$

$$\leqslant \int_0^1 \left|f(\theta)\frac{sin\theta u}{u} - \int_0^\theta \frac{\sin\lambda u}{u} f^{'}(\lambda)d\lambda\right| \left|f(\theta)\frac{sin\theta u}{u} - \int_0^\theta \frac{\sin\gamma u}{u} f^{'}(\gamma)d\gamma\right| du \leqslant$$

$$\leqslant \int_0^1 \left(|f(\theta)|\left|\frac{sin\theta u}{u}\right| + \left|\int_0^\theta \frac{\sin\lambda u}{u} f^{'}(\lambda)d\lambda\right|\right) \times$$

$$\times \left(|f(\theta)|\left|\frac{sin\theta u}{u}\right| + \left|\int_0^\theta \frac{\sin\gamma u}{u} f^{'}(\gamma)d\gamma\right|\right) du \leqslant$$

$$\leqslant \int_0^1 \left(|f(\theta)\theta| + \int_0^\theta |\lambda f^{'}(\lambda)|d\lambda\right)\left(|f(\theta)\theta| + \int_0^\theta |\gamma f^{'}(\gamma)|d\gamma\right) du =$$

$$= \left(|f(\theta)\theta| + \int_0^\theta |\lambda f^{'}(\lambda)|d\lambda\right)\left(|f(\theta)\theta| + \int_0^\theta |\gamma f^{'}(\gamma)|d\gamma\right).$$

Now if $\theta \to \infty$, then

$$\left|\int_0^1 \rho^2(u)du\right| \longrightarrow \int_0^\infty |\lambda f^{'}(\lambda)|d\lambda \int_0^\infty |\gamma f^{'}(\gamma)|d\gamma.$$

From the similar considerations as in the previous case, we find that

$$\left|\int_1^T \rho^2(u)du\right| = \left|\int_1^T \left(f(\theta)\frac{sin\theta u}{u} - \int_0^\theta \frac{\sin\lambda u}{u} f^{'}(\lambda)d\lambda\right) \times\right.$$

$$\left. \times \left(f(\theta)\frac{sin\theta u}{u} - \int_0^\theta \frac{\sin\gamma u}{u} f^{'}(\gamma)d\gamma\right) du\right| \leqslant$$

$$\leqslant \int_1^T \frac{1}{u^2}\left(|f(\theta)| + \int_0^\theta |f^{'}(\lambda)|d\lambda\right)\left(|f(\theta)| + \int_0^\theta |f^{'}(\gamma)|d\gamma\right) du.$$

If $\theta \to \infty$, then

$$\left|\int_1^T \rho^2(u)du\right| \longrightarrow \int_1^T \frac{du}{u^2} \int_0^\infty |f^{'}(\lambda)|d\lambda \int_0^\infty |f^{'}(\gamma)|d\gamma =$$

$$= \left(1 - \frac{1}{T}\right) \int_0^\infty |f^{'}(\lambda)|d\lambda \int_0^\infty |f^{'}(\gamma)|d\gamma.$$

Namely,

$$\int_0^T \rho^2(u)du \leqslant \int_0^\infty\int_0^\infty |\lambda\gamma f^{'}(\lambda) f^{'}(\gamma)|d\lambda d\gamma + \left(1 - \frac{1}{T}\right) \int_0^\infty\int_0^\infty |f^{'}(\lambda) f^{'}(\gamma)|d\lambda d\gamma.$$

We estimate $\int\limits_0^T \rho(u+\tau)\rho(u-\tau)du$. To do this, write it in the following form

$$\int\limits_0^T \rho(u+\tau)\rho(u-\tau)du = \int\limits_0^{\tau+1} \rho(u+\tau)\rho(u-\tau)du + \int\limits_{\tau+1}^T \rho(u+\tau)\rho(u-\tau)du.$$

Consider

$$\left|\int\limits_0^{\tau+1} \rho(u+\tau)\rho(u-\tau)du\right| =$$

$$= \left|\int\limits_0^{\tau+1} \left(f(\theta)\frac{sin\theta(u+\tau)}{u+\tau} - \int\limits_0^{\theta} \frac{\sin\lambda(u+\tau)}{u+\tau} f^{'}(\lambda)d\lambda\right) \times\right.$$

$$\left.\times \left(f(\theta)\frac{sin\theta(u-\tau)}{u-\tau} - \int\limits_0^{\theta} \frac{\sin\gamma(u-\tau)}{u-\tau} f^{'}(\gamma)d\gamma\right) du\right| \leqslant$$

$$\leqslant \int\limits_0^{\tau+1} \left(|f(\theta)\theta| + \int\limits_0^{\theta} |\lambda f^{'}(\lambda)|d\lambda\right)\left(|f(\theta)\theta| + \int\limits_0^{\theta} |\gamma f^{'}(\gamma)|d\gamma\right) du$$

Hence, if $\theta \to \infty$

$$\left|\int\limits_0^{\tau+1} \rho(u+\tau)\rho(u-\tau)du\right| \longrightarrow (\tau+1)\int\limits_0^{\infty} |\lambda f^{'}(\lambda)|d\lambda \int\limits_0^{\infty} |\gamma f^{'}(\gamma)|d\gamma.$$

Consider the second term

$$\left|\int\limits_{\tau+1}^T \rho(u+\tau)\rho(u-\tau)du\right| =$$

$$= \left|\int\limits_{\tau+1}^{T} \left(f(\theta)\frac{sin\theta(u+\tau)}{u+\tau} - \int\limits_0^{\theta} \frac{\sin\lambda(u+\tau)}{u+\tau} f^{'}(\lambda)d\lambda\right) \times\right.$$

$$\left.\times \left(f(\theta)\frac{sin\theta(u-\tau)}{u-\tau} - \int\limits_0^{\theta} \frac{\sin\gamma(u-\tau)}{u-\tau} f^{'}(\gamma)d\gamma\right) du\right| \leqslant$$

$$\leqslant \int\limits_{\tau+1}^{T} \frac{1}{|u+\tau||u-\tau|} \left(|f(\theta)| + \int\limits_0^{\theta} |f^{'}(\lambda)| d\lambda \right) \left(|f(\theta)| + \int\limits_0^{\theta} |f^{'}(\gamma)| d\gamma \right) du \to$$

$$\to \int\limits_{\tau+1}^{T} \frac{du}{u^2-\tau^2} \int\limits_0^{\infty} |f^{'}(\lambda)| d\lambda \int\limits_0^{\infty} |f^{'}(\gamma)| d\gamma, \theta \to \infty.$$

Since $\tau \leqslant T$, then

$$\int\limits_{\tau+1}^{T} \frac{du}{u^2-\tau^2} = \frac{1}{2\tau} \ln \left| \frac{u-\tau}{u+\tau} \right| \Bigg|_{\tau+1}^{T} = \frac{1}{2\tau} \left(\ln \left| \frac{T-\tau}{T+\tau} \right| - \ln \left| \frac{\tau+1-\tau}{\tau+1+\tau} \right| \right) =$$

$$= \frac{1}{2\tau} \ln \left| \frac{(T-\tau)(2\tau+1)}{(T+\tau)} \right| \leqslant 1.$$

This means that

$$\int\limits_0^{T} \rho(u+\tau)\rho(u-\tau) du \leqslant (\tau+1) \int\limits_0^{\infty} \int\limits_0^{\infty} |\lambda\gamma f^{'}(\lambda) f^{'}(\gamma)| d\lambda d\gamma +$$

$$+ \int\limits_0^{\infty} \int\limits_0^{\infty} |f^{'}(\lambda) f^{'}(\gamma)| d\lambda d\gamma.$$

Denote

$$I_1 = \int\limits_0^{\infty} \int\limits_0^{\infty} |\lambda\gamma f^{'}(\lambda) f^{'}(\gamma)| d\lambda d\gamma;$$

$$I_2 = \int\limits_0^{\infty} \int\limits_0^{\infty} |f^{'}(\lambda) f^{'}(\gamma)| d\lambda d\gamma.$$

Then

$$C_p \leqslant \left(\frac{4}{T} \right)^{p/2} \int\limits_0^{A} \left(I_1 + \left(1 - \frac{1}{T} \right) I_2 + (\tau+1) I_1 + I_2 \right)^{p/2} d\tau =$$

$$= \left(\frac{4}{T} \right)^{p/2} \int\limits_0^{A} \left((\tau+2) I_1 + \left(2 - \frac{1}{T} \right) I_2 \right)^{p/2} d\tau.$$

For any non-negative a, b and $p > 0$ the following inequality holds

$$(a+b)^p \leqslant d_p (a^p + b^p),$$

where

$$d_p = \begin{cases} 1, & \text{при } 0 < p \leqslant 1, \\ 2^p, & \text{при } p > 1. \end{cases}$$

From the last inequality we have that

$$C_p \leqslant \left(\frac{4}{T}\right)^{p/2} d_p \int_0^A \left(((\tau+2)I_1)^{p/2} + \left(\left(2 - \frac{1}{T}\right) I_2\right)^{p/2} \right) d\tau =$$

$$= \left(\frac{4}{T}\right)^{p/2} d_p \left(\frac{1}{p} \left(2(A+2)^{p/2+1} - 2^{p/2+2} \right) I_1^{p/2} + A \left(\left(2 - \frac{1}{T}\right) I_2\right)^{p/2} \right).$$

It is easy to see that $C_p \to 0$ if $T \to \infty$. Then, from the Remark 5.3 it follows that for test the hypothesis about the covariance function can be used the Criterion 5.1. $\diamondsuit$

5.2. Estimates of covariance functions of Gaussian stationary stochastic process in $L_p(T)$ when its value is known only in a finite set of points

Usually in practice the value of the process are observed at the certain times. And based on this data, you need to make conclusions about the behavior of the process that was considered. Therefore, we estimate the covariance function of Gaussian stationary stochastic process when we know the value of this process at the certain times, whose number is finite.

Let $X = \{X(t), t \in \mathbb{T} = [0, T+A], 0 < T < \infty, 0 < A < \infty\}$, $EX(t) = 0$ be a measurable real-valued Gaussian stationary stochastic process with the covariance function

$$\rho(\tau) = EX(t+\tau)X(t), \quad 0 \leqslant \tau \leqslant A,$$

and defined on the probability space $\{\Omega, \mathcal{B}, P\}$.

As an estimator of the covariance function $\rho(\tau)$ we choose

$$\widehat{\rho}_{T,n}(\tau) = \frac{1}{T} \sum_{i=0}^{n-1} X(t_i+\tau)X(t_i)\Delta t_i = \frac{1}{n} \sum_{i=0}^{n-1} X\left(\frac{iT}{n} + \tau\right) X\left(\frac{iT}{n}\right), \quad (5.5)$$

where $X(t_i)$ and $X(t_i+\tau)$ are independent, known values of the random process, $t_i = \frac{iT}{n}$, $i = 0, 1, \ldots, n$, $n \in \mathbb{N}$, $\Delta t_i = \frac{T}{n}$.

Remark 5.4. Since,

$$\mathbf{E}\widehat{\rho}_{T,n}(\tau) = \mathbf{E}\left(\frac{1}{n}\sum_{i=0}^{n-1} X\left(\frac{iT}{n}+\tau\right)X\left(\frac{iT}{n}\right)\right) = \frac{1}{n}\sum_{i=0}^{n-1}\rho(\tau) = \rho(\tau),$$

then $\widehat{\rho}_{T,n}(\tau)$ is unbiased estimate for $\rho(\tau)$.

Theorem 5.3. *Let X be measurable real-valued Gaussian stationary stochastic process with known values at the certain times $t_i = \frac{iT}{n}$, $i = 0, 1, \dots, n$, $n \in \mathbb{N}$. Let $\mathbf{E}X(t) = 0$ and $\rho(\tau)$ be the covariance function of this process and let*

$$C_p = \frac{1}{n^p}\int_0^A \left(\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\left[\rho^2\left(\frac{(i-j)T}{n}\right) + \right.\right.$$
$$\left.\left. + \ \rho\left(\frac{(i-j)T}{n}+\tau\right)\rho\left(\frac{(i-j)T}{n}-\tau\right)\right]\right)^{p/2} d\tau.$$

if $0 < A < \infty$. If condition

$$\varepsilon \geqslant \left(\frac{p}{\sqrt{2}} + \sqrt{(\frac{p}{2}+1)p}\right)^p C_p$$

holds, then

$$P\left\{\int_0^A (\widehat{\rho}_{T,n}|\tau) - \rho(\tau)|^p d\mu(\tau) > \varepsilon\right\} \leqslant 2\sqrt{1 + \frac{\varepsilon^{1/p}\sqrt{2}}{C_p^{\frac{1}{p}}}}\exp\left\{-\frac{\varepsilon^{\frac{1}{p}}}{\sqrt{2}C_p^{\frac{1}{p}}}\right\}.$$

Proof. Consider

$$\mathbf{E}(\widehat{\rho}_{T,n}(\tau) - \rho(\tau))^2 = \mathbf{E}(\widehat{\rho}_{T,n}(\tau))^2 - \rho^2(\tau).$$

From the Isserlis equality (see book [19]) for jointly Gaussian random variables it follows that

$$E\widehat{\rho}^2_{T,n}(\tau) =$$
$$= E\left(\frac{1}{n^2}\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} X\left(\frac{iT}{n}+\tau\right)X\left(\frac{iT}{n}\right)X\left(\frac{jT}{n}+\tau\right)X\left(\frac{jT}{n}\right)\right) =$$

$$= \frac{1}{n^2} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \left[EX\left(\frac{iT}{n}+\tau\right) X\left(\frac{iT}{n}\right) EX\left(\frac{jT}{n}+\tau\right) X\left(\frac{jT}{n}\right) + \right.$$

$$+ EX\left(\frac{iT}{n}+\tau\right) X\left(\frac{jT}{n}+\tau\right) EX\left(\frac{iT}{n}\right) X\left(\frac{jT}{n}\right) +$$

$$\left. + EX\left(\frac{iT}{n}+\tau\right) X\left(\frac{jT}{n}\right) EX\left(\frac{iT}{n}\right) X\left(\frac{jT}{n}+\tau\right) \right] =$$

$$= \frac{1}{n^2} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \left[\rho^2(\tau) + \rho^2\left(\frac{(i-j)T}{n}\right) + \rho\left(\frac{(i-j)T}{n}+\tau\right) \rho\left(\frac{(i-j)T}{n}-\tau\right) \right] =$$

$$= \rho^2(\tau) + \frac{1}{n^2} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \left[\rho^2\left(\frac{(i-j)T}{n}\right) + \rho\left(\frac{(i-j)T}{n}+\tau\right) \rho\left(\frac{(i-j)T}{n}-\tau\right) \right].$$

Namely,

$$\mathbf{E}(\widehat{\rho}_{T,n}(\tau) - \rho(\tau))^2 = \frac{1}{n^2} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \left[\rho^2\left(\frac{(i-j)T}{n}\right) + \right.$$

$$\left. \rho\left(\frac{(i-j)T}{n}+\tau\right) \rho\left(\frac{(i-j)T}{n}-\tau\right) \right]. \quad (5.6)$$

Since $\widehat{\rho}_{T,n}(\tau)$ is a quadratic form of Gaussian vectors, then by Lemma 3.1, Chapter 6 in book [19] we have that $\widehat{\rho}_{T,n}(\tau) - \rho(\tau)$, $\tau \geqslant 0$ is a square Gaussian stochastic process. From Theorem 3.4 follows that

$$P\left\{ \int_0^A |\widehat{\rho}_{T,n}(\tau) - \rho(\tau)|^p d\mu(\tau) > \varepsilon \right\} \leqslant 2\sqrt{1 + \frac{\varepsilon^{1/p}\sqrt{2}}{C_p^{\frac{1}{p}}}} \exp\left\{ -\frac{\varepsilon^{\frac{1}{p}}}{\sqrt{2}C_p^{\frac{1}{p}}} \right\}.$$

Applying equality (5.6) we obtain

$$C_p = \frac{1}{n^p} \int_0^A \left(\sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \left[\rho^2\left(\frac{(i-j)T}{n}\right) + \right.\right.$$

$$\left.\left. + \; \rho\left(\frac{(i-j)T}{n}+\tau\right) \rho\left(\frac{(i-j)T}{n}-\tau\right) \right] \right)^{p/2} d\tau.$$

The theorem is proved. ◊

Denote

$$g(\varepsilon) = 2\sqrt{1 + \frac{\varepsilon^{1/p}\sqrt{2}}{C_p^{\frac{1}{p}}}} \exp\left\{-\frac{\varepsilon^{\frac{1}{p}}}{\sqrt{2}C_p^{\frac{1}{p}}}\right\}.$$

From the Theorem 5.3 it follows that if $\varepsilon \geqslant z_p = C_p\left(\frac{p}{\sqrt{2}} + \sqrt{(\frac{p}{2}+1)p}\right)^p$, then

$$P\left\{\int_0^A |\widehat{\rho}_{T,n}(\tau) - \rho(\tau)|^p d\tau > \varepsilon\right\} \leqslant g(\varepsilon).$$

Let ε_δ be a solution of the equation $g(\varepsilon) = \delta$, where $0 < \delta < 1$. Put $S_\delta = \max\{\varepsilon_\delta, z_p\}$. Then it is obvious that $g(S_\delta) \leqslant \delta$ and

$$P\left\{\int_0^A |\widehat{\rho}_{T,n}(\tau) - \rho(\tau)|^p d\tau > S_\delta\right\} \leqslant \delta. \tag{5.7}$$

Let $\mathbb{H}$ be the hypothesis that the covariance function of a measurable real-valued stationary Gaussian stochastic process X equals to $\rho(\tau)$ if $0 \leqslant \tau \leqslant A$. As a estimation of the $\rho(\tau)$ we choose $\widehat{\rho}_{T,n}(\tau)$. From the Theorem 5.3 it follows that to test the hypothesis $\mathbb{H}$ one can use the following criterion.

Criterion 5.2. For a given level of confidence δ the hypothesis $\mathbb{H}$ is accepted if

$$\int_0^A |\widehat{\rho}_{T,n}(\tau) - \rho(\tau)|^p d\mu(\tau) < S_\delta$$

otherwise hypothesis is rejected.

Remark 5.5. By using this criteria the error of the first kind does not exceed δ.

5.3. Estimates for covariance function of a stationary Gaussian process in the norm of the space $L_p[0, A]$ with unknown mean

Let us consider a continuous real stationary Gaussian stochastic process X defined on a probability space $\{\Omega, \mathcal{B}, P\}$,

$$X = \{X(t), t \in \mathbb{T} = [0, T + A], 0 < A < T < \infty\}$$

and $\mathbf{E}X(t) = m$. We denote the covariance function of this process by

$$\rho(\tau) = E(X(t+\tau) - m)(X(t) - m), \quad \tau \in \mathbb{R}. \tag{5.8}$$

Also we use the following denotation:

$$r(\tau) = \frac{1}{T^2}\int_0^T\int_0^T \rho(s-t-\tau)dsdt. \tag{5.9}$$

We choose as an estimate of the covariance function $\rho(\tau)$ the statistics $\hat{\rho}(\tau)$ defined in (5.1).

Remark 5.6. Since the process X is a continuous one, then the right part in (5.9) contains Riemann integral.

Remark 5.7. Since

$$\mathbf{E}\hat{\rho}(\tau) = \frac{1}{T}\int_0^T \mathbf{E}\left(X(t+\tau)X(t) - \hat{m}X(t+\tau) - \hat{m}_\tau X(t) + \hat{m}\hat{m}_\tau\right)dt =$$

$$= \frac{1}{T}\int_0^T \left(\mathbf{E}X(t+\tau)X(t) - \mathbf{E}\hat{m}X(t+\tau) - \mathbf{E}\hat{m}_\tau X(t) + \mathbf{E}\hat{m}\hat{m}_\tau\right)dt =$$

$$= \rho(\tau) + m^2 - \frac{1}{T^2}\int_0^T\int_0^T \mathbf{E}X(s)X(t+\tau)dtds -$$

$$- \frac{1}{T^2}\int_0^T\int_0^T \mathbf{E}X(s+\tau)X(t)dtds + \frac{1}{T^2}\int_0^T\int_0^T \mathbf{E}X(s+\tau)X(t)dsdt =$$

$$= \rho(\tau) + m^2 - \frac{1}{T^2}\int_0^T\int_0^T \rho(s-t-\tau)dsdt - m^2 - \frac{1}{T^2}\int_0^T\int_0^T \rho(s-t+\tau)dsdt - m^2 +$$

$$+ \frac{1}{T^2}\int_0^T\int_0^T \rho(s-t+\tau)dsdt + m^2 = \rho(\tau) - \frac{1}{T^2}\int_0^T\int_0^T \rho(s-t-\tau)dsdt = \rho(\tau) - r(\tau),$$

then $\hat{\rho}(\tau)$ is a biased estimate for $\rho(\tau)$ and the bias is equal to $r(\tau)$. However, the statistics $\tilde{\rho}(\tau) = \hat{\rho}(\tau) + r(\tau)$ is an unbiased estimate. Moreover, variances of the estimates $\hat{\rho}(\tau)$ and $\tilde{\rho}(\tau)$ are equal.

Theorem 5.4. *Let $X(t)$ be a measurable stationary Gaussian process with $\mathbf{E}X(t) = m$ and the covariance function $\rho(\tau)$. Suppose that for* $0 < A < T$

and $p \geqslant 1$ the condition $C(p,T) < \infty$ holds, where

$$C(p,T) = \int_0^A \left(\frac{1}{T^4} \left[\int_0^T \int_0^T \rho(t-s)dtds \right]^2 + \frac{1}{T^4} \left[\int_0^T \int_0^T \rho(t-s+\tau)dtds \right]^2 + \right.$$

$$+ \frac{2}{T^2} \int_0^T (T-s)[\rho^2(s) + \rho(s+\tau)\rho(s-\tau)]ds -$$

$$- \frac{1}{T^3} \int_0^T \int_0^T \int_0^T [\rho(t-s+\tau)\rho(s-u+\tau) + 2\rho(t-s)\rho(s-u) +$$

$$\left. + \rho(t-s-\tau)\rho(s-u-\tau)]\, dtdsdu \right)^{\frac{p}{2}} d\tau. \quad (5.10)$$

Then for

$$\varepsilon \geqslant \left(\frac{p}{\sqrt{2}} + \sqrt{\left(\frac{p}{2}+1\right)p} \right)^p C(p,T)$$

the following inequality holds true:

$$P\left\{ \int_0^A |\hat\rho(\tau) - \mathbf{E}\hat\rho(\tau)|^p d\tau > \varepsilon \right\} \leqslant 2\sqrt{1 + \frac{\varepsilon^{1/p}\sqrt{2}}{C^{\frac{1}{p}}(p,T)}} \exp\left\{ - \frac{\varepsilon^{\frac{1}{p}}}{\sqrt{2}C^{\frac{1}{p}}(p,T)} \right\}.$$

Proof. At first, we shall calculate

$$\mathbf{D}\hat\rho(\tau) = \mathbf{E}\hat\rho^2(\tau) - (\rho(\tau) - r(\tau))^2. \quad (5.11)$$

In order to do that we consider $\mathbf{E}\hat\rho^2(\tau)$:

$$\mathbf{E}\hat\rho^2(\tau) = \mathbf{E} \left(\frac{1}{T^2} \int_0^T \int_0^T (X(t+\tau) - \hat m_\tau)(X(t) - \hat m) \times \right.$$

$$\times (X(u+\tau) - \hat m_\tau)(X(u) - \hat m)dtdu) =$$

$$= \frac{1}{T^2} \int_0^T \int_0^T [\mathbf{E}(X(t+\tau) - \hat m_\tau)(X(t) - \hat m)\mathbf{E}(X(u+\tau) - \hat m_\tau)(X(u) - \hat m) +$$

$$+ \mathbf{E}(X(t+\tau) - \hat m_\tau)(X(u+\tau) - \hat m_\tau)\mathbf{E}(X(t) - \hat m)(X(u) - \hat m) +$$

$$+\,\mathbf{E}(X(t+\tau)-\hat{m}_\tau)(X(u)-\hat{m})\mathbf{E}(X(t)-\hat{m})(X(u+\tau)-\hat{m}_\tau)]\,dtdu =$$

$$= \frac{1}{T^2}\int\limits_0^T\int\limits_0^T [I_1+I_2+I_3]\,dtdu.$$

Let's calculate each of the integrals above separately:

$$I_1 = \mathbf{E}(X(t+\tau)-\hat{m}_\tau)(X(t)-\hat{m})\mathbf{E}(X(u+\tau)-\hat{m}_\tau)(X(u)-\hat{m}) =$$
$$= (\mathbf{E}X(t+\tau)X(t)-\mathbf{E}\hat{m}X(t+\tau)-\mathbf{E}\hat{m}_\tau X(t)+\mathbf{E}\hat{m}\hat{m}_\tau)\times$$
$$\times\,(\mathbf{E}X(u+\tau)X(u)-\mathbf{E}\hat{m}X(u+\tau)-\mathbf{E}\hat{m}_\tau X(u)+\mathbf{E}\hat{m}\hat{m}_\tau) =$$
$$= \left[(\rho(\tau)+r(\tau))-\left(\frac{1}{T}\int\limits_0^T\rho(s-t-\tau)ds+\frac{1}{T}\int\limits_0^T\rho(s-t+\tau)ds\right)\right]\times$$
$$\times\left[(\rho(\tau)+r(\tau))-\left(\frac{1}{T}\int\limits_0^T\rho(s-u-\tau)ds+\frac{1}{T}\int\limits_0^T\rho(s-u+\tau)ds\right)\right] =$$
$$= (\rho(\tau)+r(\tau))^2-(\rho(\tau)+r(\tau))\left(\frac{1}{T}\int\limits_0^T\rho(s-t-\tau)ds+\right.$$
$$\left.+\,\frac{1}{T}\int\limits_0^T\rho(s-t+\tau)ds+\frac{1}{T}\int\limits_0^T\rho(s-u-\tau)ds+\frac{1}{T}\int\limits_0^T\rho(s-u+\tau)ds\right)+$$
$$+\left(\frac{1}{T}\int\limits_0^T\rho(s-t-\tau)ds\frac{1}{T}\int\limits_0^T\rho(s-u-\tau)ds+\right.$$
$$+\,\frac{1}{T}\int\limits_0^T\rho(s-t-\tau)ds\frac{1}{T}\int\limits_0^T\rho(s-u+\tau)ds+$$
$$+\,\frac{1}{T}\int\limits_0^T\rho(s-t+\tau)ds\frac{1}{T}\int\limits_0^T\rho(s-u-\tau)ds+$$
$$\left.+\,\frac{1}{T}\int\limits_0^T\rho(s-t+\tau)ds\frac{1}{T}\int\limits_0^T\rho(s-u+\tau)ds\right).$$

Now let's calculate I_2:

$$I_2 = \mathbf{E}(X(t+\tau)-\hat{m}_\tau)(X(u+\tau)-\hat{m}_\tau)\mathbf{E}(X(t)-\hat{m})(X(u)-\hat{m}) =$$
$$= \left(\mathbf{E}X(t+\tau)X(u+\tau)-\mathbf{E}\hat{m}_\tau X(t+\tau)-\mathbf{E}\hat{m}_\tau X(u+\tau)+\mathbf{E}\hat{m}^2\right)\times$$

$$\times \left(\mathbf{E}X(t)X(u) - \mathbf{E}\hat{m}X(t) - \mathbf{E}\hat{m}X(u) + \mathbf{E}\hat{m}^2\right) =$$

$$= \left[(\rho(t-u) + r(0)) - \left(\frac{1}{T}\int_0^T \rho(s-t)ds + \frac{1}{T}\int_0^T \rho(s-u)ds\right)\right] \times$$

$$\times \left[(\rho(t-u) + r(0)) - \left(\frac{1}{T}\int_0^T \rho(s-t)ds + \frac{1}{T}\int_0^T \rho(s-u)ds\right)\right] =$$

$$= (\rho(t-u) + r(0))^2 - 2\rho(t-u)\left(\frac{1}{T}\int_0^T \rho(s-t)ds + \frac{1}{T}\int_0^T \rho(s-u)ds\right) -$$

$$- 2r(0)\left(\frac{1}{T}\int_0^T \rho(s-t)ds + \frac{1}{T}\int_0^T \rho(s-u)ds\right) +$$

$$+ \left[\left(\frac{1}{T}\int_0^T \rho(s-t)ds\right)^2 + 2\frac{1}{T}\int_0^T \rho(s-t)ds\frac{1}{T}\int_0^T \rho(s-u)ds + \right.$$

$$\left. + \left(\frac{1}{T}\int_0^T \rho(s-u)ds\right)^2\right].$$

For I_3 we have the following:

$$I_3 = \mathbf{E}(X(t+\tau) - \hat{m}_\tau)(X(u) - \hat{m})\mathbf{E}(X(t) - \hat{m})(X(u+\tau) - \hat{m}_\tau) =$$

$$= (\mathbf{E}X(t+\tau)X(u) - \mathbf{E}\hat{m}X(t+\tau) - \mathbf{E}\hat{m}_\tau X(u) + \mathbf{E}\hat{m}\hat{m}_\tau) \times$$

$$\times (\mathbf{E}X(t)X(u+\tau) - \mathbf{E}\hat{m}_\tau X(t) - \mathbf{E}\hat{m}X(u+\tau) + \mathbf{E}\hat{m}\hat{m}_\tau) =$$

$$= \left[\rho(t-u+\tau) - \frac{1}{T}\int_0^T \rho(s-t-\tau)ds - \frac{1}{T}\int_0^T \rho(s-u+\tau)ds + r(\tau)\right] \times$$

$$\times \left[\rho(t-u-\tau) - \frac{1}{T}\int_0^T \rho(s-t+\tau)ds - \frac{1}{T}\int_0^T \rho(s-u-\tau)ds + r(\tau)\right] =$$

$$= \rho(t-u+\tau)\rho(t-u-\tau) - \rho(t-u+\tau)\frac{1}{T}\int_0^T \rho(s-t+\tau)ds -$$

$$-\rho(t-u+\tau)\frac{1}{T}\int_0^T \rho(s-u-\tau)ds+\rho(t-u+\tau)r(\tau)-$$

$$-\rho(t-u-\tau)\frac{1}{T}\int_0^T \rho(s-t-\tau)ds+\frac{1}{T}\int_0^T \rho(s-t-\tau)ds\frac{1}{T}\int_0^T \rho(s-t+\tau)ds+$$

$$+\frac{1}{T}\int_0^T \rho(s-t-\tau)ds\frac{1}{T}\int_0^T \rho(s-u-\tau)ds-r(\tau)\frac{1}{T}\int_0^T \rho(s-t-\tau)ds-$$

$$-\rho(t-u-\tau)\frac{1}{T}\int_0^T \rho(s-u+\tau)ds+\frac{1}{T}\int_0^T \rho(s-u+\tau)ds\frac{1}{T}\int_0^T \rho(s-t+\tau)ds+$$

$$+\frac{1}{T}\int_0^T \rho(s-u+\tau)ds\frac{1}{T}\int_0^T \rho(s-u-\tau)ds-r(\tau)\frac{1}{T}\int_0^T \rho(s-u+\tau)ds+$$

$$+r(\tau)\rho(t-u-\tau)-r(\tau)\frac{1}{T}\int_0^T \rho(s-t+\tau)ds-r(\tau)\frac{1}{T}\int_0^T \rho(s-u-\tau)ds+r^2(\tau).$$

That is, we have that

$$\mathbf{E}\hat{\rho}^2(\tau)=(\rho(\tau)-r(\tau))^2+r^2(0)+r^2(\tau)+$$

$$+\frac{1}{T^2}\int_0^T\int_0^T \left[\rho^2(t-u)dtdu+\rho(t-u+\tau)\rho(t-u-\tau)dtdu\right]-$$

$$-\frac{1}{T^3}\int_0^T\int_0^T\int_0^T [2\rho(s-t)\rho(t-u)+\rho(t-s-\tau)\rho(t-s+\tau)+$$

$$+\rho(t-s+\tau)\rho(t-u-\tau)-\rho(t-s+\tau)\rho(t-u+\tau)+\rho(t-u+\tau)\rho(s-u-\tau)+$$

$$+\rho(t-u-\tau)\rho(s-u+\tau)-\rho(t-u-\tau)\rho(s-u+\tau)]\,dsdtdu=$$

$$=\frac{1}{T^4}\left[\int_0^T\int_0^T \rho(t-s)dtds\right]^2+\frac{1}{T^4}\left[\int_0^T\int_0^T \rho(t-s+\tau)dtds\right]^2+$$

$$+\frac{2}{T^2}\int_0^T (T-s)[\rho^2(s)+\rho(s+\tau)\rho(s-\tau)]ds-$$

$$-\frac{1}{T^3}\int_0^T\int_0^T\int_0^T [\rho(t-s+\tau)\rho(s-u+\tau)+2\rho(t-s)\rho(s-u)+$$

$$+\rho(t-s-\tau)\rho(s-u-\tau)]\,dtdsdu+(\rho(\tau)-r(\tau))^2 .$$

Then from (5.11) follows

$$\mathbf{D}\hat{\rho}(\tau)=\frac{1}{T^4}\left[\int_0^T\int_0^T \rho(t-s)dtds\right]^2+\frac{1}{T^4}\left[\int_0^T\int_0^T \rho(t-s+\tau)dtds\right]^2+$$

$$+\frac{2}{T^2}\int_0^T (T-s)[\rho^2(s)+\rho(s+\tau)\rho(s-\tau)]ds-$$

$$-\frac{1}{T^3}\int_0^T\int_0^T\int_0^T [\rho(t-s+\tau)\rho(s-u+\tau)+2\rho(t-s)\rho(s-u)+$$

$$+\rho(t-s-\tau)\rho(s-u-\tau)]\,dtdsdu. \quad (5.12)$$

If in the definitions of the variables $\hat{\rho}(\tau)$, $\hat{m}$ and $\hat{m}_\tau$ we substitute the integrals by corresponding integral sums, that is

$$\hat{\rho}_n(\tau)=\frac{1}{n}\sum_{i=0}^{n-1}\left(X(t_i+\tau)-\frac{1}{n}\sum_{i=0}^{n-1}X(t_i+\tau)\right)\left(X(t_i)-\frac{1}{n}\sum_{i=0}^{n-1}X(t_i)\right),$$

where $\{t_i\}$ is a partition of the interval $[0,T]$, then it is easy to see that $\hat{\rho}(\tau)$ is a mean square limit of the $\hat{\rho}_n(\tau)$. Therefore from the Definition 1.6 and Lemma 3.1, Chapter 6 in book [19] follows that for each $\tau \geqslant 0$, the variable $\hat{\rho}(\tau)-\mathbf{E}\hat{\rho}(\tau)$ is a square Gaussian random variable. Then from the Theorem 3.4 we have that for $0<A<\infty$ the following estimate is true:

$$P\left\{\int_0^A |\hat{\rho}(\tau)-\mathbf{E}\hat{\rho}(\tau)|^p d\tau>\varepsilon\right\}\leqslant 2\sqrt{1+\frac{\varepsilon^{1/p}\sqrt{2}}{C^{\frac{1}{p}}(p,T)}}\exp\left\{-\frac{\varepsilon^{\frac{1}{p}}}{\sqrt{2}C^{\frac{1}{p}}(p,T)}\right\}.$$

From definition of the value $C(p,T)$ (see Theorem 3.4) and from formula (5.12) follows the expression (5.10) for $C(p,T)$. ♢

Corollary 5.1. *Let the conditions of the theorem 5.4 hold. Then for*

$$u\geqslant\left(\frac{p}{\sqrt{2}}+\sqrt{(\frac{p}{2}+1)p}\right)C^{1/p}(p,T),$$

where $C(p,T)$ was defined in (5.10), the following inequality holds:

$$P\left\{\left(\int_0^A |\hat{\rho}(\tau)-\mathbf{E}\hat{\rho}(\tau)|^p d\tau\right)^{1/p} > u\right\} \leqslant 2\sqrt{1+\frac{u\sqrt{2}}{C^{\frac{1}{p}}(p,T)}}\exp\left\{-\frac{u}{\sqrt{2}C^{\frac{1}{p}}(p,T)}\right\}.$$

Theorem 5.5. *Let X be a measurable stationary Gaussian process with $\mathbf{E}X(t)=m$ and with covariance function defined in (5.8). Also, let $C(p,T)$ be such as defined in (5.10). Then for*

$$\varepsilon \geqslant \left(\int_0^A |r(\tau)|^p d\tau\right)^{1/p} + \left(\frac{p}{\sqrt{2}} + \sqrt{\left(\frac{p}{2}+1\right)p}\right) C^{1/p}(p,T),$$

where A is an arbitrary number and $r(\tau)$ is defined in (5.9), the following inequality holds:

$$P\left\{\left(\int_0^A |\hat{\rho}(\tau)-\rho(\tau)|^p d\tau\right)^{1/p} > \varepsilon\right\} \leqslant$$

$$\leqslant 2\left(1+\frac{\left(\varepsilon-\left(\int_0^A |r(\tau)|^p d\tau\right)^{1/p}\right)\sqrt{2}}{C^{\frac{1}{p}}(p,T)}\right)^{1/2} \exp\left\{-\frac{\varepsilon-\left(\int_0^A |r(\tau)|^p d\tau\right)^{1/p}}{\sqrt{2}C^{\frac{1}{p}}(p,T)}\right\}.$$

Proof. It is easy to see that the following inequalities hold true:

$$\left(\int_0^A |\hat{\rho}(\tau)-\rho(\tau)|^p d\tau\right)^{1/p} = \left(\int_0^A |\hat{\rho}(\tau)-\mathbf{E}\hat{\rho}(\tau)-r(\tau)|^p d\tau\right)^{1/p} \leqslant$$

$$\leqslant \left(\int_0^A |\hat{\rho}(\tau)-\mathbf{E}\hat{\rho}(\tau)|^p d\tau\right)^{1/p} + \left(\int_0^A |r(\tau)|^p d\tau\right)^{1/p}.$$

For any $\varepsilon > 0$ we have that

$$P\left\{\left(\int\limits_0^A |\hat{\rho}(\tau) - \rho(\tau)|^p d\tau\right)^{1/p} > \varepsilon\right\} \leqslant$$

$$\leqslant P\left\{\left(\int\limits_0^A |\hat{\rho}(\tau) - \mathbf{E}\hat{\rho}(\tau)|^p d\tau\right)^{1/p} > \varepsilon - \left(\int\limits_0^A |r(\tau)|^p d\tau\right)^{1/p}\right\}.$$

Now, if we choose

$$\varepsilon \geqslant \left(\int\limits_0^A |r(\tau)|^p d\tau\right)^{1/p} + \left(\frac{p}{\sqrt{2}} + \sqrt{\left(\frac{p}{2}+1\right)p}\right) C^{1/p}(p,T),$$

where $C(p,T)$ was defined in (5.12), then form the corollary 5.1 we get the assertion of the this theorem . ♢

Let $\mathbb{H}$ be the hypothesis, which says that under $0 \leqslant \tau \leqslant A$ the covariance function of a real valued measurable stationary Gaussian process X with unknown mean is equal to $\rho(\tau)$. As an estimate for $\rho(\tau)$ we shall take $\hat{\rho}(\tau)$.

Let's define

$$g(\varepsilon) = 2\sqrt{1 + \frac{\varepsilon\sqrt{2}}{C^{\frac{1}{p}}(p,T)}} \exp\left\{-\frac{\varepsilon}{\sqrt{2}C^{\frac{1}{p}}(p,T)}\right\}.$$

Then from the theorem 5.5 follows that if

$$\varepsilon \geqslant z_p := \left(\int\limits_0^A |r(\tau)|^p d\tau\right)^{1/p} + \left(\frac{p}{\sqrt{2}} + \sqrt{(\frac{p}{2}+1)p}\right) C^{1/p}(p,T),$$

then

$$P\left\{\left(\int\limits_0^A |\widehat{\rho}(\tau) - \rho(\tau)|^p d\tau\right)^{1/p} > \varepsilon\right\} \leqslant g\left(\varepsilon - \left(\int\limits_0^A |r(\tau)|^p d\tau\right)^{1/p}\right).$$

Let ε_δ be a solution of the equation $g\left(\varepsilon - \left(\int\limits_0^A |r(\tau)|^p d\tau\right)^{1/p}\right) = \delta,$ where $0 < \delta < 1$.

Remark 5.8. Let's define $D(u) = 2\sqrt{1+\sqrt{2}u}\exp\{-\frac{u}{\sqrt{2}}\}$. Then $D(0) = 2$

and for $u \geqslant 0$, $D(u)$ is monotonically decreasing to zero, therefore the equation $D(u) = \delta$ has only one solution, which we shall denote as u_δ. The equation $g\left(\varepsilon - \left(\int\limits_0^A |r(\tau)|^p d\tau\right)^{1/p}\right) = \delta$ for $\varepsilon > \left(\int\limits_0^A |r(\tau)|^p d\tau\right)^{1/p}$ can be written as

$$D\left(\frac{\varepsilon - \left(\int\limits_0^A |r(\tau)|^p d\tau\right)^{1/p}}{C^{\frac{1}{p}}(p,T)}\right) = \delta.$$

Thus, this equation has the only solution $\varepsilon_\delta = u_\delta C^{\frac{1}{p}}(p,T) + \left(\int\limits_0^A |r(\tau)|^p d\tau\right)^{1/p}$.

Let's choose $S_\delta = \max\{\varepsilon_\delta, z_p\}$. Then it is evident that $g(S_\delta) \leqslant \delta$ and

$$P\left\{\left(\int\limits_0^A |\widehat{\rho}(\tau) - \rho(\tau)|^p d\tau\right)^{1/p} > S_\delta\right\} \leqslant \delta. \tag{5.13}$$

Then for testing the hypothesis $\mathbb{H}$ we can use the following criterion.

Criterion 5.3. For a given confidence level δ the hypothesis $\mathbb{H}$ is accepted if

$$\left(\int\limits_0^A |\hat{\rho}(\tau) - \rho(\tau)|^p d\tau\right)^{1/p} \leqslant S_\delta$$

otherwise, the hypothesis is rejected.

Remark 5.9. In this criterion, the type I error (or error of the first kind) does not exceed δ. Since in this paper we do not consider an alternative hypothesis, then we do not estimate the type II error here. But we would like to note that in the case, when neither $r(T,\tau)$, nor $C(p,T)$ tend to zero under $T \to \infty$, the type II error can not be made arbitrarily small. We plan to consider similar criterion with different alternative hypotheses and to study its asymptotic properties in the next paper.

Example 5.3. Let's consider the hypothesis that the covariance function of a stochastic process is the following:

$$\rho(\tau) = \left(1 + \frac{\tau^2}{a^2}\right)^{-\nu},$$

where $a > 0$, $\nu > 0$ are known numbers. We restrict ourselves to the case, which is the most frequently used in meteorological studies, that is

$$\rho(\tau) = \frac{a^2}{\tau^2 + a^2}, \tag{5.14}$$

and we estimate each of the integrals in the definition of the value $C(p, T)$. We have that

$$I_1 = \int_0^T \int_0^T \rho(t-s)dtds = \int_0^T \int_0^T \frac{a^2}{a^2 + (t-s)^2} dtds =$$

$$= a \int_0^T \left(arctg \frac{T-s}{a} + arctg \frac{s}{a} \right) ds \leqslant a \int_0^T \left(\left| arctg \frac{T-s}{a} \right| + \left| arctg \frac{s}{a} \right| \right) ds \leqslant \pi a T.$$

Due to the similar considerations we obtain an estimate for the next integral:

$$I_2 = \int_0^T \int_0^T \rho(t-s+\tau)dtds = a \int_0^T \left(arctg \frac{T-s+\tau}{a} + arctg \frac{\tau - s}{a} \right) ds \leqslant$$

$$\leqslant a \int_0^T \left(\left| arctg \frac{T-s+\tau}{a} \right| + \left| arctg \frac{\tau - s}{a} \right| \right) ds \leqslant \pi a T.$$

In the paper [83], it was shown that if a covariance function $\rho(\tau)$ is continuous one, then

$$I_3 = \int_0^T (T-s)(\rho^2(s) + \rho(s+\tau)\rho(s-\tau))ds \leqslant 4T\rho^2(0).$$

Since now we are considering continuous covariance function, then the estimate given above holds in our case too.

Taking into account estimates for the integrals $I_1 - I_4$ and the fact that $D\hat{\rho}(\tau) > 0$ we get an estimate for $C(p, T)$:

$$C(p, T) \leqslant A \left[\frac{2\,(\pi a)^2}{T^2} + \frac{4\rho^2(0)}{T} \right]^{p/2}. \tag{5.15}$$

Let's now estimate the value of $r(\tau)$. Due to the similar considerations as in the case of estimating I_2, we get that:

$$r(\tau) = \frac{a^2}{T^2} \int_0^T \int_0^T \frac{dsdt}{a^2 + (s - t - \tau)^2} \leqslant \frac{\pi a T}{T^2} = \frac{\pi a}{T}. \tag{5.16}$$

From the inequalities (5.15) and (5.16) follows that $C(p,T) \to 0$ and $r(\tau) \to 0$ as $T \to \infty$. This means that for testing the hypothesis about covariance function, given in (5.14),we can use the Criterion 5.3.

5.4. A criterion for testing hypotheses about the covariance function of a stationary Gaussian stochastic process when the alternative hypothesis is available

In the previous sections were formulated the criteria for testing hypotheses about the covariance function of Gaussian stationary stochastic process. Namely were proposed the hypotheses which we accepted when the certain conditions holds and otherwise the hypotheses were rejected. In additional, in this subsection we consider the problem of testing hypotheses about the covariance function of a stationary Gaussian stochastic process when the alternative hypothesis is available.

Let $X = \{X(t), t \in \mathbb{T} = [0, T + A], 0 < T < \infty, 0 < A < \infty\}$, $EX(t) = 0$ be a measurable real-valued Gaussian stationary stochastic process with the covariance function

$$\rho(\tau) = EX(t + \tau)X(t), \quad 0 \leqslant \tau \leqslant A,$$

and defined on the probability space $\{\Omega, \mathcal{B}, P\}$.

We suppose that $\mathbb{H}_1$ is the hypothesis, which consists in the fact that if $0 \leqslant \tau \leqslant A$ then the covariance function of real-valued Gaussian stationary stochastic process X equals to $\rho_1(\tau)$. As a estimation of the covariance function we choose the correlogram $\hat{\rho}(\tau)$ that is defined in (3.8). Let there exist an alternative hypothesis $\mathbb{H}_2$ which consists in the fact that if $0 \leqslant \tau \leqslant A$ then the covariance function of real-valued Gaussian stationary stochastic process X equals to $\rho_2(\tau)$. We assume that $\rho_1(\tau) \geqslant \rho_2(\tau)$.

Let ε_δ be a solution of the equation $g(\varepsilon) = \delta$ for a given $0 < \delta < 1$, where

$$g(\varepsilon) = 2\sqrt{1 + \frac{\varepsilon^{1/p}\sqrt{2}}{C_p^{\frac{1}{p}}}} \exp\left\{-\frac{\varepsilon^{\frac{1}{p}}}{\sqrt{2}C_p^{\frac{1}{p}}}\right\}.$$

From Theorem 3.4 we have that if condition $\varepsilon \geqslant z_p = C_p\left(\frac{p}{\sqrt{2}} + \sqrt{(\frac{p}{2}+1)p}\right)^p$ holds, then

$$P\left\{\int_0^A |\hat{\rho}(\tau) - \rho(\tau)|^p d\tau > \varepsilon\right\} \leqslant g(\varepsilon).$$

Denote $S_\delta = \max\{\varepsilon_\delta, z_p\}$.

Criterion 5.4. For a given level of confidence δ the hypothesis $\mathbb{H}_1$ about that the covariance function of Gaussian stationary stochastic process equals to $\rho_1(\tau)$ is accepted if

$$\int_0^A |\hat{\rho}(\tau) - \rho_1(\tau)|^p d\tau < S_\delta. \tag{5.17}$$

The hypothesis $\mathbb{H}_2$ about that the covariance function of Gaussian stationary stochastic process equals to $\rho_2(\tau)$ is accepted if

$$\int_0^A |\hat{\rho}(\tau) - \rho_2(\tau)|^p d\tau < S_\delta. \tag{5.18}$$

If both of the inequalities (5.17) and (5.18) hold true or not true, none of the inequalities, then the main and the alternative hypotheses are rejected. This is means that for the application of this criterion are not enough data. It needs to increase the upper limit of the interval, calculate all constants and check whether the mentioned inequalities hold true.

5.5. A criterion for testing hypotheses about the covariance function of non-stationary Gaussian random process

In all previous section of this chapter were considered Gaussian stationary random processes. Based on the estimates that were obtained in the section 1.6 we can prove theorem that is similar to the previous cases and with it help formulate a criteria for testing hypotheses about the covariance function of non-stationary Gaussian stochastic process.

Consider measurable real-valued Gaussian non-stationary stochastic process $X = \{X(t), t \in \mathbb{T} = [0,T], 0 < T < \infty\}$, $EX(t) = 0$ with the covariance function

$$\rho(t,s) = EX(t)X(s), \quad 0 \leqslant t \leqslant T, \quad 0 \leqslant s \leqslant T,$$

and defined on the probability space $\{\Omega, \mathcal{B}, P\}$.

As an estimator of the covariance function $\rho(t,s)$ we choose

$$\widehat{\rho}_n(t,s) = \frac{1}{N}\sum_{i=1}^{N} X(t_i)X(s_i), \tag{5.19}$$

where $X_i(t)$, $X_i(s)$, $i = 1,\dots,N$ are observed independent trajectories of the process X.

Remark 5.10. Since,

$$\mathbf{E}\widehat{\rho}_n(t,s) = \mathbf{E}\left(\frac{1}{N}\sum_{i=1}^{N} X_i(t)\,X_i(s)\right) = \frac{1}{N}\sum_{i=1}^{N}\rho(t,s) = \rho(t,s),$$

then $\widehat{\rho}_n(t,s)$ is unbiased estimate for $\rho(t,s)$.

Theorem 5.6. *Let X be a measurable real-valued non-stationary Gaussian stochastic process, $X_i(t)$, $0 \leqslant t \leqslant T$ and $X_i(s)$, $0 \leqslant s \leqslant T$, $i = 1,2,\dots,N$ are observed independent trajectories of the process X. Let $\mathbf{E}X(t) = 0$, $\rho(t,s)$ be a covariance function of this process and*

$$C_p = \frac{1}{N^p}\int_0^T\int_0^T \left(\rho(t,t)\rho(s,s) + \rho^2(t,s)\right)^p dtds.$$

If condition

$$\varepsilon \geqslant \left(\frac{p}{\sqrt{2}} + \sqrt{(\frac{p}{2}+1)p}\right)^p C_p$$

holds, then

$$P\left\{\int_0^T\int_0^T (\widehat{\rho}_n(t,s) - \rho(t,s))^p dtds > \varepsilon\right\} \leqslant 2\sqrt{1 + \frac{\varepsilon^{1/p}\sqrt{2}}{C_p^{\frac{1}{p}}}}\exp\left\{-\frac{\varepsilon^{\frac{1}{p}}}{\sqrt{2}C_p^{\frac{1}{p}}}\right\}.$$

Proof. Consider

$$\mathbf{E}(\widehat{\rho}_n(t,s) - \rho(t,s))^2 = \mathbf{E}(\widehat{\rho}_n(t,s))^2 - \rho^2(t,s).$$

From the Isserlis equality for jointly Gaussian random variables it follows

$$\mathbf{E}\widehat{\rho}_n^2(t,s) = \mathbf{E}\left(\frac{1}{N^2}\sum_{i=1}^{N}\sum_{k=1}^{N} X_i(t)\,X_i(s)\,X_k(t)\,X_k(s)\right) =$$

$$= \frac{1}{N^2} \sum_{i=1}^{N} \sum_{k=1}^{N} \mathbf{E} X_i(t) X_i(s) X_k(t) X_k(s) =$$

$$= \frac{1}{N^2} \sum_{i=1}^{N} \sum_{k=1}^{N} (\mathbf{E} X_i(t) X_i(s) \mathbf{E} X_k(t) X_k(s) + \mathbf{E} X_i(t) X_k(t) \mathbf{E} X_k(s) X_i(s) +$$

$$+ \mathbf{E} X_k(t) X_i(s) \mathbf{E} X_k(s) X_i(t)) = \rho^2(t,s) +$$

$$+ \frac{1}{N^2} \sum_{i=1}^{N} \sum_{k=1}^{N} (\mathbf{E} X_i(t) X_k(t) \mathbf{E} X_k(s) X_i(s) + \mathbf{E} X_k(t) X_i(s) \mathbf{E} X_k(s) X_i(t)).$$

Since $\rho(t,s) = \rho(s,t)$ and

$$\mathbf{E} X_i(t) X_k(s) = \begin{cases} 0, & \text{при} i \neq k, \\ \rho(t,s), & \text{i} = k, \end{cases}$$

then

$$\mathbf{E} \widehat{\rho}_n^2(t,s) = \rho^2(t,s) + \frac{1}{N} \left(\rho(t,t)\rho(s,s) + \rho^2(t,s) \right).$$

Hence

$$\mathbf{E}(\widehat{\rho}_n(t,s) - \rho(t,s))^2 = \frac{1}{N} \left(\rho(t,t)\rho(s,s) + \rho^2(t,s) \right). \tag{5.20}$$

$\diamondsuit$

From the values of $\widehat{\rho}_n(t,s)$, $\rho(t,s)$, Definition 1.6 and Lemma 3.1, Chapter 6 in book [19] it follows that for all $t, s \in \mathbb{T}$ $\widehat{\rho}_n(t,s) - \rho(t,s)$ is a square Gaussian stochastic process. From Theorem 3.4 follows that

$$P\left\{ \int_0^T \int_0^T |\widehat{\rho}_n(t,s) - \rho(t,s)|^p dtds > \varepsilon \right\} \leqslant 2\sqrt{1 + \frac{\varepsilon^{1/p}\sqrt{2}}{C_p^{\frac{1}{p}}}} \exp\left\{ -\frac{\varepsilon^{\frac{1}{p}}}{\sqrt{2} C_p^{\frac{1}{p}}} \right\},$$

Applying equality (5.20) we obtain

$$C_p = \frac{1}{N^p} \int_0^T \int_0^T \left(\rho(t,t)\rho(s,s) + \rho^2(t,s) \right)^p dtds.$$

Denote

$$g(\varepsilon) = 2\sqrt{1 + \frac{\varepsilon^{1/p}\sqrt{2}}{C_p^{\frac{1}{p}}}} \exp\left\{ -\frac{\varepsilon^{\frac{1}{p}}}{\sqrt{2} C_p^{\frac{1}{p}}} \right\}.$$

From the Theorem 5.6 it follows that if $\varepsilon \geqslant z_p = C_p \left(\frac{p}{\sqrt{2}} + \sqrt{(\frac{p}{2} + 1)p} \right)^p$,

then

$$P\left\{\int_0^T\int_0^T |\widehat{\rho}_n(t,s)-\rho(t,s)|^p dtds > \varepsilon\right\} \leqslant g(\varepsilon).$$

Let ε_δ be a solution of the equation $g(\varepsilon) = \delta$, where $0 < \delta < 1$. Put $S_\delta = \max\{\varepsilon_\delta, z_p\}$. Then it is obvious that $g(S_\delta) \leqslant \delta$ and

$$P\left\{\int_0^T\int_0^T |\widehat{\rho}_n(t,s)-\rho(t,s)|^p dtds > \varepsilon\right\} \leqslant \delta. \tag{5.21}$$

Let $\mathbb{H}$ be the hypothesis that the covariance function of a measurable real-valued non-stationary Gaussian stochastic process X equals to $\rho(t,s)$ if $0 \leqslant t \leqslant T$, $0 \leqslant s \leqslant T$. As a estimation of the $\rho(t,s)$ we choose $\widehat{\rho}_n(t,s)$. From the Theorem 5.6 it follows that to test the hypothesis one can use the following criterion.

Criterion 5.5. For a given level of confidence δ the hypothesis $\mathbb{H}$ is accepted if

$$\int_0^T\int_0^T |\widehat{\rho}_n(t,s)-\rho(t,s)|^p dtds < S_\delta$$

otherwise hypothesis is rejected.

Remark 5.11. By using this criteria the error of the first kind does not exceed δ.

Chapter 6
Square Gaussian stochastic processes defined on $\mathbb{R}^+$.

Orlicz space, considered in the first chapter, contains a wide class of random variables and processes. An important role among them occupy square Gaussian random variables and processes. This is the class we use for the estimation of correlation function of Gaussian stochastic process. Square Gaussian random processes were first introduced by Ryzhov Yu.M. and were investigated by Kozachenko Yu.V., Moklyachuk O.M.,Oleshko T., Stadnik A.

This chapter is devoted to investigation Square Gaussian stochastic processes defined on $\mathbb{R}^+$. The problem of estimation for distribution of supremuma for such processes is considered. Obtained inequalities are applied for investigation of stationary in wide sense square Gaussian stochastic processes. For the real-valued stationary Gaussian stochastic process the estimations for correlogram deviation from correlation function in uniform metric are obtained and criterion for testing of hypothesis about correlation function of Gaussian stationary stochastic processes is constructed.

Using both, criterion constructed in previous chapter and criterion which was constructed in this chapter, enable us significantly reduce the probability of the second type's error.

6.1. The estimations for distribution of supremuma Square Gaussian stochastic processes

Let $\{\Omega, \mathcal{B}, P\}$ be a common probability space, $U(x) = e^{|x|} - 1$.

Suppose that function $R_1(s)$, $-A_1 < s < A_2$, $A_1 > 0$, $A_2 > 0$ (is possible that $A_1 = \infty$ and $A_2 = \infty$) is continuous and such that $R_1(0) = 1$ and $R_1(s)$ increases monotonically with $s > 0$ and decreases monotonically with $s < 0$.

Suppose also that $R_2(s)$, $|s| < A$, $A > 0$ (is possible that $A = \infty$) is even function, such, that $R_2(0) = 1$ and $R_2(s)$ increases monotonically with $s > 0$.

Definition 6.1. [52] Will say that stochastic process $X = \{X(t), t \in \mathbf{T}\}$, where $\mathbf{T}$ is some parametric set from the space $L_U(\Omega)$, $U(x) = e^{|x|} - 1$, belongs to the class $O(R_1, R_2)$, if in space $L_U(\Omega)$ exist such norm $\langle\langle\cdot\rangle\rangle$, that the following conditions hold true:

1) one can found such constants K_1 and K_2, that for all $t \in \mathbf{T}, u \in \mathbf{T}$,

we will have

$$K_2\|X(t)\| \leqslant \langle\langle X(t)\rangle\rangle \leqslant K_1\|X(t)\|,$$
$$K_2\|X(t)-X(u)\| \leqslant \langle\langle X(t)-X(u)\rangle\rangle \leqslant K_1\|X(t)-X(u)\|,$$

where $\|\cdot\|$ is Luxemburg norm in the space $L_U(\Omega)$,

2) for $-A_1 < s < A_2, t \in \mathbf{T}$ we have

$$E\exp\left\{s\frac{X(t)}{\langle\langle X(t)\rangle\rangle}\right\} \leqslant R_1(s),$$

3) for $|s| < A, t, u \in \mathbf{T}$ we have

$$E\exp\left\{s\frac{X(t)-X(u)}{\langle\langle X(t)-X(s)\rangle\rangle}\right\} \leqslant R_2(s),$$

4)

$$\sup_{m(t,s)<h} \langle\langle X(t)-X(s)\rangle\rangle \leqslant \sigma(h),$$

where $\sigma = \{\sigma(h), h \geqslant 0\}$, is continuous monotone increasing function, $\sigma(0) = 0$ and $\sigma(h) \longrightarrow 0$ for $h \longrightarrow 0$, $m(t,s)$ is a metric in the space $(\mathbf{T}, m)$.

Lemma 6.1. *[52] Square Gaussian stochastic process $X = \{X(t), t \in \mathbf{T}\}$ belongs to the class $O(R,R)$, where*

$$R(s) = \exp\left\{-\frac{|s|}{2}\right\}(1-|s|)^{-\frac{1}{2}}, \quad |s| < 1,$$
$$\langle\langle X(t)\rangle\rangle = \sqrt{2}(E|X(t)|^2)^{\frac{1}{2}},$$
$$\langle\langle X(t)-X(s)\rangle\rangle = \sqrt{2}(E((X(t)-X(s))^2)^{\frac{1}{2}}.$$

Let $(\mathbf{T}, m)$ be a compact metric space with metrics m and let $X = \{X(t), t \in \mathbf{T}\}$ be separable square-Gaussian stochastic process.

Suppose, that exist continuous function $\sigma = \{\sigma(h), h > 0\}$, strictly monotone increasing such that $\sigma(h) \longrightarrow 0$ for $h \longrightarrow 0$ and the following inequality holds

$$\sup_{m(t,s)<h, t,s\in\mathbf{T}} (E(X(t)-X(s))^2)^{\frac{1}{2}} \leqslant \sigma(h).$$

Remark 6.1. If process $X(t)$ is continuous in L_2-norm, then the function

$$\sigma(h) = \sup_{m(t,s)<h, t,s\in\mathbf{T}} (E(X(t)-X(s))^2)^{\frac{1}{2}},$$

has this property if it is continuous and strictly monotone increasing.
We introduce the following notation:

- $\varepsilon_0 = \inf\limits_{t\in\mathbf{T}} \sup\limits_{s\in\mathbf{T}} m(t,s)$,
- $\delta_0 = \sup\limits_{t\in\mathbf{T}} \left(E|X(t)|^2\right)^{\frac{1}{2}}$,
- $\sigma^{(-1)}(h)$-inverse to $\sigma(u)$ function,
- $t_0 = \sigma(\varepsilon_0)$,
- $N(\varepsilon)$-the smallest number of closed balls of radius ε, which cover $(\mathbf{T}, m)$,
- $r(u) > 0$, $u \geqslant 1$ is monotone increasing function, $r(u) \longrightarrow \infty$ for $u \longrightarrow \infty$, such, that $r(e^t)$ is convex function for $t \geqslant 0$.

Theorem 6.1. *If the condition*

$$\int\limits_0^{t_0} r(N(\sigma^{(-1)}(v)))dv < \infty,$$

holds, then for all real p, $0 < p < 1$, and u such that

$$0 < u < \frac{1-p}{\sqrt{2}} \min\left\{\frac{1}{\delta_0}, \frac{1}{t_0}\right\}$$

we have the inequality

$$E\exp\left\{u\sup_{t\in\mathbf{T}}|X(t)|\right\} \leqslant 2\left(R\left(\frac{u\sqrt{2}\delta_0}{1-p}\right)\right)^{1-p}\left(R\left(\frac{u\sqrt{2}t_0}{1-p}\right)\right)^{p}\times$$

$$\times r^{(-1)}\left(\frac{1}{t_0 p}\int\limits_0^{t_0 p} r(N(\sigma^{(-1)}(v)))dv\right), \tag{6.1}$$

where

$$R(z) = (1-z)^{-\frac{1}{2}}\exp\left\{-\frac{z}{2}\right\}, \tag{6.2}$$

$0 \leqslant z < 1$

Proof. The proof of the theorem follows from lemma 4.1 [59] if $M = 1, A^+ = 1$. ♢

Corollary 6.1. *Let the conditions of theorem 6.1 hold and* $z_0 = \max(\delta_0, t_0)$. *Then for* $0 < p < 1$ *and for all*

$$0 < u < \frac{1-p}{z_0\sqrt{2}}$$

inequality

$$E\exp\left\{u\sup_{t\in\mathbf{T}}|X(t)|\right\} \leqslant 2R\left(\frac{u\sqrt{2}z_0}{1-p}\right) r^{(-1)}\left(\frac{1}{t_0p}\int\limits_0^{t_0p} r(N(\sigma^{(-1)}(v)))dv\right). \tag{6.3}$$

holds.

Proof. Since the function $R(z)$ is monotone increasing for $0 < z < 1$, then corollary follows from the theorem 6.1. ♢

Let $(\mathbf{T}, m)$-be separable finite dimensional metric space. Suppose, that space $(\mathbf{T}, m)$ can be represented as a countable union of compact sets B_k, $k = 1, 2, \ldots$, namely $\mathbf{T} = \bigcup_{k=1}^{\infty} B_k$. Consider a separable square Gaussian stochastic process $X = \{X(t), t \in \mathbf{T}\}$.

Assume the existence of such continuous strictly monotone increasing functions $\sigma_k = \{\sigma_k(h), h > 0\}$, that $\sigma_k(h) \longrightarrow 0$ when $h \longrightarrow 0$, for which the next inequalities hold

$$\sup_{m(t,s)<h, t,s\in B_k} (E(X(t) - X(s))^2)^{\frac{1}{2}} \leqslant \sigma_k(h).$$

We denote

- $\varepsilon_{0k} = \inf\limits_{t\in B_k} \sup\limits_{s\in B_k} m(t,s)$,
- $\delta_{0k} = \sup\limits_{t\in B_k} \left(E|X(t)|^2\right)^{1/2}$,
- $\sigma_k^{(-1)}$-inverse to σ_k function
- $t_{0k} = \sigma_k(\varepsilon_{0k})$,
- $z_{0k} = \max(\delta_{0k}, t_{0k})$,
- $N_k(u)$ - the smallest number of closed balls of radius u, which cover B_k,
- $r(u) > 0$, $u \geqslant 1$ - monotone increasing function, $r(u) \longrightarrow \infty$ for $u \longrightarrow \infty$, such, that function $r(e^t)$ is convex for $t \geqslant 0$.

From the theorem 6.1 the next theorem follows.

Theorem 6.2. *If for all k condition*

$$\int\limits_0^{t_{0k}} r(N_k(\sigma_k^{(-1)}(v)))dv < \infty,$$

holds, then for all real p, $0 < p < 1$ and u such, that $0 < u < \frac{1-p}{\sqrt{2}} \min\left\{\frac{1}{\delta_{0k}}, \frac{1}{t_{0k}}\right\}$ inequality

$$E\exp\left\{u \sup_{t\in B_k} |X(t)|\right\} \leqslant 2\left(R\left(\frac{u\sqrt{2}\delta_{0k}}{1-p}\right)\right)^{1-p}\left(R\left(\frac{u\sqrt{2}t_{0k}}{1-p}\right)\right)^{p} \times$$

$$\times r^{(-1)}\left(\frac{1}{t_{0k}p}\int\limits_0^{t_{0k}p} r(N_k(\sigma_k^{(-1)}(v)))dv\right), \tag{6.4}$$

is true, where $R(z)$ defined in (6.2), $0 \leqslant z < 1$.

Theorem 6.3. *Let $c(t)$, $t \in \mathbf{T}$ be continuous function and $0 < c(t) < 1$ for all $t \in T$. We denote $\gamma_k = \sup_{t\in B_k} |c(t)|$. If for $0 < p < 1$ conditions*

1) $d = \sum_{k=1}^{\infty} \gamma_k z_{0k} < \infty$,

2) $\int_0^{pt_{0k}} r(N_k(\sigma_k^{(-1)}(v)))dv < \infty$,

3) $\prod_{k=1}^{\infty}\left(r^{(-1)}\left(\frac{1}{pt_{0k}}\int_0^{pt_{0k}} r(N_k(\sigma_k^{(-1)}(v)))dv\right)\right)^{\frac{\gamma_k z_{ok}}{d}} < \infty$

hold, then for all $0 < u < \frac{1-p}{d\sqrt{2}}$ inequality

$$E\exp\left\{u\sup_{t\in\mathbf{T}}|c(t)X(t)|\right\} \leqslant 2R\left(\frac{ud\sqrt{2}}{1-p}\right)\times$$

$$\times\prod_{k=1}^{\infty}\left(r^{(-1)}\left(\frac{1}{t_{0k}p}\int\limits_0^{t_{0k}p} r(N_k(\sigma_k^{(-1)}(v)))dv\right)\right)^{\frac{\gamma_k z_{ok}}{d}}, \tag{6.5}$$

is true, where $R(s) = \exp\left\{-\frac{|s|}{2}\right\}(1-|s|)^{-\frac{1}{2}}$, $|s| < 1$.

Proof. Obviously, that

$$\sup_{t\in\mathbf{T}}|c(t)X(t)| \leqslant \sup_k \sup_{t\in B_k} |c(t)||X(t)| = \sum_{k=1}^{\infty}\gamma_k \sup_{t\in B_k}|X(t)|.$$

Then for $u > 0$

$$E \exp\left\{ u \sup_{t \in \mathbf{T}} |c(t)X(t)| \right\} \leqslant E \exp\left\{ u \sum_{k=1}^{\infty} \gamma_k \sup_{t \in B_k} |X(t)| \right\}. \tag{6.6}$$

Will choose $z_{0k} = \max(\delta_{0k}, t_{0k})$. Since $R(z)$ is monotone increasing function for $0 < z < 1$, then from theorem 6.2 and corollary 6.1 for all k and

$$0 < u < \frac{1-p}{\sqrt{2}} \frac{1}{z_{0k}}$$

the next inequality holds

$$E \exp\left\{ u \sup_{t \in B_k} |X(t)| \right\} \leqslant 2R \left(\frac{u\sqrt{2} z_{0k}}{1-p} \right) r^{(-1)} \left(\frac{1}{t_{0k} p} \int_0^{t_{0k} p} r(N_k(\sigma_k^{(-1)}(v))) dv \right). \tag{6.7}$$

$\diamondsuit$

Assume that $\{q_k\}$ is such sequence, that $q_k > 1$, $k = 1, 2, \ldots$ and $\sum_{k=1}^{\infty} \frac{1}{q_k} = 1$.

From (6.6), (6.7) and Holder inequality [19] follows, that for $u > 0$ for which all inequalities $0 < u\gamma_k q_k < \frac{1-p}{\sqrt{2}} \frac{1}{z_{0k}}$, $k = 1, 2, \ldots$ hold, we have

$$E \exp\left\{ u \sup_{t \in \mathbf{T}} |c(t)X(t)| \right\} \leqslant E \prod_{k=1}^{\infty} \exp\left\{ u\gamma_k \sup_{t \in B_k} |X(t)| \right\} \leqslant$$

$$\leqslant \prod_{k=1}^{\infty} \left(E \exp\left\{ u\gamma_k q_k \sup_{t \in B_k} |X(t)| \right\} \right)^{1/q_k} \leqslant$$

$$\leqslant \prod_{k=1}^{\infty} \left(2R \left(\frac{u\gamma_k q_k \sqrt{2} z_{0k}}{1-p} \right) r^{(-1)} \left(\frac{1}{t_{0k} p} \int_0^{t_{0k} p} r(N_k(\sigma_k^{(-1)}(v))) dv \right) \right)^{1/q_k} =$$

$$= \prod_{k=1}^{\infty} 2^{1/q_k} \prod_{k=1}^{\infty} \left(R \left(\frac{u\gamma_k q_k \sqrt{2} z_{0k}}{1-p} \right) \right)^{1/q_k} \times$$

$$\times \prod_{k=1}^{\infty} \left(r^{(-1)} \left(\frac{1}{t_{0k} p} \int_0^{t_{0k} p} r(N_k(\sigma_k^{(-1)}(v))) dv \right) \right)^{1/q_k}.$$

Last inequality holds for all u, for wich inequalities

$$0 < u < \frac{1-p}{\sqrt{2}} \frac{1}{z_{0k} \gamma_k q_k}$$

hold for all $k = 1, 2, \ldots$. We denote $d = \sum_{k=1}^{\infty} \gamma_k z_{0k}$ and choose $q_k = \frac{d}{\gamma_k z_{0k}}$ $\left(q_k > 1, \sum_{k=1}^{\infty} \frac{1}{q_k} = 1\right)$.

Than for $0 < u < \frac{1-p}{d\sqrt{2}}$ the following conditions hold true

$$R\left(u\frac{q_k \gamma_k z_{0k}\sqrt{2}}{1-p}\right) = R\left(u\frac{d\sqrt{2}}{1-p}\right)$$

and

$$\prod_{k=1}^{\infty}\left(R\left(u\frac{q_k \gamma_k z_{0k}\sqrt{2}}{1-p}\right)\right)^{\frac{1}{q_k}} =$$

$$= \left(R\left(u\frac{d\sqrt{2}}{1-p}\right)\right)^{\sum_{k=1}^{\infty}\frac{1}{q_k}} = R\left(u\frac{d\sqrt{2}}{1-p}\right).$$

Since

$$\prod_{k=1}^{\infty} 2^{\frac{1}{q_k}} = 2^{\sum_{k=1}^{\infty}\frac{1}{q_k}} = 2,$$

we will have

$$E\exp\left\{u\sup_{t\in\mathbf{T}}|c(t)X(t)|\right\} \leqslant 2R\left(\frac{ud\sqrt{2}}{1-p}\right)\times$$

$$\times\prod_{k=1}^{\infty}\left(r^{(-1)}\left(\frac{1}{t_{0k}p}\int_0^{t_{0k}p} r(N_k(\sigma_k^{(-1)}(v)))dv\right)\right)^{\frac{\gamma_k Z_{0k}}{d}}$$

for

$$0 < u < \frac{1-p}{d\sqrt{2}}.$$

Theorem 6.4. *If the conditions of the theorem 6.3 hold, than for an arbitrary $x > 0$ and $0 < p < 1$ the following inequality holds*

$$P\left\{\sup_{t\in T}|c(t)X(t)| > x\right\} \leqslant 2\exp\left\{-\frac{x(1-p)}{d\sqrt{2}}\right\}\left(1+\frac{\sqrt{2}x(1-p)}{d}\right)^{1/2}\widetilde{\Phi}(p),$$

where

$$\widetilde{\Phi}(p) = \prod_{k=1}^{\infty} r^{(-1)}\left(\frac{1}{t_{0k}p}\int_0^{t_{0k}p} r\left(N_k(\sigma_k^{(-1)}(v))\right)dv\right)^{\frac{\gamma_k Z_{0k}}{d}}.$$

Proof. From the theorem 6.3 and Chebyshev inequality follows, that for $x > 0$ and $0 < u < \frac{1-p}{d\sqrt{2}}$ we will obtain

$$P\left\{\sup_{t\in T}|c(t)X(t)| > x\right\} \leqslant \frac{E\exp\{u\sup\limits_{t\in T}|c(t)X(t)|\}}{\exp\{ux\}} \leqslant$$

$$\leqslant 2R\left(\frac{ud\sqrt{2}}{(1-p)}\right)\exp\{-ux\}\widetilde{\Phi}(p) =$$

$$= 2\left(1-\frac{ud\sqrt{2}}{(1-p)}\right)^{-1/2}\exp\left\{-\frac{ud\sqrt{2}}{2(1-p)}\right\}\exp\{-ux\}\widetilde{\Phi}(p).$$

Let us denote $D = \frac{d\sqrt{2}}{1-p}$. Then

$$P\left\{\sup_{t\in T}|c(t)X(t)| > x\right\} \leqslant Z(u,x)\widetilde{\Phi}(p),$$

for $0 < u < \frac{1}{D}$, where

$$Z(u,x) = 2(1-uD)^{-1/2}\exp\left\{-\frac{u}{2}(D+2x)\right\}.$$

It is easy to verify, that for $0 < u < \frac{1}{D}$ infinum of this function is achieved in the point $u = \frac{1}{D} - \frac{1}{D+2x} < \frac{1}{D}$. Therefore,

$$\inf_{0<u<\frac{1}{D}} Z(u,x) = 2\exp\left\{-\frac{x}{D}\right\}\left(\frac{D}{D+2x}\right)^{-1/2}.$$

The last equality proves the theorem. ◊

6.2. Stationary square Gaussian stochastic processes.

Let $X = \{X(t), t \geqslant 0\}$ be a stationary in the wide sense square Gaussian stochastic process and let for all $t, s \geqslant 0$,

$$(E(X(t)-X(s))^2)^{\frac{1}{2}} = \widetilde{\sigma}(|t-s|)$$

be true.

Assume that $\widetilde{\sigma}(|t-s|) < \sigma(|t-s|)$, where $\sigma = \{\sigma(u), u > 0\}$ is strictly monotone increasing continuous function, $\sigma(u) \longrightarrow 0$ for $u \longrightarrow 0$ and

$\lim\limits_{u \longrightarrow \infty} \sigma(u) = c_\sigma < \infty$. Since

$$\left(E(X(t) - X(s))^2\right)^{1/2} \leqslant \left(E(X(t))^2\right)^{1/2} + \left(E(X(s))^2\right)^{1/2},$$

then $c_\sigma = 2\left(E(X(t))^2\right)^{1/2}$.

Let $\lambda = \{t_0, t_1, \ldots, t_k, \ldots\}$, $k = 1, 2, \ldots$ be such partition of $\mathbb{R}^+$, that $t_0 = 0$, $t_{k-1} < t_k$, $t_k - t_{k-1} \geqslant 1$ and $t_k \longrightarrow \infty$ for $k \longrightarrow \infty$ and let $c(t) > 0$ be some continuous function, that $0 < c(t) < 1$.

Let us denote $B_k = [t_{k-1}, t_k]$, $k = 1, 2, \ldots$, $\gamma_k = \sup\limits_{t \in B_k} |c(t)|$.

As before, we denote

- $\varepsilon_{0k} = \inf\limits_{t \in B_k} \sup\limits_{s \in B_k} m(t, s)$,
- $\delta_{0k} = \sup\limits_{t \in B_k} \left(E|X(t)|^2\right)^{1/2}$,
- $t_{0k} = \sigma(\varepsilon_{0k})$,
- $z_{0k} = \max(\delta_{0k}, t_{0k})$,
- $N_k(u)$ - the smallest number of closed balls of radius u, which cover B_k,
- $r(u) > 0$, $u \geqslant 1$- monotone increasing function, $r(u) \longrightarrow \infty$ for $u \longrightarrow \infty$, such, that function $r(e^t)$ is convex for $t \geqslant 0$.

We will use this denotation throughout this section.

Lemma 6.2. *Let $X = \{X(t), t \geqslant 0\}$ be a stationary separable square Gaussian stochastic processes.*

If the following conditions hold

1) $\sum_{k=1}^{\infty} \gamma_k < \infty$,

2) $\sum_{k=1}^{\infty} \gamma_k \ln(t_k - t_{k-1}) < \infty$,

3) for some $\alpha > 0$ and any $\varepsilon > 0$

$$\int\limits_0^\varepsilon \left(\sigma^{(-1)}(u)\right)^{-\alpha} du < \infty,$$

then for $0 < p < 1$ *and* $0 < u < \frac{1-p}{d\sqrt{2}}$ *inequality*

$$E\exp\left\{u\sup_{t\in\mathbf{R}^+}|c(t)X(t)|\right\}\leqslant 2R\left(\frac{ud\sqrt{2}}{1-p}\right)\times$$

$$\times\exp\left\{\frac{1}{d}\sum_{k=1}^{\infty}\gamma_k z_{0k}\ln\left[\left(\frac{1}{pt_{0k}}\int\limits_0^{pt_{0k}}\left(1+\frac{t_k-t_{k-1}}{2\sigma^{(-1)}(v)}\right)^{\alpha}dv\right)^{1/\alpha}\right]\right\}\leqslant$$

$$\leqslant 2R\left(\frac{ud\sqrt{2}}{1-p}\right)\exp\left\{\frac{1}{d}\sum_{k=1}^{\infty}\gamma_k z_{0k}\left[\ln(t_k-t_{k-1})+\frac{1}{\alpha}\ln R^*(p)\right]\right\}$$

holds true, where
$d=\sum_{k=1}^{\infty}\gamma_k z_{0k}$,
$R^*(p)=\sup\limits_k\left\{\left(\frac{1}{pt_{0k}}\int_0^{pt_{0k}}\left(1+\frac{1}{2\sigma^{(-1)}(v)}\right)^{\alpha}dv\right)^{\frac{1}{\alpha}}\right\}.$

Proof. Lemma follows from the theorem 6.3. We put $r(v)=v^{\alpha}$, $v>0$, $\alpha>0$. In this case $T=\mathbb{R}$, $m(t,s)=|t-s|$. Since $B_k=[t_{k-1},t_k]$, then

$$N_k(u)\leqslant\frac{t_k-t_{k-1}}{2u}+1$$

and

$$\frac{1}{pt_{0k}}\int\limits_0^{pt_{0k}}r\left(N_k\left(\sigma_k^{(-1)}(u)\right)\right)du\leqslant\frac{1}{pt_{0k}}\int\limits_0^{pt_{0k}}r\left(1+\frac{t_k-t_{k-1}}{2\sigma^{(-1)}(u)}\right)du$$

(for stationary process $\sigma_k(u)=\sigma(u)$, $k=1,2,\ldots$).

Since for $x\geqslant 1$ and $y>0$ the inequality

$$1+xy\leqslant x(1+y)$$

holds and taking into account that $r(v)=v^{\alpha}$, $\alpha>0$ is an increasing function, we have

$$r\left(1+\frac{t_k-t_{k-1}}{2\sigma^{(-1)}(u)}\right)\leqslant r\left((t_k-t_{k-1})\left(1+\frac{1}{2\sigma^{(-1)}(u)}\right)\right)=$$

$$=(t_k-t_{k-1})^{\alpha}\left(1+\frac{1}{2\sigma^{(-1)}(u)}\right)^{\alpha}.$$

Then

$$r^{(-1)}\left(\frac{1}{pt_{0k}}\int\limits_0^{pt_{0k}} r(N_k(\sigma^{(-1)}(u)))du\right) = \left(\frac{1}{pt_{0k}}\int\limits_0^{pt_{0k}} r(N_k(\sigma^{(-1)}(u)))du\right)^{\frac{1}{\alpha}} \leqslant$$

$$\leqslant (t_k - t_{k-1})\left(\frac{1}{pt_{0k}}\int\limits_0^{pt_{0k}}\left(1+\frac{1}{2\sigma^{(-1)}(u)}\right)^{\alpha} dv\right)^{\frac{1}{\alpha}}.$$

Thus,

$$\prod_{k=1}^{\infty}\left(r^{(-1)}\left(\frac{1}{pt_{0k}}\int\limits_0^{pt_{0k}} r\left(N_k\left(\sigma^{(-1)}(v)\right)\right)dv\right)\right)^{\frac{\gamma_k z_{0k}}{d}} =$$

$$= \exp\left\{\frac{1}{d}\sum_{k=1}^{\infty}\gamma_k z_{0k}\ln\left[r^{(-1)}\left(\frac{1}{pt_{0k}}\int\limits_0^{pt_{0k}} r\left(N_k\left(\sigma^{(-1)}(v)\right)\right)dv\right)\right]\right\} =$$

$$= \exp\left\{\frac{1}{d}\sum_{k=1}^{\infty}\gamma_k z_{0k}\ln\left[\left(\frac{1}{pt_{0k}}\int\limits_0^{pt_{0k}}\left(1+\frac{t_k - t_{k-1}}{2\sigma^{(-1)}(v)}\right)^{\alpha} dv\right)^{1/\alpha}\right]\right\} \leqslant$$

$$\leqslant \exp\left\{\frac{1}{d}\sum_{k=1}^{\infty}\gamma_k z_{0k}\ln\left[(t_k - t_{k-1})\left(\frac{1}{pt_{0k}}\int\limits_0^{pt_{0k}}\left(1+\frac{1}{2\sigma^{(-1)}(v)}\right)^{\alpha} dv\right)^{\frac{1}{\alpha}}\right]\right\} =$$

$$= \exp\left\{\frac{1}{d}\left[\sum_{k=1}^{\infty}\gamma_k z_{0k}\ln(t_k - t_{k-1}) + \right.\right.$$

$$\left.\left. + \frac{1}{\alpha}\sum_{k=1}^{\infty}\gamma_k z_{0k}\ln\left(\frac{1}{pt_{0k}}\int\limits_0^{pt_{0k}}\left(1+\frac{1}{2\sigma^{(-1)}(v)}\right)^{\alpha} dv\right)\right]\right\}. \qquad (6.8)$$

$\diamondsuit$

Since $z_{0k} \leqslant c_\sigma$, then

$$\sum_{k=1}^{\infty}\gamma_k z_{0k} \leqslant c_\sigma \sum_{k=1}^{\infty}\gamma_k < \infty,$$

and

$$\sum_{k=1}^{\infty}\gamma_k z_{0k}\ln(t_k - t_{k-1}) \leqslant c_\sigma \sum_{k=1}^{\infty}\gamma_k \ln(t_k - t_{k-1}) < \infty.$$

For $\varepsilon > 0$

$$\frac{1}{\varepsilon}\int\limits_0^{\varepsilon}\left(1+\frac{1}{2\sigma^{(-1)}(v)}\right)^{\alpha} dv < \infty,$$

because

$$\int_0^{\varepsilon}\left(\sigma^{(-1)}(v)\right)^{-\alpha}dv<\infty,$$

therefore all integrals in (6.8) is convergent.

Consider a function $z(u)=\frac{1}{u}\int_0^u\left(1+\frac{1}{2\sigma^{(-1)}(v)}\right)^{\alpha}dv$ and denote $f(v)=\left(1+\frac{1}{2\sigma^{(-1)}(v)}\right)^{\alpha}$, $\alpha>0$.

$$z'(u)=-\frac{1}{u^2}\int_0^u f(v)dv+\frac{1}{u}f(u)=\frac{1}{u}\left(f(u)-\frac{1}{u}\int_0^u f(v)dv\right).$$

Function $f(v)$ decreases for $v>0$ and $\alpha>0$, therefore $\frac{1}{u}\int_0^u f(v)dv>f(u)$, that is $z'(u)<0$, and therefore, $z(u)$ - decreases.

$\varepsilon_{0k}\geqslant 1$, $t_{0k}=\sigma_k(\varepsilon_{0k})=\sigma(\varepsilon_{0k})\geqslant\sigma(1)$, this means R^* exist, moreover,

$$R^*(p)\leqslant\frac{1}{p\sigma(1)}\int_0^{p\sigma(1)}\left(1+\frac{1}{2\sigma^{(-1)}(v)}\right)^{\alpha}dv.$$

Therefore

$$R^*(p)=\sup_k\left\{\frac{1}{pt_{0k}}\int_0^{pt_{0k}}\left(1+\frac{1}{2\sigma^{(-1)}(v)}\right)^{\alpha}dv\right\}<\infty.$$

Then

$$\sum_{k=1}^{\infty}\gamma_k z_{0k}\ln\left(\frac{1}{pt_{0k}}\int_0^{pt_{0k}}\left(1+\frac{1}{2\sigma^{(-1)}(v)}\right)^{\alpha}dv\right)^{\frac{1}{\alpha}}=$$

$$=\frac{1}{\alpha}\ln R^*(p)\sum_{k=1}^{\infty}\gamma_k z_{0k}\leqslant\frac{1}{\alpha}\ln R^*(p)c_\sigma\sum_{k=1}^{\infty}\gamma_k<\infty.$$

Therefore,

$$E\exp\left\{u\sup_{t\in\mathbf{R}^+}|c(t)X(t)|\right\}\leqslant 2R\left(\frac{ud\sqrt{2}}{1-p}\right)\times$$

$$\exp\left\{\frac{1}{d}\sum_{k=1}^{\infty}\gamma_k z_{0k}\left[\ln(t_k-t_{k-1})+\frac{1}{\alpha}\ln R^*(p)\right]\right\}.$$

Theorem 6.5. *Let $X=\{X(t),t\geqslant 0\}$ be a stationary separable square Gaussian stochastic processes and let the following conditions hold*

1) $\sum_{k=1}^{\infty} \gamma_k < \infty$,

2) $\sum_{k=1}^{\infty} \gamma_k \ln(t_k - t_{k-1}) < \infty$,

3) for some $\alpha > 0$ *and for any* $\varepsilon > 0$

$$\int_0^{\varepsilon} \left(\sigma^{(-1)}(u)\right)^{-\alpha} du < \infty.$$

Then for an arbitrary $x > 0$ *and* $0 < p < 1$ *the following inequality holds true*

$$P\left\{\sup_{t \in \mathbf{R}^+} |c(t)X(t)| > x\right\} \leqslant 2\exp\left\{-\frac{x(1-p)}{d\sqrt{2}}\right\}\left(1 + \frac{\sqrt{2}x(1-p)}{d}\right)^{1/2} \times$$

$$\times \widetilde{\Phi}_1(p) \leqslant 2\exp\left\{-\frac{x(1-p)}{d\sqrt{2}}\right\}\left(1 + \frac{\sqrt{2}x(1-p)}{d}\right)^{1/2} \widetilde{\Phi}_2(p),$$

where

$$d = \sum_{k=1}^{\infty} \gamma_k z_{0k},$$

$$\widetilde{\Phi}_1(p) = \exp\left\{\frac{1}{d}\sum_{k=1}^{\infty} \gamma_k z_{0k} \ln\left[\left(\frac{1}{pt_{0k}} \int_0^{pt_{0k}} \left(1 + \frac{t_k - t_{k-1}}{2\sigma^{(-1)}(v)}\right)^{\alpha} dv\right)^{1/\alpha}\right]\right\}, \tag{6.9}$$

$$\widetilde{\Phi}_2(p) = \exp\left\{\frac{1}{d}\sum_{k=1}^{\infty} \gamma_k z_{0k} \left[\ln(t_k - t_{k-1}) + \frac{1}{\alpha}\ln R^*(p)\right]\right\}, \tag{6.10}$$

$$R^*(p) = \sup_k \left\{\left(\frac{1}{pt_{0k}} \int_0^{pt_{0k}} \left(1 + \frac{1}{2\sigma^{(-1)}(v)}\right)^{\alpha} dv\right)^{\frac{1}{\alpha}}\right\}.$$

Proof. The proof follows from lemma 6.2 and Chebyshev inequality. ♢

Example 6.1. Let $X = \{X(t), t \in T\}$ be a stationary square Gaussian stochastic processes and assume also, that $\sigma(v) \leqslant cv^{\beta}$, $v > 0$, $c > 0$, $\beta \leqslant 1$ та $\sigma(v) \leqslant c_{\sigma}$.

Consider $c(t) = \frac{1}{(\ln(e+t))^{\gamma}}$, $t > 0$, $\gamma > 2$. Let us verify, that conditions of the theorem 6.5 hold for $0 < \alpha < \beta$ in this case and let us estimate the probabilities

$$P\left\{\sup_{t\in\mathbb{R}^+}|c(t)X(t)|>x\right\}.$$

Will choose the points t_k in this way:
$t_k=e^{k+1}-e$, $k=0,1,\dots,$
$t_k-t_{k-1}=e^k(e-1)>1$, $k=1,2,\dots.$

1) $c(t)>0$ is monotone decreasing function, therefore

$$\gamma_k=\max_{t\in[t_{k-1},t_k]}c(t)=c(t_{k-1})=\frac{1}{(\ln(e+e^k-e))^\gamma}=\frac{1}{k^\gamma},\quad k=1,2,\dots$$

Hence,

$$\sum_{k=1}^{\infty}\gamma_k=\sum_{k=1}^{\infty}c(t_{k-1})=\sum_{k=1}^{\infty}\frac{1}{k^\gamma}<\infty,$$

this means that condition 1) of the theorem 6.5 holds.

2)

$$\sum_{k=1}^{\infty}\gamma_k\ln(t_k-t_{k-1})=\sum_{k=1}^{\infty}\frac{\ln(e^k(e-1))}{k^\gamma}\leqslant\sum_{k=1}^{\infty}\frac{k+1}{k^\gamma}<\infty,$$

i.e. condition 2) of the theorem 6.5 holds.

Let us verify that condition 3) of the theorem 6.5 holds true.

Since $\sigma(v)\leqslant cv^\beta$, $v>0$, then $\sigma^{(-1)}(v)\geqslant\left(\frac{v}{c}\right)^{1/\beta}$. Whereas $\alpha<1$

$$\int_0^\varepsilon\left(1+\frac{1}{2\sigma^{(-1)}(v)}\right)^\alpha dv\leqslant\int_0^\varepsilon\left(1+\frac{c^{1/\beta}}{2v^{1/\beta}}\right)^\alpha dv\leqslant$$
$$\leqslant\int_0^\varepsilon\left(1+\frac{c^{\alpha/\beta}}{2^\alpha v^{\alpha/\beta}}\right)dv=\varepsilon+\frac{c^{\alpha/\beta}}{2^\alpha}\int_0^\varepsilon v^{-\frac{\alpha}{\beta}}dv<\infty,$$

for $\alpha<\beta$, we have that condition 3) holds also.

Let us estimate $\widetilde{\Phi}_1(p)$ from the theorem 6.5.

Since $t_{0k}=\sigma(\varepsilon_{0k})=\sigma\left(\frac{t_k-t_{k-1}}{2}\right)$, than for $v<t_{0k}$

$$\sigma^{(-1)}(v)\leqslant\sigma^{(-1)}(t_{0k})=\frac{t_k-t_{k-1}}{2},$$

namely

$$\frac{t_k-t_{k-1}}{2\sigma^{(-1)}(v)}\geqslant 1.$$

Hence,

$$\int\limits_0^{pt_{0k}} \left(1+\frac{t_k-t_{k-1}}{2\sigma^{(-1)}(v)}\right)^\alpha dv \leqslant \int\limits_0^{pt_{0k}} \left(\frac{t_k-t_{k-1}}{\sigma^{(-1)}(v)}\right)^\alpha dv =$$

$$=(t_k-t_{k-1})^\alpha \int\limits_0^{pt_{0k}} \left(\frac{1}{\sigma^{(-1)}(v)}\right)^\alpha dv \leqslant (t_k-t_{k-1})^\alpha \int\limits_0^{pt_{0k}} \left(\frac{c}{v}\right)^{\frac{\alpha}{\beta}} dv =$$

$$= c^{\frac{\alpha}{\beta}} (t_k-t_{k-1})^\alpha \frac{(pt_{0k})^{1-\frac{\alpha}{\beta}}}{1-\frac{\alpha}{\beta}}.$$

Therefore

$$\left(\frac{1}{pt_{0k}} \int\limits_0^{pt_{0k}} \left(1+\frac{t_k-t_{k-1}}{2\sigma^{(-1)}(v)}\right)^\alpha dv\right)^{1/\alpha} \leqslant$$

$$\leqslant \frac{c^{1/\beta}(t_k-t_{k-1})(pt_{0k})^{-1/\beta}}{\left(1-\frac{\alpha}{\beta}\right)^{1/\alpha}} = \frac{2}{p^{1/\beta}\left(1-\frac{\alpha}{\beta}\right)^{1/\alpha}}.$$

From the last inequality and (6.9) follows

$$\widetilde{\Phi}_1(p) \leqslant \exp\left\{\frac{1}{d}\sum_{k=1}^{\infty} \gamma_k z_{0k} \ln\left(\frac{2}{p^{1/\beta}\left(1-\frac{\alpha}{\beta}\right)^{1/\alpha}}\right)\right\} = \frac{2}{p^{1/\beta}\left(1-\frac{\alpha}{\beta}\right)^{1/\alpha}}.$$

From previous inequality for $\alpha \longrightarrow 0$, we will have

$$\widetilde{\Phi}_1(p) \leqslant \frac{2e^{1/\beta}}{p^{1/\beta}}.$$

So, from the theorem 6.5 for $x>0$, $0<p<1$ we have estimates

$$P\left\{\sup_{t\in\mathbb{R}^+} |c(t)X(t)| > x\right\} \leqslant 4\frac{e^{1/\beta}}{p^{1/\beta}} \exp\left\{-\frac{x(1-p)}{d\sqrt{2}}\right\}\left(1+\frac{\sqrt{2}x(1-p)}{d}\right)^{1/2}.$$

$z_{0k} = \max\left\{\sigma\left(\frac{t_k - t_{k-1}}{2}\right), \delta_{0k}\right\}$, and $\delta_{0k} = \sup\limits_{t \in B_k} \left(E|X(t)|^2\right)^{1/2} = \delta_0$, because the process is stationary and $\sigma\left(\frac{t_{k+1} - t_k}{2}\right) < c_\sigma$.

Suppose that $c_\sigma \leqslant \delta_0$. Then $z_{0k} = \delta_0$ and

$$d = \sum_{k=1}^{\infty} \gamma_k z_{0k} = \delta_0 \sum_{k=1}^{\infty} \frac{1}{k^\gamma}.$$

Put $p = \frac{d\sqrt{2}}{x}$. Then for $x > d\sqrt{2}$ we will obtain

$$P\left\{\sup_{t \in \mathbb{R}^+} |c(t)X(t)| > x\right\} \leqslant 4e^{1+\frac{1}{\beta}} \exp\left\{-\frac{x}{d\sqrt{2}}\right\} \left(\frac{\sqrt{2}x}{d}\right)^{\frac{1}{2}} \left(\frac{x}{d\sqrt{2}}\right)^{\frac{1}{\beta}},$$

or

$$P\left\{\sup_{t \in \mathbb{R}^+} |c(t)X(t)| > x\right\} \leqslant \exp\left\{-\frac{x}{d\sqrt{2}}\right\} \left(\frac{x}{d\sqrt{2}}\right)^{\frac{1}{2}+\frac{1}{\beta}} C(\beta),$$

where $C(\beta) = 4\sqrt{2}e^{1+\frac{1}{\beta}}$.

Lemma 6.3. *Let $X = \{X(t), t \geqslant 0\}$ be a stationary separable square Gaussian stochastic process.*

If the following conditions hold

1) $\sum_{k=1}^{\infty} \gamma_k < \infty$,

2) $\sum_{k=1}^{\infty} \gamma_k \ln(t_k - t_{k-1}) < \infty$,

3) for some $\alpha \geqslant 1$ and for any $\varepsilon > 0$

$$\int_0^\varepsilon \left(\ln\left(1 + \frac{1}{2\sigma^{(-1)}(u)}\right)\right)^\alpha du < \infty,$$

then for $0 < p < 1$ and $0 < u < \frac{1-p}{d\sqrt{2}}$ inequality

$$E \exp\left\{u \sup_{t \in \mathbf{R}^+} |c(t)X(t)|\right\} \leqslant 2R\left(\frac{ud\sqrt{2}}{1-p}\right) \times$$

$$\exp\left\{\frac{2^{1-\frac{1}{\alpha}}}{d} \sum_{k=1}^{\infty} \gamma_k z_{0k} \left[\ln(t_k - t_{k-1})\right)+\right.$$

$$+\left(\frac{1}{pt_{0k}}\int\limits_0^{pt_{0k}}\left(\ln\left(1+\frac{1}{2\sigma^{(-1)}(v)}\right)\right)^{\alpha}dv\right)^{\frac{1}{\alpha}}\Bigg]\Bigg\}\leqslant$$

$$\leqslant 2R\left(\frac{ud\sqrt{2}}{1-p}\right)\exp\left\{\frac{2^{1-\frac{1}{\alpha}}}{d}\left[\sum_{k=1}^{\infty}\gamma_k z_{0k}\left(\ln(t_k-t_{k-1})+\widetilde{R}(p)^{\frac{1}{\alpha}}\right)\right]\right\}$$

holds true, where
$d=\sum_{k=1}^{\infty}\gamma_k z_{0k}$,
$\widetilde{R}(p)=\max\limits_k\left\{\left(\frac{1}{pt_{0k}}\int_0^{pt_{0k}}\left(\ln\left(1+\frac{1}{2\sigma^{(-1)}(v)}\right)\right)^{\alpha}dv\right)\right\}.$

Proof. Consider $r(v)=(\ln v)^{\alpha}$, $\alpha\geqslant 1$, $v\geqslant 1$. Whereas

$$r(xy)=(\ln x+\ln y)^{\alpha}\leqslant 2^{\alpha-1}\left((\ln x)^{\alpha}+(\ln y)^{\alpha}\right),$$

we will obtaine

$$\frac{1}{pt_{0k}}\int\limits_0^{pt_{0k}}r(N_k(\sigma^{(-1)}(v)))dv\leqslant\frac{1}{pt_{0k}}\int\limits_0^{pt_{0k}}r\left(1+\frac{t_{k+1}-t_k}{2\sigma^{(-1)}(v)}\right)dv\leqslant$$

$$\leqslant\frac{1}{pt_{0k}}\int\limits_0^{pt_{0k}}r\left((t_{k+1}-t_k)\left(1+\frac{1}{2\sigma^{(-1)}(v)}\right)\right)dv\leqslant$$

$$\leqslant\frac{2^{\alpha-1}}{pt_{0k}}\int\limits_0^{pt_{0k}}\left[(\ln(t_{k+1}-t_k))^{\alpha}+\left(\ln\left(1+\frac{1}{2\sigma^{(-1)}(v)}\right)\right)^{\alpha}\right]dv=$$

$$=2^{\alpha-1}\left[(\ln(t_{k+1}-t_k))^{\alpha}+\frac{1}{pt_{0k}}\int\limits_0^{pt_{0k}}\left(\ln\left(1+\frac{1}{2\sigma^{(-1)}(v)}\right)\right)^{\alpha}dv\right].$$

since for our case $\ln r^{(-1)}(z)=z^{\frac{1}{\alpha}}$, then

$$\prod_{k=1}^{\infty}\left(r^{(-1)}\left(\frac{1}{pt_0k}\int\limits_0^{pt_{0k}}r(N_k(\sigma^{(-1)}(v)))dv\right)\right)^{\frac{\gamma_k z_{0k}}{d}}=$$

$$=\exp\left\{\frac{1}{d}\sum_{k=1}^{\infty}\gamma_k z_{0k}\left(\frac{1}{pt_{0k}}\int\limits_0^{pt_{0k}}r(N_k(\sigma^{(-1)}(v)))dv\right)^{\frac{1}{\alpha}}\right\}\leqslant$$

$$\leqslant\exp\left\{\frac{2^{1-\frac{1}{\alpha}}}{d}\sum_{k=1}^{\infty}\gamma_k z_{0k}\left[(\ln(t_k-t_{k-1}))^{\alpha}+\right.\right.$$

$$+\frac{1}{pt_{0k}}\int_0^{pt_{0k}}\left(\ln\left(1+\frac{1}{2\sigma^{(-1)}(v)}\right)\right)^{\alpha}dv\Bigg]^{\frac{1}{\alpha}}\Bigg\}\leqslant$$

$$\leqslant \exp\left\{\frac{2^{1-\frac{1}{\alpha}}}{d}\sum_{k=1}^{\infty}\gamma_k z_{0k}\left[\ln(t_k-t_{k-1})+\right.\right.$$

$$\left.\left.+\left(\frac{1}{pt_{0k}}\int_0^{pt_{0k}}\left(\ln\left(1+\frac{1}{2\sigma^{(-1)}(v)}\right)\right)^{\alpha}dv\right)^{\frac{1}{\alpha}}\right]\right\}.$$

Function

$$f(v)=\left(\ln\left(1+\frac{1}{2\sigma^{(-1)}(v)}\right)\right)^{\alpha}$$

decreases for $v>0$, therefore $z(u)=\frac{1}{u}\int_0^u f(v)dv$ also decreases for $u>0$.

Denote

$$\widetilde{R}(p)=\max_k\left\{\frac{1}{pt_{0k}}\int_0^{pt_{0k}}\left(\ln\left(1+\frac{1}{2\sigma^{(-1)}(v)}\right)\right)^{\alpha}dv\right\}<\infty.$$

Using (6.5) we obtain

$$E\exp\left\{u\sup_{t\in\mathbf{R}^+}|c(t)X(t)|\right\}\leqslant 2R\left(\frac{ud\sqrt{2}}{1-p}\right)\times$$

$$\times\exp\left\{\frac{2^{1-\frac{1}{\alpha}}}{d}\left[\sum_{k=1}^{\infty}\gamma_k z_{0k}\left(\ln(t_k-t_{k-1})+\left(\widetilde{R}(p)\right)^{\frac{1}{\alpha}}\right)\right]\right\}.$$

Theorem 6.6. *Let $X=\{X(t),t\geqslant 0\}$ be a stationary separable square Gaussian stochastic process.*

If the following conditions hold

1) $\sum_{k=1}^{\infty}\gamma_k<\infty$,

2) $\sum_{k=1}^{\infty}\gamma_k\ln(t_k-t_{k-1})<\infty$,

3) for some $\alpha\geqslant 1$ and any $\varepsilon>0$

$$\int_0^{\varepsilon}\left(\ln\left(1+\frac{1}{2\sigma^{(-1)}(u)}\right)\right)^{\alpha}du<\infty,$$

then for arbitrary $x > 0$, $0 < p < 1$ inequality

$$P\left\{\sup_{t\in\mathbf{R}^+}|c(t)X(t)| > x\right\} \leqslant 2\exp\left\{-\frac{x(1-p)}{d\sqrt{2}}\right\}\left(1+\frac{\sqrt{2}x(1-p)}{d}\right)^{1/2}\times$$

$$\times\widetilde{\Phi}_3(p) \leqslant 2\exp\left\{-\frac{x(1-p)}{d\sqrt{2}}\right\}\left(1+\frac{\sqrt{2}x(1-p)}{d}\right)^{1/2}\widetilde{\Phi}_4(p)$$

holds true where

$$d = \sum_{k=1}^{\infty}\gamma_k z_{0k},$$

$$\widetilde{\Phi}_3(p) = \exp\left\{\frac{2^{1-\frac{1}{\alpha}}}{d}\sum_{k=1}^{\infty}\gamma_k z_{0k}\left[\ln(t_k - t_{k-1}) + \right.\right.$$

$$\left.\left. + \left(\frac{1}{pt_{0k}}\int_0^{pt_{0k}}\left(\ln\left(1+\frac{1}{2\sigma^{(-1)}(v)}\right)\right)^{\alpha}dv\right)^{\frac{1}{\alpha}}\right]\right\},$$

$$\widetilde{\Phi}_4(p) = \exp\left\{\frac{2^{1-\frac{1}{\alpha}}}{d}\left[\sum_{k=1}^{\infty}\gamma_k z_{0k}\left(\ln(t_k - t_{k-1}) + \left(\widetilde{R}(p)\right)^{\frac{1}{\alpha}}\right)\right]\right\},$$

$$\widetilde{R}(p) = \max_k\left\{\left(\frac{1}{pt_{0k}}\int_0^{pt_{0k}}\left(\ln\left(1+\frac{1}{2\sigma^{(-1)}(v)}\right)\right)^{\alpha}dv\right)\right\}.$$

Proof. The proof follows from lemma 6.3 and Chebyshev inequality. ♢

6.3. The estimates for correlogram deviation from correlation function of stationary Gaussian stochastic process in uniform metric

Assume that $\xi = \{\xi(t), t \geqslant 0\}$ is a real-valued continuous in mean square stationary Gaussian stochastic process with $E\xi(t) = 0$ and correlation function $\rho(\tau) = E\xi(t+\tau)\xi(t)$.

Consider correlogram

$$\widehat{\rho}_T(\tau) = \frac{1}{T}\int_0^T \xi(t+\tau)\xi(t)dt.$$

as an estimate of correlation function $\rho(\tau)$. $\widehat{\rho}_T(\tau)$ is unbiased estimate of $\rho(\tau)$: $E\widehat{\rho}_T(\tau) = \rho(\tau)$.

Let the process $\xi(t)$ have a square integrable spectral density $f = \{f(\lambda), \lambda \in$

$\mathbb{R}\}$ $(f \in L_2(R))$, that is

$$\int_{-\infty}^{+\infty} f^2(\lambda)d\lambda < \infty.$$

By the definition of spectral density, the function f is Lebesque integrable ($f \in L_1(R)$). In this case correlation function $\rho(\tau)$ of stochastic process is square integrable, namely $\|\rho\|_2^2 = \int_{-\infty}^{+\infty} \rho^2(\tau)d\tau < \infty$.

Let us denote $X(T,\tau) = \widehat{\rho}_T(\tau) - \rho(\tau)$. As before, $X(T,\tau)$ is square Gaussian stochastic process. Let us estimate $E(X(T,\tau))^2$ and $E(X(T,\tau) - X(T_1,\tau_1))^2$.

Assume that space $(\mathbf{T}, m)$ is defined as follows

$$\mathbf{T} = \{(T,\tau) : A < T < \infty, a < \tau < b, 0 < a < b, A > 0\},$$

$$m\left((T_1,\tau_1),(T_2,\tau_2)\right) = \max_{(T_1,\tau_1),(T_2,\tau_2)\in\mathbf{T}} \{|T_1 - T_2|, |\tau_1 - \tau_2|\}.$$

Lemma 6.4. *If the condition*

$$\int_{-\infty}^{+\infty} f^2(\lambda)d\lambda < \infty. \tag{6.11}$$

holds true, then

$$\sup_{(T,\tau)\in\boldsymbol{T}} E(X(T,\tau))^2 = \frac{C_1}{T},$$

where $C_1 = (1+\sqrt{2})\|\rho\|_2^2$.

Proof. By the Isserlis formula [19] and by the well-known formula

$$\int_0^T\int_0^T f(t-s)dtds = 2\int_0^T (T-u)f(u)du$$

for the even function f, we will get

$$E(X(T,\tau))^2 = E\widehat{\rho}^2(\tau) - (E\widehat{\rho}(\tau))^2 =$$

$$= \frac{2}{T^2}\int_0^T (T-u)(\rho^2(u) + \rho(u-\tau)\rho(u+\tau))du.$$

From the condition $\rho \in L_2(R)$ and fact that $\rho(\tau)$ is "even"follows, that

$$\int_0^{+\infty} \rho^2(u)du = \frac{1}{2}\int_{-\infty}^{+\infty} \rho^2(u)du < \infty.$$

Then ($\tau > 0$ for definiteness)

$$\int\limits_0^{+\infty} \rho(u-\tau)\rho(u+\tau)du \leqslant \left(\int\limits_0^{+\infty} \rho^2(u-\tau)du\right)^{\frac{1}{2}} \left(\int\limits_0^{+\infty} \rho^2(u+\tau)du\right)^{\frac{1}{2}} =$$

$$= \left(\int\limits_{-\tau}^{+\infty} \rho^2(v)dv\right)^{\frac{1}{2}} \left(\int\limits_{\tau}^{+\infty} \rho^2(v)dv\right)^{\frac{1}{2}} \leqslant \left(\int\limits_{-\infty}^{+\infty} \rho^2(v)dv\right)^{\frac{1}{2}} \left(\int\limits_{0}^{+\infty} \rho^2(v)dv\right)^{\frac{1}{2}} =$$

$$= \sqrt{2} \int\limits_0^{+\infty} \rho^2(v)dv < \infty,$$

and we will get

$$E(X(T,\tau))^2 \leqslant \frac{2+2\sqrt{2}}{T} \int\limits_0^{+\infty} \rho^2(u)du = \frac{(1+\sqrt{2})\|\rho\|_2^2}{T}.$$

Consider a partition of the space $\mathbf{T}$: $\mathbf{T} = \bigcup_{k=1}^{\infty} B_k$, where

$$B_k = \{(T,\tau) : T_k \leqslant T \leqslant T_{k+1}, \quad a \leqslant \tau \leqslant b\},$$

$T_k < T_{k+1}$, $T_{k+1} - T_k > 1$, $T_k \longrightarrow \infty$ if $k \longrightarrow \infty$.

Lemma 6.5. *Assume that condition*

$$\int\limits_{-\infty}^{+\infty} f^2(\lambda) \left(\ln(1+|\lambda|)\right)^{2\alpha} d\lambda < \infty. \tag{6.12}$$

holds for $\alpha > 0$. Then

$$\sup_{(T,\tau),(T_1,\tau_1)\in B_k, m((T,\tau),(T_1,\tau_1))<h} \left(E\left(X(T,\tau) - X(T_1,\tau_1)\right)^2\right)^{\frac{1}{2}} \leqslant \sigma_k(h),$$

where

$\sigma_k(h) = \frac{C_2^{\frac{1}{2}}}{T_k^{\frac{1}{2}}\left(\ln\left(e^\alpha + \frac{C}{h}\right)\right)^{\frac{\alpha}{2}}}$, $C > 0$ *is an arbitrary constant,*

$\widetilde{f} = \int_{-\infty}^{+\infty} f^2(\lambda) \left(\ln\left(e^\alpha + \frac{C|\lambda|}{2}\right)\right)^{2\alpha} d\lambda,$

$$C_2 = 8\pi \left[\widetilde{f}\left(\ln\left(e^\alpha + \frac{C}{b-a}\right)\right)^{-\alpha} + \|f\|_2 \widetilde{f}^{\frac{1}{2}}\right] + 2\|\rho\|_2^2 \times$$

$$\times \left(1 + \frac{6T_{k+1}}{T_k} + \frac{T_{k+1}^2}{T_k^2}\right) \frac{T_{k+1} - T_k}{T_k} \left(\ln\left(e^\alpha + \frac{1}{T_{k+1} - T_k}\right)\right)^{\alpha}. \tag{6.13}$$

Proof. Consider (T,τ) and (T',τ') with B_k and suppose that $T \leqslant T'$.

$$E\left(X(T,\tau)-X(T',\tau')\right)^2 =$$

$$= \Bigg| \frac{1}{T^2}\int\limits_0^T\int\limits_0^T \left[\rho^2(t-s)+\rho(t-s-\tau)\rho(t-s+\tau)\right] dtds$$

$$-\frac{2}{TT'}\int\limits_0^T\int\limits_0^{T'} \left[\rho(t-s+\tau-\tau')\rho(t-s)+\rho(t-s+\tau)\rho(t-s-\tau')\right] dtds \leqslant$$

$$+\frac{1}{T'^2}\int\limits_0^{T'}\int\limits_0^{T'} \left[\rho^2(t-s)+\rho(t-s-\tau')\rho(t-s+\tau')\right] dtds\Bigg|$$

$$\leqslant \Bigg| \frac{1}{T^2}\int\limits_0^T\int\limits_0^T \left[\rho^2(t-s)+\rho(t-s+\tau)\rho(t-s-\tau)\right] dtds-$$

$$-\frac{2}{T^2}\int\limits_0^T\int\limits_0^T \left[\rho(t-s)\rho(t-s+\tau-\tau')+\rho(t-s+\tau)\rho(t-s-\tau')\right] dtds+$$

$$+\frac{1}{T^2}\int\limits_0^T\int\limits_0^T \left[\rho^2(t-s)+\rho(t-s+\tau')\rho(t-s-\tau')\right] dtds\Bigg|$$

$$+\Bigg| \frac{2}{T^2}\int\limits_0^T\int\limits_0^T \left[\rho(t-s)\rho(t-s+\tau-\tau')+\rho(t-s+\tau)\rho(t-s-\tau')\right] dtds+$$

$$-\frac{2}{TT'}\int\limits_0^T\int\limits_0^{T'} \left[\rho(t-s)\rho(t-s+\tau-\tau')+\rho(t-s+\tau)\rho(t-s-\tau')\right] dtds\Bigg|+$$

$$+\Bigg| \frac{1}{T'^2}\int\limits_0^{T'}\int\limits_0^{T'} \left[\rho^2(t-s)+\rho(t-s+\tau')\rho(t-s-\tau')\right] dtds-$$

$$-\frac{1}{T^2}\int\limits_0^T\int\limits_0^T \left[\rho^2(t-s)+\rho(t-s+\tau')\rho(t-s-\tau')\right] dtds\Bigg| = I + A_1 + A_2.$$

Let us estimate A_1, A_2 та I.

$$A_1 = \Bigg| \frac{2}{T^2}\int\limits_0^T\int\limits_0^T \left[\rho(t-s)\rho(t-s+\tau-\tau')+\rho(t-s+\tau)\rho(t-s-\tau')\right] dtds-$$

$$-\frac{2}{TT'}\int_0^T\int_0^{T'}\left[\rho(t-s)\rho(t-s+\tau-\tau')+\rho(t-s+\tau)\rho(t-s-\tau')\right]dtds\Bigg| =$$

$$=\Bigg|\frac{2}{T^2}\int_0^T\int_0^T\left[\rho(t-s)\rho(t-s+\tau-\tau')+\rho(t-s+\tau)\rho(t-s-\tau')\right]dtds-$$

$$-\frac{2}{T^2}\int_0^T\int_0^{T'}\left[\rho(t-s)\rho(t-s+\tau-\tau')+\rho(t-s+\tau)\rho(t-s-\tau')\right]dtds+$$

$$+\frac{2}{T^2}\int_0^T\int_0^{T'}\left[\rho(t-s)\rho(t-s+\tau-\tau')+\rho(t-s+\tau)\rho(t-s-\tau')\right]dtds-$$

$$-\frac{2}{TT'}\int_0^T\int_0^{T'}\left[\rho(t-s)\rho(t-s+\tau-\tau')+\rho(t-s+\tau)\rho(t-s-\tau')\right]dtds\Bigg| \leqslant$$

$$\leqslant\left|\frac{2}{T^2}\int_T^{T'}\int_0^T\left[\rho(t-s)\rho(t-s+\tau-\tau')+\rho(t-s+\tau)\rho(t-s-\tau')\right]dtds\right|+$$

$$+\Bigg|\left(\frac{2}{T^2}-\frac{2}{TT'}\right)\int_0^T\int_0^{T'}\left[\rho(t-s)\rho(t-s+\tau-\tau')+\right.$$

$$\left.+\rho(t-s+\tau)\rho(t-s-\tau')\right]dtds\Bigg| \leqslant$$

$$\leqslant\frac{2}{T^2}\Bigg|\int_T^{T'}\left(\int_0^T\rho^2(t-s)dt\int_0^T\rho^2(t-s+\tau-\tau')dt\right)^{\frac{1}{2}}ds+$$

$$+\int_T^{T'}\left(\int_0^T\rho^2(t-s+\tau)dt\int_0^T\rho^2(t-s-\tau')dt\right)^{\frac{1}{2}}ds\Bigg|+$$

$$+\Bigg|\left(\frac{2}{T^2}-\frac{2}{TT'}\right)\Bigg[\int_0^T\left(\int_0^{T'}\rho^2(t-s)dt\int_0^{T'}\rho^2(t-s+\tau-\tau')dt\right)^{\frac{1}{2}}ds+$$

$$+\int_0^T\left(\int_0^{T'}\rho^2(t-s+\tau)dt\int_0^{T'}\rho^2(t-s-\tau')dt\right)^{\frac{1}{2}}ds\Bigg]\Bigg| \leqslant$$

$$\leqslant\frac{4}{T^2}|T'-T|\int_{-\infty}^{+\infty}\rho^2(u)du+\left|\frac{2}{T^2}-\frac{2}{TT'}\right|2T\int_{-\infty}^{+\infty}\rho^2(u)du=$$

$$= \frac{4(T'+T)|T'-T|}{T^2T'}\|\rho\|_2^2 \leqslant 4\|\rho\|_2^2 \frac{2T_{k+1}}{T_k}\frac{|T'-T|}{T^2}.$$

In a similar way, we obtain

$$A_2 = \left| \frac{1}{T^2}\int_0^T\int_0^T \left[\rho^2(t-s) + \rho(t-s+\tau')\rho(t-s-\tau')\right] dtds - \right.$$

$$\left. - \frac{1}{T'^2}\int_0^{T'}\int_0^{T'} \left[\rho^2(t-s) + \rho(t-s+\tau')\rho(t-s-\tau')\right] dtds \right| \leqslant$$

$$\leqslant \left| \left(\frac{1}{T^2} - \frac{1}{T'^2}\right)\int_0^T\int_0^T \left[\rho^2(t-s) + \rho(t-s+\tau')\rho(t-s-\tau')\right] dtds \right| +$$

$$+ \frac{1}{T'^2}\left| \int_0^T\int_0^T \left[\rho^2(t-s) + \rho(t-s+\tau')\rho(t-s-\tau')\right] dtds - \right.$$

$$\left. - \int_0^{T'}\int_0^{T'} \left[\rho^2(t-s) + \rho(t-s+\tau')\rho(t-s-\tau')\right] dtds \right| \leqslant$$

$$\leqslant 2\|\rho\|_2^2 \frac{|T'-T|}{T^2}\left(1 + \frac{T'+T}{T'} + \frac{T^2}{T'^2}\right) \leqslant 2\|\rho\|_2^2 \frac{|T'-T|}{T^2} \times$$

$$\times \left(1 + \frac{2T_{k+1}}{T_k} + \frac{T_{k+1}^2}{T_k^2}\right).$$

We use the following lemma, proven in [19].

Lemma A. Let $Y_T(\tau) = \sqrt{T}(\widehat{\rho}_T(\tau) - \rho(\tau))$, $\tau \geqslant 0$. For any $T > 0$ and $\tau, \tau_1 \geqslant 0$ the inequality

$$E|Y_T(\tau) - Y_T(\tau_1)|^2 \leqslant 8\pi \left[\int_{-\infty}^{+\infty} f^2(\lambda)\sin^2\frac{\lambda(\tau-\tau_1)}{2}d\lambda\right] +$$

$$+8\pi\|f\|_2 \left[\int_{-\infty}^{+\infty} f^2(\lambda)\sin^2(\frac{\lambda(\tau-\tau_1)}{2})d\lambda\right]^{\frac{1}{2}}$$

holds, where $\|f\|_2^2 = \int_{-\infty}^{+\infty} f^2(\lambda)d\lambda < \infty$.

For our case from lemma [19], we will get

$$E(X(T,\tau) - X(T,\tau'))^2 \leqslant \frac{8\pi}{T}\left(\left[\int_{-\infty}^{+\infty} f^2(\lambda)\sin^2\frac{\lambda(\tau-\tau')}{2}d\lambda\right] + \right.$$

$$+\|f\|_2 \left[\int\limits_{-\infty}^{+\infty} f^2(\lambda) \sin^2 \frac{\lambda(\tau-\tau')}{2} d\lambda \right]^{\frac{1}{2}} \Bigg). \tag{6.14}$$

Since the inequality [73]

$$\left| \sin \frac{u}{v} \right| \leqslant \left(\frac{\ln(e^\alpha + u)}{\ln(e^\alpha + v)} \right)^\alpha$$

holds for all $u \geqslant 0$, $v > 0$ and $\alpha > 0$, then

$$\left| \sin \frac{\lambda(\tau-\tau')}{2} \right| \leqslant \frac{\left(\ln\left(e^\alpha + \frac{C|\lambda|}{2} \right) \right)^\alpha}{\left(\ln\left(e^\alpha + \frac{C}{|\tau-\tau'|} \right) \right)^\alpha},$$

where $C > 0$ is an arbitrary constant, and (6.14) can be rewritten as

$$I \leqslant \frac{8\pi}{T} \left[\int\limits_{-\infty}^{+\infty} f^2(\lambda) \left(\ln\left(e^\alpha + \frac{C|\lambda|}{2} \right) \right)^{2\alpha} d\lambda \frac{1}{\left(\ln\left(e^\alpha + \frac{C}{|\tau-\tau'|} \right) \right)^{2\alpha}} + \right.$$

$$\left. +\|f\|_2 \left(\int\limits_{-\infty}^{+\infty} f^2(\lambda) \left(\ln\left(e^\alpha + \frac{C|\lambda|}{2} \right) \right)^{2\alpha} d\lambda \right)^{\frac{1}{2}} \frac{1}{\left(\ln\left(e^\alpha + \frac{C}{|\tau-\tau'|} \right) \right)^{\alpha}} \right].$$

From the inequality (6.12) follows that

$$\widetilde{f} = \int\limits_{-\infty}^{+\infty} f^2(\lambda) \left(\ln\left(e^\alpha + \frac{C|\lambda|}{2} \right) \right)^{2\alpha} d\lambda < \infty.$$

Then

$$I \leqslant \frac{\widetilde{C_2}}{T \left(\ln\left(e^\alpha + \frac{C}{|\tau-\tau'|} \right) \right)^{\alpha}},$$

where

$$\widetilde{C_2} = 8\pi \left[\widetilde{f} \left(\ln\left(e^\alpha + \frac{C}{b-a} \right) \right)^{-\alpha} + \|f\|_2 \widetilde{f}^{\frac{1}{2}} \right].$$

Hence,

$$E(X(T,\tau) - X(T',\tau'))^2 \leqslant \frac{\widetilde{C_2}}{T \left(\ln\left(e^\alpha + \frac{C}{|\tau-\tau'|} \right) \right)^{\alpha}} + \frac{C^*|T'-T|}{T^2},$$

$C^* = 2\|\rho\|_2^2 \left(1 + \frac{6T_{k+1}}{T_k} + \frac{T_{k+1}^2}{T_k^2}\right).$
Whereas

$$\sup_{h<T_{k+1}-T_k} h\left(\ln\left(e^\alpha + \frac{1}{h}\right)\right)^\alpha = (T_{k+1}-T_k)\left(\ln\left(e^\alpha + \frac{1}{T_{k+1}-T_k}\right)\right)^\alpha,$$

then

$$\sup_{(T,\tau),(T',\tau')\in B_k, m((T,\tau),(T',\tau'))<h} \left(E\left(X(T,\tau)-X(T',\tau')\right)^2\right)^{\frac{1}{2}} \leqslant \sigma_k(h),$$

where

$$\sigma_k(h) = \frac{C_2^{\frac{1}{2}}}{T_k^{\frac{1}{2}}\left(\ln\left(e^\alpha + \frac{C}{h}\right)\right)^{\frac{\alpha}{2}}},$$

$C_2 = \widetilde{C_2} + C^* \frac{T_{k+1}-T_k}{T_k}\left(\ln\left(e^\alpha + \frac{1}{T_{k+1}-T_k}\right)\right)^\alpha.$ $\diamondsuit$

As before, we denote

- $\varepsilon_{0k} = \inf_{(T_1,\tau_1)\in B_k} \sup_{(T_2,\tau_2)\in B_k} m\left((T_1,\tau_1),(T_2,\tau_2)\right) = \max\left\{\frac{T_{k+1}-T_k}{2}, \frac{b-a}{2}\right\},$
- $\delta_{0k} = \sup_{(T,\tau)\in B_k} \left(E(X(T,\tau))^2\right)^{\frac{1}{2}} = \frac{C_1^{\frac{1}{2}}}{T_k^{\frac{1}{2}}},$
- $\sigma_k^{(-1)}(v)$ - inverse to $\sigma_k(h)$ function,

$$\sigma_k^{(-1)}(v) = \frac{C}{\exp\left\{\left(\frac{C_2}{v^2 T_k}\right)^{\frac{1}{\alpha}}\right\} - \exp\{\alpha\}},$$

- $t_{0k} = \sigma_k(\varepsilon_{0k}),$
- $N_k(\varepsilon)$-the smallest number of closed balls of radius ε, which cover B_k,
- $r(u) > 0$, $u \geqslant 1$ is monotone increasing function, $r(u) \longrightarrow \infty$ for $u \longrightarrow \infty$, such, that function $r(e^t)$ is convex for $t \geqslant 0$.

Since C is an arbitrary constant, then let us choose for simplicity $C = \sqrt{b-a}$, and put $T_{k+1} - T_k > b - a$ for our partition.

Lemma 6.6. *Assume that* $X(T,\tau) = \widehat{\rho}_T(\tau) - \rho(\tau)$, $c = \{c(T), T \in [A; +\infty)\}$ *is some continuous function with* $0 < c(T) < 1$. *Let us denote*
$\gamma_k = \max_{T\in[T_k;T_{k+1}]} c(T)$.

If conditions

1) $\sum_{k=1}^{\infty} \gamma_k z_{0k} \ln(T_{k+1} - T_k) < \infty$,

2) $\int_{-\infty}^{+\infty} f^2(\lambda) \left(\ln(1+|\lambda|)\right)^{2\alpha} d\lambda < \infty$, $\alpha > 2$

hold, then for $0 < p < 1$ *and* $0 < u < \frac{1-p}{d\sqrt{2}}$ *the next inequality*

$$E \exp\left\{ u \sup_{(T,\tau)\in \mathbf{T}} |c(T)X(T,\tau)| \right\} \leqslant 2R\left(\frac{ud\sqrt{2}}{1-p}\right) \times$$

$$\times \exp\left\{ \frac{1}{d} \sum_{k=1}^{\infty} \gamma_k z_{0k} \ln(T_{k+1} - T_k) + \frac{2\widetilde{P}}{p^{\frac{2}{\alpha}}\left(1 - \frac{2}{\alpha}\right)} \right\}, \tag{6.15}$$

holds true, where
$d = \sum_{k=1}^{\infty} \gamma_k z_{0k}$,
$\widetilde{P} = \sup_k \left(\ln \left(e^{\alpha} + \frac{2\sqrt{b-a}}{T_{k+1}-T_k} \right) \right)$.

Proof. Let us put $r(v) = (\ln v)^f$, $v > e$, $1 < f < \frac{\alpha}{2}$. For our case

$$r(xy) = (\ln x + \ln y)^f \leqslant 2^{f-1}\left((\ln x)^f + (\ln y)^f\right)$$

and

$$N_k(\sigma_k^{(-1)}(v)) \leqslant \left(\frac{T_{k+1} - T_k}{2\sigma_k^{(-1)}(v)} + 1\right)\left(\frac{b-a}{2\sigma_k^{(-1)}(v)} + 1\right) \leqslant$$
$$\leqslant \frac{(T_{k+1} - T_k)(b-a)}{\left(\sigma_k^{(-1)}(v)\right)^2},$$

for $v < t_{0k}$. Then

$$\frac{1}{pt_{0k}} \int_0^{pt_{0k}} r(N_k(\sigma_k^{(-1)}(v)))dv \leqslant \frac{1}{pt_{0k}} \int_0^{pt_{0k}} r\left(\frac{(b-a)(T_{k+1} - T_k)}{\left(\sigma_k^{(-1)}(v)\right)^2}\right) dv \leqslant$$

$$\leqslant \frac{1}{pt_{0k}} \int_0^{pt_{0k}} r\left(\frac{(b-a)(T_{k+1} - T_k)}{C^2} \exp\left\{2\left(\frac{C_2}{v^2 T_k}\right)^{\frac{1}{\alpha}}\right\}\right) dv =$$

$$= \frac{1}{pt_{0k}} \int_0^{pt_{0k}} \left(\ln\frac{(b-a)(T_{k+1} - T_k)}{C^2} + 2\left(\frac{C_2}{v^2 T_k}\right)^{\frac{1}{\alpha}}\right)^f dv \leqslant$$

$$\leqslant 2^{f-1}\left[\left(\ln\frac{(b-a)(T_{k+1} - T_k)}{C^2}\right)^f + \frac{2^f}{pt_{0k}}\left(\frac{C_2}{T_k}\right)^{\frac{f}{\alpha}} \int_0^{pt_0 k} v^{-\frac{2f}{\alpha}} dv\right] =$$

$$= 2^{f-1}\left[\left(\ln\frac{(b-a)(T_{k+1}-T_k)}{C^2}\right)^f + \frac{2^f C_2^{\frac{f}{\alpha}}}{T_k^{\frac{f}{\alpha}} pt_{0k}} \frac{(pt_{0k})^{1-\frac{2f}{\alpha}}}{1-\frac{2f}{\alpha}}\right] =$$

$$= 2^{f-1}\left[\left(\ln\frac{(b-a)(T_{k+1}-T_k)}{C^2}\right)^f + \frac{2^f C_2^{\frac{f}{\alpha}}}{T_k^{\frac{f}{\alpha}}} \frac{(pt_{0k})^{-\frac{2f}{\alpha}}}{1-\frac{2f}{\alpha}}\right].$$

Since

$$t_{0k} = \sigma_k(\varepsilon_{0k}) = \frac{C_2^{\frac{1}{2}}}{T_k^{\frac{1}{2}}\left(\ln\left(e^\alpha + \frac{2C}{T_{k+1}-T_k}\right)\right)^{\frac{\alpha}{2}}},$$

then

$$\frac{1}{pt_{0k}}\int\limits_0^{pt_{0k}} r(N_k(\sigma_k^{(-1)}(v)))dv \leqslant$$

$$\leqslant 2^{f-1}\left[(\ln(T_{k+1}-T_k))^f + \frac{2^f C_2^{\frac{f}{\alpha}}}{T_k^{\frac{f}{\alpha}} p^{\frac{2f}{\alpha}}\left(1-\frac{2f}{\alpha}\right)} \frac{T_k^{\frac{f}{\alpha}}\left(\ln\left(e^\alpha + \frac{2\sqrt{b-a}}{T_{k+1}-T_k}\right)\right)^f}{C_2^{\frac{f}{\alpha}}}\right] =$$

$$= 2^{f-1}\left[(\ln(T_{k+1}-T_k))^f + \frac{2^f}{p^{\frac{2f}{\alpha}}\left(1-\frac{2f}{\alpha}\right)}\left(\ln\left(e^\alpha + \frac{2\sqrt{b-a}}{T_{k+1}-T_k}\right)\right)^{\frac{f}{\alpha}}\right],$$

$1 < f < \frac{\alpha}{2}$.

In our case $\ln r^{(-1)}(z) = z^{\frac{1}{f}}$, therefore, if f converge to one, we will get

$$\prod_{k=1}^{\infty}\left(r^{(-1)}\left(\frac{1}{pt_{0k}}\int\limits_0^{pt_{0k}} r(N_k(\sigma_k^{(-1)}(v)))dv\right)\right)^{\frac{\gamma_k z_{0k}}{d}} =$$

$$= \exp\left\{\sum_{k=1}^{\infty}\frac{\gamma_k z_{0k}}{d}\left(\frac{1}{pt_{0k}}\int\limits_0^{pt_{0k}} r(N_k(\sigma_k^{(-1)}(v)))dv\right)^{\frac{1}{f}}\right\} \leqslant$$

$$\leqslant \exp\left\{2^{1-\frac{1}{f}}\sum_{k=1}^{\infty}\frac{\gamma_k z_{0k}}{d}\left[\ln(T_{k+1}-T_k) + \right.\right.$$

$$\left.\left. + \frac{2}{p^{\frac{2}{\alpha}}\left(1-\frac{2f}{\alpha}\right)^{\frac{1}{f}}}\left(\ln\left(e^\alpha + \frac{2\sqrt{b-a}}{T_{k+1}-T_k}\right)\right)\right]\right\} \leqslant$$

$$\leqslant \exp\left\{\frac{1}{d}\sum_{k=1}^{\infty}\gamma_k z_{0k}\ln(T_{k+1}-T_k) + \frac{2\widetilde{P}}{p^{\frac{2}{\alpha}}\left(1-\frac{2}{\alpha}\right)}\right\}, \tag{6.16}$$

where $\widetilde{P} = \sup_k \left(\ln \left(e^{\alpha} + \frac{2\sqrt{b-a}}{T_{k+1}-T_k} \right) \right)$.

The proof of the theorem follows from (6.5) and the last inequality. $\diamondsuit$

Theorem 6.7. *Assume that* $X(T,\tau) = \widehat{\rho}_T(\tau) - \rho(\tau)$, $c = \{c(T), T \in [A;+\infty)\}$ *is some continuous function and* $0 < c(T) < 1$.
Let us denote $\gamma_k = \max_{T\in[T_k;T_{k+1}]} c(T)$.

If the next conditions hold

1) $\sum_{k=1}^{\infty} \gamma_k z_{0k} \ln(T_{k+1} - T_k) < \infty$,

2) $\int_{-\infty}^{+\infty} f^2(\lambda) \left(\ln(1 + |\lambda|)\right)^{2\alpha} d\lambda < \infty$, $\alpha > 2$,

than for arbitrary $x > d\sqrt{2}$ *inequality*

$$P\left\{ \sup_{(T,\tau)\in \mathbf{T}} |c(T)X(T,\tau)| > x \right\} \leqslant$$

$$\leqslant 2e \exp\left\{ -\frac{x}{d\sqrt{2}} + \frac{2\widetilde{P}}{(1-\frac{2}{\alpha})} \left(\frac{x}{d\sqrt{2}}\right)^{\frac{2}{\alpha}} \right\} \left(\frac{x\sqrt{2}}{d}\right)^{\frac{1}{2}} \widetilde{\Phi}_5,$$

holds, where
$d = \sum_{k=1}^{\infty} \gamma_k z_{0k}$,
$\widetilde{P} = \sup_k \left(\ln \left(e^{\alpha} + \frac{2\sqrt{b-a}}{T_{k+1}-T_k} \right) \right)$,
$\widetilde{\Phi}_5 = \exp\left\{ \frac{1}{d} \sum_{k=1}^{\infty} \gamma_k z_{0k} \ln(T_{k+1} - T_k) \right\}$.

Proof. We will easily get our inequality, if we put $p = \frac{d\sqrt{2}}{x}$ $(x > d\sqrt{2})$ in (6.16) and use Chebyshev inequality and theorem 6.4. Indeed,

$$P\left\{ \sup_{(T,\tau)\in \mathbf{T}} |c(T)X(T,\tau)| > x \right\} \leqslant 2\exp\left\{ -\frac{x(1-p)}{d\sqrt{2}} \right\} \times$$

$$\times \left(1 + \frac{x\sqrt{2}(1-p)}{d}\right)^{\frac{1}{2}} \exp\left\{ \frac{2\widetilde{P}}{p^{\frac{2}{\alpha}}\left(1-\frac{2}{\alpha}\right)} \right\} \widetilde{\Phi}_5 =$$

$$= 2\exp\left\{ -\frac{x}{d\sqrt{2}} \left(1 - \frac{d\sqrt{2}}{x}\right) \right\} \left(1 + \frac{x\sqrt{2}}{d}\left(1 - \frac{d\sqrt{2}}{x}\right)\right)^{\frac{1}{2}} \times$$

$$\times \exp\left\{ \frac{2\widetilde{P}}{(1-\frac{2}{\alpha})} \left(\frac{x}{d\sqrt{2}}\right)^{\frac{2}{\alpha}} \right\} \leqslant$$

$$\leqslant 2e\exp\left\{-\frac{x}{d\sqrt{2}}+\frac{2\widetilde{P}}{(1-\frac{2}{\alpha})}\left(\frac{x}{d\sqrt{2}}\right)^{\frac{2}{\alpha}}\right\}\left(\frac{x\sqrt{2}}{d}\right)^{\frac{1}{2}}\widetilde{\Phi}_5.$$

Theorem 6.8. *Assume that* $X(T,\tau)=\widehat{\rho}_T(\tau)-\rho(\tau)$ *and let* $c(T)=\frac{T^{\frac{1}{2}}}{(\ln T)^{\beta}}$ *be the function defined for all* $T>e^m$*, where* m *- some fixed number ,* $m>4$ *and* $2<\beta<\frac{m}{2}$*.*

If for some $\alpha>2$ *the next condition holds*

$$\int_{-\infty}^{+\infty} f^2(\lambda)\left(\ln(1+|\lambda|)\right)^{2\alpha}d\lambda<\infty,$$

then for arbitrary $x>d\sqrt{2}$ *inequality*

$$P\left\{\sup_{(T,\tau)\in\mathbf{T}}|c(T)X(T,\tau)|>x\right\}\leqslant$$

$$\leqslant 2e\exp\left\{-\frac{x}{d\sqrt{2}}+D_\alpha\left(\frac{x}{d\sqrt{2}}\right)^{\frac{2}{\alpha}}\right\}\left(\frac{x\sqrt{2}}{d}\right)^{\frac{1}{2}}D$$

holds, where
$d=C_0e^{\frac{1}{2}}\Sigma_{k=1}^{\infty}\frac{1}{(m+k+1)^{\beta}}$,
C_0 *is known constant, which is determined through* C_1 *and* $\overline{C}_2$ *:*
$C_1=(1+\sqrt{2})\|\rho\|_2^2$,
$\overline{C}_2=8\pi\left[\widetilde{f}\left(\ln\left(e^{\alpha}+\frac{C}{b-a}\right)\right)^{-\alpha}+\|f\|_2\widetilde{f}^{\frac{1}{2}}\right]+2\|\rho\|_2^2(1+6e+e^2)\times$
$\times(e-1)\left(\ln\left(e^{\alpha}+\frac{1}{e^m(e-1)}\right)\right)^{\alpha}$,

$$D_\alpha=\frac{2\left(\ln\left(e^{\alpha}+\frac{2\sqrt{b-a}}{e^m(e-1)}\right)\right)}{\left(1-\frac{2}{\alpha}\right)},\qquad D=\exp\left\{\frac{\sum_{k=1}^{\infty}\frac{1}{(m+k+1)^{\beta-1}}}{\sum_{k=1}^{\infty}\frac{1}{(m+k+1)^{\beta}}}\right\},$$

Proof. Theorem follows from the previous one. Function $c(T)>0$ is monotone increasing function with $\beta<\frac{\ln T}{2}$. Since $\beta>2$, then we choose for simplicity $A=e^m$, $m>4$. Let us verify if conditions of the theory 6.7 is done and let us find the estimations for distribution

$$P\left\{\sup_{(T,\tau)\in\mathbf{T}}|c(T)X(T,\tau)|>x\right\}.$$

Let us choose the points T_k of partition in the following way:
$T_k = e^{m+k}$, $k = 1, 2, \dots$. In this case $T_{k+1} - T_k = e^{m+k}(e-1) > 1$.

$$\gamma_k = c(T_{k+1}) = \frac{T_{k+1}^{\frac{1}{2}}}{(\ln T_{k+1})^\beta} = \frac{e^{\frac{m+k+1}{2}}}{(m+k+1)^\beta}, \quad k = 1, 2, \dots,$$

$$z_{0k} = \max\{\delta_{0k}, t_{0k}\} = \max\left\{ \frac{c_1^{\frac{1}{2}}}{T_k^{\frac{1}{2}}}, \frac{c_2^{\frac{1}{2}}}{T_k^{\frac{1}{2}} \left(\ln\left(e^\alpha + \frac{2\sqrt{b-a}}{T_{k+1}-T_k}\right)\right)^{\frac{\alpha}{2}}} \right\} = \frac{C_0}{e^{\frac{m+k}{2}}},$$

where

$$C_0 = \max\left\{ C_1^{\frac{1}{2}}, \frac{\overline{C}_2^{\frac{1}{2}}}{\left(\ln\left(e^\alpha + \frac{2\sqrt{b-a}}{e^m(e-1)}\right)\right)^{\frac{\alpha}{2}}} \right\},$$

$\overline{C}_2 = 8\pi \left[\widetilde{f} \left(\ln\left(e^\alpha + \frac{C}{b-a}\right)\right)^{-\alpha} + \|f\|_2 \widetilde{f}^{\frac{1}{2}} \right] + 2\|\rho\|_2^2 (1 + 6e + e^2) \times$
$\times (e-1) \left(\ln\left(e^\alpha + \frac{1}{e^m(e-1)}\right)\right)^\alpha$.

Thus,

$$d = \sum_{k=1}^\infty \gamma_k z_{0k} = C_0 e^{\frac{1}{2}} \sum_{k=1}^\infty \frac{1}{(m+k+1)^\beta} < \infty, \quad \text{при} \quad \beta > 1,$$

$$\sum_{k=1}^\infty \gamma_k z_{0k} \ln(T_{k+1} - T_k) \leqslant C_0 e^{\frac{1}{2}} \sum_{k=1}^\infty \frac{1}{(m+k+1)^{\beta-1}} < \infty, \quad \text{for} \quad \beta > 2,$$

i.e. condition 1) of the theorem 6.7 is done.

Let us estimate $\widetilde{\Phi}_5$ and $\widetilde{P}$.

$$\widetilde{\Phi}_5 = \exp\left\{ \frac{1}{d} \sum_{k=1}^\infty \gamma_k z_{0k} \ln(T_{k+1} - T_k) \right\} \leqslant \exp\left\{ \frac{\sum_{k=1}^\infty \frac{1}{(m+k+1)^{\beta-1}}}{\sum_{k=1}^\infty \frac{1}{(m+k+1)^\beta}} \right\},$$

$$\widetilde{P} = \max_{k \geqslant m} \left(\ln\left(e^\alpha + \frac{2\sqrt{b-a}}{e^k(e-1)}\right)\right) = \left(\ln\left(e^\alpha + \frac{2\sqrt{b-a}}{e^m(e-1)}\right)\right).$$

Therefore,

$$P\left\{ \sup_{(T,\tau)\in\mathbf{T}} |c(T)X(T,\tau)| > x \right\} \leqslant$$

$$\leqslant 2e \exp\left\{-\frac{x}{d\sqrt{2}} + D_\alpha \left(\frac{x}{d\sqrt{2}}\right)^{\frac{2}{\alpha}}\right\} \left(\frac{x\sqrt{2}}{d}\right)^{\frac{1}{2}} D,$$

where

$$D_\alpha = \frac{2\left(\ln\left(e^\alpha + \frac{2\sqrt{b-a}}{e^m(e-1)}\right)\right)}{\left(1-\frac{2}{\alpha}\right)}, \qquad D = \exp\left\{\frac{\sum_{k=1}^\infty \frac{1}{(m+k+1)^{\beta-1}}}{\sum_{k=1}^\infty \frac{1}{(m+k+1)^\beta}}\right\}.$$

Theorem 6.8 enable us to construct criterion for testing of hypothesis about correlation function of stochastic process.

6.4. Construction of the criterion for testing hypotheses about the covariance functions of stationary Gaussian stochastic process

Assume that$\xi = \{\xi(t), t \geqslant 0\}$ is a real-valued continuous in mean square stationary Gaussian stochastic process with spectral density $f(\lambda)$, $E\xi(t) = 0$ and correlation function $\rho(\tau) = E\xi(t+\tau)\xi(t)$, $a \leqslant \tau \leqslant b$.
As an estimate of $\rho(\tau)$ we consider $\widehat{\rho}_T(\tau) = \frac{1}{T}\int_0^T \xi(t+\tau)\xi(t)dt$ and we assume, that $T > e^m$ $(m > 4)$.

Let H be the hypothesis that for $a \leqslant \tau \leqslant b$ the correlation function of stochastic process $\xi(t)$ equals $\rho(\tau)$. To test the hypothesis H one can use the following criterion.

Criterion 6.1. For some level of confidence γ, $0 < \gamma < 1$, one can find such x_γ, that

$$A(x_\gamma) = 2e \exp\left\{-\frac{x_\gamma}{d\sqrt{2}} + D_\alpha \left(\frac{x_\gamma}{d\sqrt{2}}\right)^{\frac{2}{\alpha}}\right\} \left(\frac{x_\gamma\sqrt{2}}{d}\right)^{\frac{1}{2}} D = \gamma,$$

where $\alpha > 2$ is such, that $\int_{-\infty}^{+\infty} f^2(\lambda)(\ln(1+|\lambda|))^{2\alpha} < \infty$,
$d = C_0 e^{\frac{1}{2}} \Sigma_{k=1}^\infty \frac{1}{(m+k+1)^\beta}$,
C_0 is known constant, which is determined through C_1 and $\overline{C}_2$,

$$D_\alpha = \frac{2\left(\ln\left(e^\alpha + \frac{2\sqrt{b-a}}{e^m(e-1)}\right)\right)}{\left(1-\frac{2}{\alpha}\right)}, \qquad D = \exp\left\{\frac{\sum_{k=1}^\infty \frac{1}{(m+k+1)^{\beta-1}}}{\sum_{k=1}^\infty \frac{1}{(m+k+1)^\beta}}\right\},$$

$2 < \beta < \frac{m}{2}$.

The hypothesis H is accepted if for $T > e^m$

$$\sup_{a<\tau<b} \frac{T^{\frac{1}{2}}}{(\ln T)^{\beta}} |\widehat{\rho}_T(\tau) - \rho(\tau)| < x_\gamma$$

and hypothesis is rejected otherwise.

Remark 6.2. Note, that this criterion can be used for large enough T (for simplicity we consider $T > e^m$, where $m > 4$), and the probability of the first type's error does not exceed γ in this case. Using both, criterion 6.1 and criterion which was constructed earlier, enable us significantly reduce the probability of the second type's error.

Chapter 7
Estimation of correlation function of homogeneous and isotropic Gaussian random field.

In the previous chapters, the problem of estimation of correlation function of Gaussian stochastic process was considered. In this chapter we will consider the similar problem for Gaussian random field. Estimates of the correlation functions of random fields were considered in the works of Dychovychnyj A.A. [27], Rakhimov G.M. [116], Revenko A.O. [117]. In the work of Dychovychnyj, for example, a random field is considered on a ball and on a cube in $\mathbb{R}^n$.

In this chapter homogeneous and isotropic mean-square continuous Gaussian random field $\xi(x)$ defined in $\mathbb{R}^n$ with $E\xi(x) = 0$ is considered. The spherical correlogram of random field is chosen as estimator of correlation function. For this field the inequalities for distribution of spherical mean deviation from its correlation function in L_2-metric are obtained. Based on these inequalities the new criterion for testing of hypotheses about its correlation function is constructed. Random field is observed on the ball in $\mathbb{R}^n$.

7.1. The estimates for distribution of spherical mean deviation from its correlation function in L_2-metric

Assume that $\xi(x)$ is homogeneous in wide sense random field defined in $\mathbb{R}^n$ (suppose that $E\xi(x) = 0$). It means that $E|\xi(x)|^2 < +\infty$ and $E\xi(x)\overline{\xi(y)}$ depends only on the distance $|x - y|$ between x and y. This implies that $B(x, y) = E\xi(x)\overline{\xi(y)} = B(|x - y|)$.

Definition 7.1. [141] Let $SO(n)$ be a group of rotations $\mathbb{R}^n$ around the origin. A homogeneous random field $\xi(x)$ is called isotropic if $E\xi(x)\overline{\xi(y)} = E\xi(gx)\overline{\xi(gy)}$ for all $g \in SO(n)$.

Correlation function $B(x, y)$ of homogeneous and isotropic random field depends only on the distance between x and y and is known that

$$B(|x - y|) = \int_{\mathbb{R}^n} e^{i(\lambda, x-y)} F(d\lambda), \tag{7.1}$$

where $F(\cdot)$ is a finite measure on σ-algebra B_n Borel sets of $\mathbb{R}^n$.
Move to spherical coordinates in (7.1). We obtaine [141]

$$B(r) = 2^{\frac{n-2}{2}}\Gamma\left(\frac{n}{2}\right)\int\limits_0^{+\infty}\frac{J_{\frac{n-2}{2}}(\lambda r)}{(\lambda r)^{\frac{n-2}{2}}}d\Phi(\lambda),$$

where
$r = |x-y|$ is a distance between x and y, $\Phi(\lambda) = \int\limits_{\sqrt{v_1^2+\cdots+v_n^2}\leqslant\lambda} F(dv)$,
therefore $\Phi(\lambda)$ is nondecreasing function on $[0,+\infty)$ and $\int_0^{+\infty} d\Phi(\lambda) = F(\mathbb{R}^n) < +\infty$.

Consider spherical Bessel function

$$Y_n(z) = 2^{\frac{n-2}{2}}\Gamma\left(\frac{n}{2}\right)\frac{J_{\frac{n-2}{2}}(z)}{z^{\frac{n-2}{2}}}. \tag{7.2}$$

Then (7.1) can be written as

$$B(r) = \int\limits_0^{+\infty} Y_n(\lambda r)d\Phi(\lambda). \tag{7.3}$$

In this section we deal with homogeneous and isotropic mean-square continuous Gaussian random field $\xi(x)$ defined in $\mathbb{R}^n$ with $E\xi(x) = 0$. Assume that **sample paths of the field are continuous with probability 1 on any bounded and closed area**. The necessary and sufficient conditions of this fact are considered in [141].

In the theorem 7.1 we will give sufficient conditions which are close to the necessary conditions.

Theorem 7.1. *[141] Suppose that for some $\varepsilon > 0$ condition*

$$\int\limits_0^{\infty}\ln^{1+\varepsilon}(1+\lambda)d\Phi(\lambda) < +\infty$$

holds. Then the random field $\xi(x)$ is continuous with probability one on any bounded and closed area.

We denote by $S_R(x)$ and $V_R(x)$ sphere and ball of radius R centered at a point x respectively.Let $m_n^{(R)}(\cdot)$ be a Lebesgue measure on $S_R(x)$.

Then

$$U_n(R) = \frac{R^n\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)}, \quad \omega_n(R) = \frac{2R^{n-1}\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}$$

are the volume of ball and the surface area of the sphere of radius R in $\mathbb{R}^n$ respectively.

Consider a random field

$$\eta_R(x) = \frac{1}{\omega_n(R)} \int\limits_{S_R(x)} \xi(y) m_n^{(R)}(dy).$$

Theorem 7.2. *[141] Random field $\eta_R(x)$ is homogeneous and isotropic. Homogeneous and isotropic random fields $\eta_R(x)$ and $\xi(x)$ are related each other and the following equalities hold*

$$E\eta_{R_1}(x_1)\eta_{R_2}(x_2) = \int\limits_0^{+\infty} Y_n(\lambda R_1) Y_n(\lambda R_2) Y_n(\lambda r_{x_1 x_2}) d\Phi(\lambda), \tag{7.4}$$

$$E\eta_R(x_1)\xi(x_2) = \int\limits_0^{+\infty} Y_n(\lambda R) Y_n(\lambda r_{x_1 x_2}) d\Phi(\lambda), \tag{7.5}$$

where

- *$Y_n(z)$ is defined in (7.2),*
- *$r_{x_1 x_2} = |x_1 - x_2|$ is a distance between the points x_1 and x_2.*

Let the random field $\xi(x)$ be observed on the ball $V_{R+r}(0)$, $r \geqslant 0$, and let the spectral function $\Phi(\lambda)$ of the field $\xi(x)$ be absolutely continuous.

Let a spherical correlogram [14]

$$\widehat{B}(r) = \frac{1}{U_n(R)} \int\limits_{V_R(0)} \xi(x) \left[\frac{1}{\omega_n(r)} \int\limits_{S_r(x)} \xi(t) m_n^{(r)}(dt) \right] dx =$$

be an estimator of correlation function in point r.

$$= \frac{1}{U_n(R)} \int\limits_{V_R(0)} \xi(x) \eta_r(x) dx. \tag{7.6}$$

Using (7.3) and the theorem 7.2, we obtain that $\widehat{B}(r)$ is unbiased estimate of $B(r)$:

$$E\widehat{B}(r) = \frac{1}{U_n(R)} \int\limits_{V_R(0)} E\xi(x)\eta_r(x) dx =$$

$$= \frac{1}{U_n(R)} \int\limits_{V_R(0)} \int\limits_0^{+\infty} Y_n(\lambda r) Y_n(0) d\Phi(\lambda) dx =$$

$$= \frac{1}{U_n(R)} \int\limits_{V_R(0)} \int\limits_0^{+\infty} Y_n(\lambda r) d\Phi(\lambda) dx = B(r),$$

since $Y_n(0) = 1$.

Calculate $E\widehat{B}^2(r)$:

$$E\widehat{B}^2(r) = E\left(\frac{1}{U_n(R)} \int\limits_{V_R(0)} \xi(x)\eta_r(x)dx\right)^2 =$$

$$= E\frac{1}{U_n^2(R)} \int\limits_{V_R(0)} \int\limits_{V_R(0)} \xi(x)\eta_r(x)\xi(y)\eta_r(y)dxdy.$$

By the Isserlis equality [19] and relationships (7.4),(7.5) we have

$$E\widehat{B}^2(r) = \frac{1}{U_n^2(R)} \int\limits_{V_R(0)} \int\limits_{V_R(0)} [E\xi(x)\eta_r(x)E\xi(y)\eta_r(y)+$$

$$+E\xi(x)\xi(y)E\eta_r(x)\eta_r(y) + E\xi(x)\eta_r(y)E\xi(y)\eta_r(x)]\, dxdy =$$

$$= \frac{1}{U_n^2(R)} \int\limits_{V_R(0)} \int\limits_{V_R(0)} \left[\left(\int\limits_0^{+\infty} Y_n(\lambda r)Y_n(0)d\Phi(\lambda)\right)^2 + \right.$$

$$+B(|x-y|) \int\limits_0^{+\infty} Y_n^2(\lambda r)Y_n(\lambda|x-y|)d\Phi(\lambda)+$$

$$\left. + \int\limits_0^{+\infty} Y_n(\lambda r)Y_n(\lambda|x-y|)d\Phi(\lambda) \int\limits_0^{+\infty} Y_n(\lambda r)Y_n(\lambda|x-y|)d\Phi(\lambda)\right] dxdy =$$

$$= \frac{1}{U_n^2(R)} \int\limits_{V_R(0)} \int\limits_{V_R(0)} \left[B^2(r) + B(|x-y|) \int\limits_0^{+\infty} Y_n^2(\lambda r)Y_n(\lambda|x-y|)d\Phi(\lambda)+\right.$$

$$\left. + \left(\int\limits_0^{+\infty} Y_n(\lambda r)Y_n(\lambda|x-y|)d\Phi(\lambda)\right)^2\right] dxdy =$$

$$+B^2(r) + \frac{1}{U_n^2(R)} \int\limits_{V_R(0)} \int\limits_{V_R(0)} \left[B(|x-y|) \int\limits_0^{+\infty} Y_n^2(\lambda r)Y_n(\lambda|x-y|)d\Phi(\lambda)+\right.$$

$$\left. + \left(\int\limits_0^{+\infty} Y_n(\lambda r)Y_n(\lambda|x-y|)d\Phi(\lambda)\right)^2\right] dxdy.$$

Therefore,

$$E\left(\widehat{B}(r)-B(r)\right)^2 = E\widehat{B}^2(r)-B^2(r) =$$

$$= \frac{1}{U_n^2(R)} \int\limits_{V_R(0)} \int\limits_{V_R(0)} \left[B(|x-y|) \int\limits_0^{+\infty} Y_n^2(\lambda r) Y_n(\lambda|x-y|) d\Phi(\lambda) + \right.$$
$$\left. + \left(\int\limits_0^{+\infty} Y_n(\lambda r) Y_n(\lambda|x-y|) d\Phi(\lambda) \right)^2 \right] dxdy.$$

Consider $\zeta(r) = \widehat{B}(r) - B(r)$, $0 \leqslant r \leqslant B$, $0 < B < +\infty$.
$\zeta(r)$ is a square Gaussian random process, since $\widehat{B}(r)$ is a limit of integral sums

$$\frac{1}{U_n^2(R)} \sum_k \eta_k(x_k)\xi(x_k)\Delta x_k,$$

$E\zeta(r) = 0$.

Let $\eta = \int_0^B \left(\widehat{B}(r)-B(r)\right)^2 dr$, $0 \leqslant r \leqslant T$.
It is clear that $\eta = l.i.m._{k\longrightarrow\infty} \sum_k \zeta^2(r_k)\Delta r_k$.

$$E\eta = \int\limits_0^B E\left(\widehat{B}(r)-B(r)\right)^2 dr =$$

$$= \frac{1}{U_n^2(R)} \int\limits_0^B \int\limits_{V_R(0)} \int\limits_{V_R(0)} \left[B(|x-y|) \int\limits_0^{+\infty} Y_n^2(\lambda r) Y_n(\lambda|x-y|) d\Phi(\lambda) + \right.$$
$$\left. + \left(\int\limits_0^{+\infty} Y_n(\lambda r) Y_n(\lambda|x-y|) d\Phi(\lambda) \right)^2 \right] dxdydr. \tag{7.7}$$

Theorem 7.3. *For estimator $\widehat{B}(r)$ of correlation function $B(r)$ of homogeneous and isotropic continuous in mean square random field $\xi(x)$ the following inequalities hold*

$$P\left\{ \int\limits_0^B \left(\widehat{B}(r)-B(r)\right)^2 dr > x \int\limits_0^B D\widehat{B}(r)dr \right\} \geqslant 1 - g(u)\exp\left\{\frac{u^2 x}{2}\right\} \tag{7.8}$$

for $u > 0$, $0 < x < -\frac{2\ln g(u)}{u^2}$, *where* $g(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp\left\{-\frac{s^2}{2}\right\} \frac{ds}{(1+s^2u^2)^{\frac{1}{4}}}$ *and*

$$P\left\{\int\limits_0^B \left(\widehat{B}(r) - B(r)\right)^2 dr > y \int\limits_0^B D\widehat{B}(r)dr\right\} \leqslant \frac{2^{\frac{1}{4}} y^{\frac{1}{4}}}{\operatorname{ch}\left(\sqrt{\frac{y}{2}} - \frac{1}{2}\right)} \tag{7.9}$$

for $y > \frac{1}{2}$.

Remark 7.1. The inequalities (7.8),(7.9) enable us to construct confidence sets for correlation function $B(r)$ in $L_2(0, B)$ space.

Let H be the hypothesis that the covariance function of homogeneous and isotropic continuous in mean square Gaussian random field $\xi(x)$ equals $B(r)$, for $0 \leqslant r \leqslant B$. As an estimator for $B(r)$ we choose $\widehat{B}(r)$ defined in (7.6). To test the hypothesis H one can use the following criterion.

Criterion 7.1. For some level of confidence α, $0 < \alpha < 1$, we can find such positive x_α and y_α, that

$$s(x_\alpha, u) + f(y_\alpha) = \alpha,$$

where
$s(x, u) = g(u) \exp\left\{\frac{u^2 x}{2}\right\}$, $u > 0$, $f(x) = \frac{2^{\frac{1}{4}} x^{\frac{1}{4}}}{\operatorname{ch}\left(\sqrt{\frac{x}{2}} - \frac{1}{2}\right)}$.

The hypothesis H is accepted if

$$x_\alpha < \frac{\int_0^B \left(\widehat{B}(r) - B(r)\right)^2 dr}{E \int_0^B \left(\widehat{B}(r) - B(r)\right)^2 dr} < y_\alpha$$

and hypothesis is rejected otherwise.

Remark 7.2. The probability of the first type's error does not exceed α when we use this criterion.

7.2. Construction criterion for testing hypothesis about the covariance function of the homogeneous and isotropic random field

Let $\xi(x)$ be a continuous in mean square homogeneous and isotropic Gaussian random field in $\mathbb{R}^n$ with zero-mean. Without any loss of generality, we can assume that the sample paths of the field $\xi(x)$ are continuous with probability one on any bounded and closed set.

Let the random field $\xi(x)$ be observed on the ball $V_{R+\tau}(0)$, $\tau \geqslant 0$ and let the spectral function of the field $\Phi(\lambda)$ be absolutely continuous.

Theorem 7.4. *Let a spherical correlogram*

$$\hat{B}(\tau) = \frac{1}{U_n(R)} \int\limits_{V_R(0)} \xi(x) \left(\frac{1}{\omega_n(r)} \int\limits_{S_r(x)} \xi(t) m_n^{(\tau)}(dt) \right) dx =$$
$$= \frac{1}{U_n(R)} \int\limits_{V_R(0)} \xi(x)\eta_\tau(x)dx \quad (7.10)$$

be an estimator of the covariance function $B(\tau)$. Then the following inequality holds for all $\varepsilon \geqslant \left(\frac{p}{\sqrt{2}} + \sqrt{(\frac{p}{2}+1)p}\right)^p C_p$

$$P\left\{ \int\limits_0^A (\hat{B}(\tau) - B(\tau))^p d\tau > \varepsilon \right\} \leqslant 2\sqrt{1 + \frac{\varepsilon^{1/p}\sqrt{2}}{C_p^{\frac{1}{p}}}} \exp\left\{ -\frac{\varepsilon^{\frac{1}{p}}}{\sqrt{2}C_p^{\frac{1}{p}}} \right\},$$

where

$$C_p = \frac{1}{U_n^2(R)} \int\limits_0^A \int\limits_{V_R(0)} \int\limits_{V_R(0)} \left(B(|x-y|) \int\limits_0^\infty Y_n^2(\lambda\tau) Y_n(\lambda|x-y|) d\Phi(\lambda) + \right.$$
$$\left. + \left[\int\limits_0^\infty Y_n(\lambda\tau) Y_n(\lambda|x-y|) d\Phi(\lambda) \right] \right) dx dy d\tau$$

and $0 < A < \infty$.

Remark 7.3. Since the sample paths of the field $\xi(x)$ are continuous with probability one on the ball $V_{R+\tau}(0)$, $\hat{B}(\tau)$ is a Riemann integral.

Proof. Consider

$$\mathbf{E}(\hat{B}(\tau) - B(\tau))^2 = \mathbf{E}(\hat{B}(\tau))^2 - B^2(\tau).$$

From the Isserlis equality for jointly Gaussian random variables and relationships 7.4 and 7.5 it follows that

$$\mathbf{E}\hat{B}^2(\tau) = \frac{1}{U_n^2(R)} \int\limits_{V_R(0)} \int\limits_{V_R(0)} (\mathbf{E}\xi(x)\eta_\tau(x)\mathbf{E}\xi(x)\eta_\tau(x)+$$

$$+ \mathbf{E}\xi(x)\xi(y)\mathbf{E}\eta_\tau(x)\eta_\tau(y) + \mathbf{E}\xi(x)\eta_\tau(y)\mathbf{E}\xi(y)\eta_\tau(x))\, dxdy =$$

$$= \frac{1}{U_n^2(R)} \int\limits_{V_R(0)} \int\limits_{V_R(0)} \left(\left[\int\limits_0^\infty Y_n(\lambda\tau)Y_n(0)d\Phi(\lambda) \right]^2 + \right.$$

$$+ B(|x-y|) \int\limits_0^\infty Y_n^2(\lambda\tau)Y_n(\lambda|x-y|)d\Phi(\lambda)+$$

$$\left. + \int\limits_0^\infty Y_n(\lambda\tau)Y_n(\lambda|x-y|)d\Phi(\lambda) \int\limits_0^\infty Y_n(\lambda\tau)Y_n(\lambda|x-y|)d\Phi(\lambda) \right) dxdy =$$

$$= \frac{1}{U_n^2(R)} \int\limits_{V_R(0)} \int\limits_{V_R(0)} \left(B^2(\tau) + B(|x-y|) \int\limits_0^\infty Y_n^2(\lambda\tau)Y_n(\lambda|x-y|)d\Phi(\lambda)+ \right.$$

$$\left. + \left[\int\limits_0^\infty Y_n(\lambda\tau)Y_n(\lambda|x-y|)d\Phi(\lambda) \right]^2 \right) dxdy = B^2(\tau)+$$

$$\frac{1}{U_n^2(R)} \int\limits_{V_R(0)} \int\limits_{V_R(0)} \left(B(|x-y|) \int\limits_0^\infty Y_n^2(\lambda\tau)Y_n(\lambda|x-y|)d\Phi(\lambda)+ \right.$$

$$\left. + \left[\int\limits_0^\infty Y_n(\lambda\tau)Y_n(\lambda|x-y|)d\Phi(\lambda) \right] \right) dxdy.$$

Therefore,

$$\mathbf{E}(\hat{B}(\tau) - B(\tau))^2 = \frac{1}{U_n^2(R)} \int\limits_{V_R(0)} \int\limits_{V_R(0)} \left(B(|x-y|) \int\limits_0^\infty Y_n^2(\lambda\tau)Y_n(\lambda|x-y|)d\Phi(\lambda)+ \right.$$

$$\left. + \left[\int\limits_0^\infty Y_n(\lambda\tau)Y_n(\lambda|x-y|)d\Phi(\lambda) \right] \right) dxdy. \quad (7.11)$$

Since $\hat{B}(\tau) - B(\tau)$ is a square Gaussian random field (see Lemma 3.1, Chapter 6 in book [19]), then it follows from the Theorem 3.4 that

$$P\left\{\int_0^A (\hat{B}(\tau) - B(\tau))^p d\tau > \varepsilon\right\} \leqslant 2\sqrt{1 + \frac{\varepsilon^{1/p}\sqrt{2}}{C_p^{\frac{1}{p}}}} \exp\left\{-\frac{\varepsilon^{\frac{1}{p}}}{\sqrt{2}C_p^{\frac{1}{p}}}\right\}.$$

Applying equality (7.11) we get

$$C_p = \frac{1}{U_n^2(R)} \int_0^A \int_{V_R(0)} \int_{V_R(0)} \left(B(|x-y|) \int_0^\infty Y_n^2(\lambda\tau) Y_n(\lambda|x-y|) d\Phi(\lambda) + \right.$$

$$\left. + \left[\int_0^\infty Y_n(\lambda\tau) Y_n(\lambda|x-y|) d\Phi(\lambda)\right]\right) dx dy d\tau. \quad \diamondsuit$$

Denote

$$g(\varepsilon) = 2\sqrt{1 + \frac{\varepsilon^{1/p}\sqrt{2}}{C_p^{\frac{1}{p}}}} \exp\left\{-\frac{\varepsilon^{\frac{1}{p}}}{\sqrt{2}C_p^{\frac{1}{p}}}\right\}.$$

From the Theorem 3 it follows that if $\varepsilon \geqslant z_p = C_p \left(\frac{p}{\sqrt{2}} + \sqrt{(\frac{p}{2}+1)p}\right)^p$ then

$$P\left\{\int_0^A (\hat{B}(\tau) - B(\tau))^p d\tau > \varepsilon\right\} \leqslant g(\varepsilon).$$

Let ε_δ be a solution of the equation $g(\varepsilon) = \delta$, $0 < \delta < 1$. Put $S_\delta = \max\{\varepsilon_\delta, z_p\}$. It is obviously that $g(S_\delta) \leqslant \delta$ and

$$P\left\{\int_0^A (\hat{B}(\tau) - B(\tau))^p d\tau > S_\delta\right\} \leqslant \delta. \tag{7.12}$$

Let $\mathbb{H}$ be the hypothesis that the covariance function of homogeneous and isotropic continuous in mean square Gaussian random field $\xi(x)$ equals $B(\tau)$ for $0 \leqslant \tau \leqslant A$. From the Theorem 7.4 and (7.12) it follows that to test the hypothesis $\mathbb{H}$ one can use the following criterion.

Criterion 7.2. For a given level of confidence δ the hypothesis $\mathbb{H}$ is accepted if

$$\int_0^A (\hat{B}(\tau) - B(\tau))^p d\mu(\tau) < S_\delta$$

otherwise hypothesis is rejected.

Remark 7.4. The equation $g(\varepsilon) = \delta$ has a solution for any $\delta > 0$, since $g(\varepsilon)$

is a monotonically decreasing function. We can find the solution of equation using numerical methods.

Remark 7.5. One can easily see that Criterion 7.2 can be used if $C_p \to 0$ as $R \to \infty$.

Example 7.1. Let the hypothesis $\mathbb{H}$ is such that the covariance function of a homogeneous and isotropic Gaussian stochastic field $\xi(x)$ equals to $B(\tau) = 9\sqrt{\pi}\frac{J_{3/2}(c\tau)}{(c\tau)^{3/2}}$, where $J_{3/2}(c\tau)$ is Bessel functions of the first kind, $c > 0$, $0 \leqslant \tau \leqslant A$. It is known that for $B(\tau)$ exist the spectral function in the following form

$$\Phi(\lambda) = \begin{cases} \left(\frac{\lambda}{c}\right)^3, & \text{as } 0 < \lambda \leqslant c, \\ 1, & \text{as } \lambda > c. \end{cases}$$

We will estimate the value of C_p from the Theorem 7.4. Consider the following integrals $I_1 = \int\limits_0^\infty Y_n(\lambda\tau)Y_n(\lambda|x-y|)d\Phi(\lambda)$ and $I_2 = \int\limits_0^\infty Y_n^2(\lambda\tau)Y_n(\lambda|x-y|)d\Phi(\lambda)$. We will choose $n = 3$ and we will evaluate the integral I_1

$$|I_1| = \left|\int\limits_0^\infty Y_3(\lambda\tau)Y_3(\lambda|x-y|)d\Phi(\lambda)\right| = \left|\frac{3}{c^3}\int\limits_0^c Y_3(\lambda\tau)Y_3(\lambda|x-y|)\lambda^2 d(\lambda)\right| \leqslant$$

$$\leqslant \frac{3}{c^3}\int\limits_0^c |Y_3(\lambda\tau)|\,|Y_3(\lambda|x-y|)|\,\lambda^2 d\lambda.$$

We will estimate the value of $|Y_3(\lambda\tau)|$. We will use an estimate of Bessel functions of the first kind that was obtained in the paper [133], namely

$$|J_k(u)| \leqslant 2^{1-\alpha}|u|^\alpha \pi^\alpha \frac{1}{k^\alpha}. \tag{7.13}$$

For simplicity we choose $\alpha = 1$. Then

$$|Y_3(\lambda\tau)| = \left|\sqrt{2}\Gamma(\frac{3}{2})\frac{J_{1/2}(\lambda\tau)}{(\lambda\tau)^{3/2}}\right| \leqslant 2\sqrt{2}\frac{|\lambda\tau|}{|\lambda\tau|^{3/2}}\pi\Gamma\left(\frac{3}{2}\right) = 2\sqrt{2}\pi\Gamma\left(\frac{3}{2}\right)\sqrt{\lambda\tau}.$$

Similarly, we obtain that

$$\left|Y_3(\lambda\sqrt{|x-y|})\right| \leqslant= 2\sqrt{2}\pi\Gamma\left(\frac{3}{2}\right)\sqrt{\lambda|x-y|}.$$

Then

$$|I_1| \leqslant \frac{3}{c^3}\int\limits_0^c 2\sqrt{2}\pi\Gamma\left(\frac{3}{2}\right)\sqrt{\lambda\tau}2\sqrt{2}\pi\Gamma\left(\frac{3}{2}\right)\sqrt{\lambda|x-y|}\lambda^2 d\lambda =$$

$$= \frac{24}{c^3}\pi^2\Gamma^2\left(\frac{3}{2}\right)\sqrt{\tau}\sqrt{|x-y|}\int\limits_0^c \lambda^3 d\lambda = 6\pi^2\Gamma^2\left(\frac{3}{2}\right)c\sqrt{\tau}\sqrt{|x-y|}.$$

Consider the integer інтеграл I_2

$$|I_2| = \left|\int\limits_0^\infty Y_3^2(\lambda\tau)Y_3(\lambda|x-y|)d\Phi(\lambda)\right| = \left|\frac{3}{c^3}\int\limits_0^c Y_3^2(\lambda\tau)Y_3(\lambda|x-y|)\lambda^2 d(\lambda)\right| \leqslant$$

$$\leqslant \frac{3}{c^3}\int\limits_0^c \left|Y_3^2(\lambda\tau)\right|\left|Y_3(\lambda|x-y|)\right|\lambda^2 d\lambda.$$

Using the similar estimates as in calculating of the integral I_1 we obtained that

$$|I_2| \leqslant \frac{3}{c^3}\int\limits_0^c 2\left(\sqrt{2}\pi\Gamma\left(\frac{3}{2}\right)\sqrt{\lambda\tau}\right)^2 2\sqrt{2}\pi\Gamma\left(\frac{3}{2}\right)\sqrt{\lambda|x-y|}\lambda^2 d\lambda =$$

$$= \frac{48\sqrt{2}}{c^3}\pi^3\Gamma^3\left(\frac{3}{2}\right)\tau\sqrt{|x-y|}\int\limits_0^c \lambda^{7/2}d\lambda = \frac{96\sqrt{2}}{9}\pi^3\Gamma^3\left(\frac{3}{2}\right)\tau\sqrt{|x-y|}c^{3/2}.$$

Now we consider the following integral

$$|I_3| = 27\sqrt{\pi}\left|\int\limits_0^R\int\limits_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}}\left(\frac{J_{3/2}(|x-y|)}{|x-y|^{3/2}}\int\limits_0^\infty Y_3^2(\lambda\tau)Y_3(\lambda|x-y|)d\Phi(\lambda)+\right.\right.$$

$$\left.\left.\int\limits_0^\infty Y_3(\lambda\tau)Y_3(\lambda|x-y|)d\Phi(\lambda)\right)dxdy\right| \leqslant$$

$$27\sqrt{\pi}\int\limits_0^R\int\limits_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}}\left(\left|\frac{J_{3/2}(|x-y|)}{|x-y|^{3/2}}\right|\int\limits_0^c\left|Y_3^2(\lambda\tau)\right|\left|Y_3(\lambda|x-y|)\right|\lambda^2 d\lambda+\right.$$

$$\left.\int\limits_0^c\left|Y_3(\lambda\tau)\right|\left|Y_3(\lambda|x-y|)\right|\lambda^2 d\lambda\right)dxdy.$$

Taking into account the estimates for I_1 and I_2, and (7.13) we will get

$$|I_3| \leqslant 27\sqrt{\pi} \int\limits_0^R \int\limits_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \frac{|x-y|}{|x-y|^{3/2}} \frac{2\pi}{3} \left(\frac{96\sqrt{2}}{9} \pi^3 \Gamma^3 \left(\frac{3}{2}\right) \tau \sqrt{|x-y|} c^{3/2} + \right.$$
$$\left. + 6\pi^2 \Gamma^2 \left(\frac{3}{2}\right) c\sqrt{\tau}\sqrt{|x-y|} \right) dydx = 36\pi^3\sqrt{\pi} c \Gamma^2 \left(\frac{3}{2}\right) \times$$
$$\times \left(\frac{96\sqrt{2}}{9} \pi \Gamma \left(\frac{3}{2}\right) \tau\sqrt{c} + 6\tau \right) \int\limits_0^R \int\limits_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} dydx.$$

in this case we will obtained

$$2\int\limits_0^R \sqrt{R^2-x^2}dx = 2R^2 \int\limits_0^{\pi/2} \sqrt{1-\cos^2 t}\sin t dt = 2R^2 \int\limits_0^{\pi/2} \sin^2 t dt =$$
$$= 2R^2 \int\limits_0^{\pi/2} \frac{1-\cos 2t}{2} dt = \frac{\pi R^2}{2}.$$

Then

$$|I_3| \leqslant 18R^2\pi^4\sqrt{\pi} c \Gamma^2 \left(\frac{3}{2}\right) \left(\frac{96\sqrt{2}}{9} \pi \Gamma \left(\frac{3}{2}\right) \tau\sqrt{c} + 6\tau \right)$$

We will estimate the value of C_p. From the Theorem 7.4 and above mentioned it follows that

$$C_p \leqslant \frac{1}{U_3^p(R)} \int\limits_0^A \left(18R^2\pi^4\sqrt{\pi} c \Gamma^2 \left(\frac{3}{2}\right) \left(\frac{96\sqrt{2}}{9} \pi \Gamma \left(\frac{3}{2}\right) \tau\sqrt{c} + 6\tau \right) \right)^{p/2} d\tau \leqslant$$
$$\frac{1}{U_3^p(R)} 18R^p \left(\pi^4\sqrt{\pi} c \Gamma^2 \left(\frac{3}{2}\right) \right)^{p/2} D_p \int\limits_0^A \left(\left(\frac{96\sqrt{2}}{9} \pi \Gamma \left(\frac{3}{2}\right) \sqrt{c} \right)^{p/2} \tau^{p/2} + \right.$$
$$\left. + 6^{p/2} \tau^{\frac{p+1}{2}} \right) d\tau = \frac{18R^p}{U_3^p(R)} \left(\pi^4\sqrt{\pi} c \Gamma^2 \left(\frac{3}{2}\right) \right)^{p/2} D_p \times$$
$$\times \left(\left(\frac{96\sqrt{2}}{9} \pi \Gamma \left(\frac{3}{2}\right) \sqrt{c} \right)^{p/2} \frac{2A^{\frac{p+2}{2}}}{p+2} + 6^{p/2} \frac{2A^{\frac{p+3}{2}}}{p+3} \right),$$

where

$$D_p = \begin{cases} 1, & \text{as } 0 < p \leqslant 1, \\ 2^p, & \text{as } p > 1. \end{cases}$$

Taking into account the value of $U_3^p(R)$ we will get

$$C_p \leqslant \frac{18\Gamma\left(\frac{5}{2}\right)}{R^{2p}\pi^{3/2}} \left(\pi^4 \sqrt{\pi} c \Gamma^2 \left(\frac{3}{2}\right)\right)^{p/2} D_p \times$$

$$\times \left(\left(\frac{96\sqrt{2}}{9} \pi \Gamma \left(\frac{3}{2}\right) \sqrt{c} \right)^{p/2} \frac{2A^{\frac{p+2}{2}}}{p+2} + 6^{p/2} \frac{2A^{\frac{p+3}{2}}}{p+3} \right).$$

7.3. Estimation of homogeneous and isotropic Gaussian random field's correlation function when the values of field are observed on a ball

As before, we will use the following notations:

- $S_R(x)$, $V_R(x)$ sphere and ball of radius R centered at a point x;
- $U_n(R,)$ $\omega_n(R)$ the volume of ball and the surface area of the sphere of radius R;
- $m_n^{(R)}(\cdot)$ a Lebesgue measure on $S_R(x)$;
- $\Phi(\lambda)$ the spectral function $\Phi(\lambda)$ of the field $\xi(x)$.

Consider a random field

$$\widehat{\eta}_r(x) = \frac{1}{\widehat{\omega}_n(r)} \sum_{k=1}^{m} \xi(x_k) \Delta S_k, \tag{7.14}$$

where

x_k points on the sphere $S_r(x)$;

ΔS_k square of the k-th element of sphere's partition;

$\widehat{\omega}_n(r) = \sum_{k=1}^{m} \Delta S_k$.

Theorem 7.5. *Random field $\widehat{\eta}_r(x)$ is homogeneous and isotropic. Homogeneous and isotropic random fields $\widehat{\eta}_r(x)$ and $\xi(x)$ are homogeneously and isotropically related each other and the following equalities hold*

$$E\widehat{\eta}_{r_1}(x_1)\widehat{\eta}_{r_2}(x_2) = \int_0^{+\infty} Y_n(\lambda r_1) Y_n(\lambda r_2) Y_n(\lambda r_{x_1 x_2}) d\Phi(\lambda), \tag{7.15}$$

$$E\widehat{\eta}_r(x_1)\xi(x_2) = \int\limits_0^{+\infty} Y_n(\lambda r)Y_n(\lambda r_{x_1x_2})d\Phi(\lambda), \tag{7.16}$$

where

$r_{x_1x_2} = |x_1 - x_2|$ *is a distance between the points* x_1 *and* x_2.
Y_n *the spherical Bessel function, introduced in (7.2).*

Proof. The proof of this theorem is similar to corresponding theorem in [141] ♢

Consider homogeneous and isotropic continuous in mean square Gaussian random field $\xi(x)$ in $\mathbb{R}^n$ with mean zero and correlation function $B(r)$. Assume that $B(r)$ can be presented as (7.3). Suppose that the spectral function $\Phi(\lambda)$ of this field is absolutely continuous, random field is observed on a ball $V_{R+r}(0)$, $0 \leqslant r \leqslant B$, and value of field is known only in some a points on the ball.

Consider N spheres on the ball $V_R(x)$ with radiuses $\frac{iR}{N}$, $1 \leqslant i \leqslant N$, $\frac{R}{N} < r$, and centers in 0.

As an estimator of correlation function in point r will use

$$\widehat{B}_N(r) = \frac{1}{\widehat{U}_n(R)} \sum_{i=1}^{N}\sum_{j=1}^{M} \xi(x_{ij})\widehat{\eta}_r(x_{ij})\Delta S_{ij}, \tag{7.17}$$

where

- x_{ij} - points on the sphere $S_{\frac{iR}{N}}(0)$;
- ΔS_{ij} - the surface area of the j-th element of sphere's $S_{\frac{iR}{N}}(0)$ partition;
- $\widehat{\omega}_n\left(\frac{iR}{N}\right) := \sum_{j=1}^{M} \Delta S_{ij}$;
- $\widehat{U}_n(R) := \sum_{i=1}^{N}\sum_{j=1}^{M} \Delta S_{ij}$.

$\widehat{B}_N(r)$ is unbiased estimate:

$$E\widehat{B}_N(r) = \frac{1}{\widehat{U}_n(R)} \sum_{i=1}^{N}\sum_{j=1}^{M} E\xi(x_{ij})\widehat{\eta}_r(x_{ij})\Delta S_{ij} =$$

$$= \frac{1}{\widehat{U}_n(R)} \sum_{i=1}^{N}\sum_{j=1}^{M} \int\limits_0^{+\infty} Y_n(\lambda r)Y_n(0)d\Phi(\lambda)\Delta S_{ij} =$$

$$= \frac{1}{\widehat{U}_n(R)} \sum_{i=1}^{N}\sum_{j=1}^{M} \int\limits_0^{+\infty} Y_n(\lambda r)d\Phi(\lambda)\Delta S_{ij} = B(r).$$

Using the Isserlis formula and equalities (7.15),(7.16) one can calculate

$E\widehat{B}_N^2(r)$.

$$E\widehat{B}_N^2(r) = E\left(\frac{1}{\widehat{U}_n(R)}\sum_{i=1}^{N}\sum_{j=1}^{M}\xi(x_{ij})\widehat{\eta}_r(x_{ij})\Delta S_{ij}\right)^2 =$$

$$= \frac{1}{\widehat{U}_n^2(R)}\sum_{i=1}^{N}\sum_{j=1}^{M}\sum_{p=1}^{N}\sum_{l=1}^{M} E\xi(x_{ij})\widehat{\eta}_r(x_{ij})\xi(x_{pl})\widehat{\eta}_r(x_{pl})\Delta S_{ij}\Delta S_{pl} =$$

$$= \frac{1}{\widehat{U}_n^2(R)}\sum_{i=1}^{N}\sum_{j=1}^{M}\sum_{p=1}^{N}\sum_{l=1}^{M} [E\xi(x_{ij})\widehat{\eta}_r(x_{ij})E\xi(x_{pl})\widehat{\eta}_r(x_{pl})+$$

$$+E\xi(x_{ij})\xi(x_{pl})E\widehat{\eta}_r(x_{ij})\widehat{\eta}_r(x_{pl}) + E\xi(x_{ij})\widehat{\eta}_r(x_{pl})E\xi(x_{pl})\widehat{\eta}_r(x_{ij})]\,\Delta S_{ij}\Delta S_{pl} =$$

$$= \frac{1}{\widehat{U}_n^2(R)}\sum_{i=1}^{N}\sum_{j=1}^{M}\sum_{p=1}^{N}\sum_{l=1}^{M}\left[\left(\int_0^{+\infty} Y_n(\lambda r)Y_n(0)d\Phi(\lambda)\right)^2 +\right.$$

$$+ \int_0^{+\infty} B(|x_{ij}-x_{pl}|)\int_0^{+\infty} Y_n^2(\lambda r)Y_n(\lambda|x_{ij}-x_{pl}|)d\Phi(\lambda)+$$

$$\left.+ \int_0^{+\infty} Y_n(\lambda r)Y_n(\lambda|x_{ij}-x_{pl}|)d\Phi(\lambda)\int_0^{+\infty} Y_n^2(\lambda r)Y_n(\lambda|x_{ij}-x_{pl}|)d\Phi(\lambda)\right]\times$$

$$\times\Delta S_{ij}\Delta S_{pl} = B^2(r) + \frac{1}{\widehat{U}_n^2(R)}\sum_{i=1}^{N}\sum_{j=1}^{M}\sum_{p=1}^{N}\sum_{l=1}^{M}\left[\int_0^{+\infty} B(|x_{ij}-x_{pl}|)\times\right.$$

$$\times\int_0^{+\infty} Y_n^2(\lambda r)Y_n(\lambda|x_{ij}-x_{pl}|)d\Phi(\lambda)+$$

$$\left.+\left(\int_0^{+\infty} Y_n(\lambda r)Y_n(\lambda|x_{ij}-x_{pl}|)d\Phi(\lambda)\right)^2\right]\Delta S_{ij}\Delta S_{pl}.$$

Then

$$E\left(\widehat{B}_N(r) - B(r)\right)^2 = E\widehat{B}_N^2(r) - B^2(r) =$$

$$= \frac{1}{\widehat{U}_n^2(R)}\sum_{i=1}^{N}\sum_{j=1}^{M}\sum_{p=1}^{N}\sum_{l=1}^{M}\left[\int_0^{+\infty} B(|x_{ij}-x_{pl}|)\times\right.$$

$$\times\int_0^{+\infty} Y_n^2(\lambda r)Y_n(\lambda|x_{ij}-x_{pl}|)d\Phi(\lambda)+$$

$$+\left(\int_0^{+\infty} Y_n(\lambda r)Y_n(\lambda|x_{ij}-x_{pl}|)d\Phi(\lambda)\right)^2\Bigg]\Delta S_{ij}\Delta S_{pl}.$$

Consider $\zeta(r)=\widehat{B}_N(r)-B(r)$, $0\leqslant r\leqslant B$, $0<B<+\infty$.
Since $\widehat{B}_N(r)$ is a quadratic form of Gaussian vectors, therefore $\zeta(r)$ is Square Gaussian random process and $E\zeta(r)=0$.

Let $\eta=\int_0^B\left(\widehat{B}_N(r)-B(r)\right)^2 dr$. Since η is a mean square limit of quadratic forms of the type $\sum_k \zeta^2(r_k)\Delta r_k$, where $r_k\in[0,B]$, then the next theorem holds.

Theorem 7.6. *For the estimator $\widehat{B}_N(r)$ of correlation function $B(r)$ homogeneous and isotropic continuous in mean square Gaussian random field $\xi(x)$ the following inequalities hold*

$$P\left\{\int_0^B\left(\widehat{B}_N(r)-B(r)\right)^2 dr > x\int_0^B D\widehat{B}_N(r)dr\right\}\geqslant 1-g(u)\exp\left\{\frac{u^2x}{2}\right\}$$

for $u>0$, $0<x<-\frac{2\ln g(u)}{u^2}$,
where $g(u)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\exp\left\{-\frac{s^2}{2}\right\}\frac{ds}{(1+s^2u^2)^{\frac14}}$, *and*

$$P\left\{\int_0^B\left(\widehat{B}_N(r)-B(r)\right)^2 dr > y\int_0^B D\widehat{B}_N(r)dr\right\}\leqslant \frac{2^{\frac14}y^{\frac14}}{\operatorname{ch}\left(\sqrt{\frac{y}{2}}-\frac12\right)}$$

for $y>\frac12$.

Let H be the hypothesis that for $0\leqslant r\leqslant B$ the covariance function of homogeneous and isotropic continuous in mean square Gaussian random field $\xi(x)$ equals $B(r)$. As an estimator for $B(r)$ we choose $\widehat{B}_N(r)$, defined in (7.17). To test the hypothesis H one can use the following criterion.

Criterion 7.3. For some level of confidence α, $0<\alpha<1$, one can find such positive x_α and y_α, that

$$s(x_\alpha,u)+f(y_\alpha)=\alpha,$$

where
$s(x,u)=g(u)\exp\left\{\frac{u^2x}{2}\right\}$, $u>0$, $f(x)=\frac{2^{\frac14}x^{\frac14}}{\operatorname{ch}\left(\sqrt{\frac{x}{2}}-\frac12\right)}$.

The hypothesis H is accepted if

$$x_\alpha < \frac{\int_0^B \left(\widehat{B}_N(r) - B(r)\right)^2 dr}{E\int_0^B \left(\widehat{B}_N(r) - B(r)\right)^2 dr} < y_\alpha$$

and hypothesis is rejected otherwise.

Remark 7.6. The probability of the first type's error does not exceed α when we use this criterion.

Bibliography

1. Antonini R.G., Kozachenko Yu.V., Tegza A.M. *Accuracy of simulation in L_p of Gaussian random processes.* Bulletin of the University of Kiev, Series: Physics & Mathem. **vol.5**, 7-14 (2002)

2. Antonini R.G., Kozachenko Yu.V.,Sorokulov V.V. *On accuracy and reliability of simulation of some random processes from the space $Sub_\varphi(\Omega)$.* Theory of Stochastic Processes. Vol.9 (25), № 3-4, 50-57. (2003)

3. Armstrong M., Dowd P.A. editors. *Geostatistical Simulations.* Kluwer, Dordrecht (1994)

4. Bandut G., Pirsol A. *Application of correlation and spectral analysis.* "Mir Moscov (1983) (in Russian)

5. Barrasa de la Krus E., Kozachenko Yu.V. *Boundary-value problems for equations of mathematical physics with strictly Orlicz random initial conditions.* Random Oper. and Stoch. Eq. **3**, N 3, 201–220 (1995)

6. Bateman H., Erdelyi A. *Higher transcendental functions*, NY Toronto London MC Craw-hill book company, INC (1953)

7. J Bendat, A Pirsol. *Applications of Correlation and Spectral Analysis* [Russian translation] - Mir, Moscow, 1983

8. Brillindger D. Time series. *Data processing and theory.* "Mir Moscov (1980) (in Russian)

9. Brockwell P. J. and Davis R.A. *Time Series: Theory and Methods*, New York: Springer, 2nd ed. (1991)

10. Brooker P., Stewart M.A. *A comparative study of simulation techniques for two dimensional data honoring specified exponential semivariograms.* J.Austr. Math. SOc.B. **36**, N 3, 249–260 (1995)

11. Buldygin V., *Properties of an empirical correlogram of a Gaussian process with squar-integrable spectral density*, Ukrain. Math. J. **47**, (1995), no. 7, 1006-1024.

12. Buldygin V.V. *Random elements convergence in topological spaces.* "Naukova dumka Kiev (1980)

13. Buldygin V.V. *Sub-Gaussian processes and convergence random series in functional spaces.* Ukrainian Mathem. Journal **29**, № 4, 443-454 (1977) (in Russian)

14. Buldygin V.V., Dykhovichny O.O. On some singular integrals and their applications to problems of statistical estimation. // Theor.Probability and Math. Statist, №53, P.19–31 (1996).

15. Buldygin V.V. and Kozachenko Yu.V. *On sub-Gaussian random variables.* Ukrainian Mathem. Journal (1980)

16. Buldygin V.V. and Kozachenko Yu.V. *On local properties of realizations of some random processes and fields.* Theory probability and mathematical statististics **10**, 39-47, (1974) (in Russian)

17. Buldygin V.V. and Kozachenko Yu.V. *Sub-Gaussian random vectors and processes.* Theory probability and mathematical statististics **36**, 10-22, (1987) (in Russian)

18. Buldygin V.V. and Kozachenko Yu.V. *Bershtein funtionals and exponential inequalities for the sum of ramdom variables.* Mathematica today, 55-79, (1994) (in Russian)

19. Buldygin V.V., Kozachenko Yu.V. *Metric characterization of random variables and random processes.* Amer. Math. Soc., Providence, RI (2000)

20. Buldygin V. , Zayats V., *On asymptotic normality of estimates for correlation functions of stationary Gaussian processes in the spaces of continuous functions*, Ukrain. Math.l J. **47**, no.11, 1696-1710, (1995)

21. Bykov V.V. *Numeric simulation in statistical radio engineering.* Soviet radio, Moscow (1978) (in Russian)

22. Chiles J.P., Delfiner P. *Geostatistics: Modelling Spatial Uncertainty.* John Wiley & Sons, Inc. New York, Toronto (1999)

23. Choi H., Ombao H., Ray B., *Sequential change-point detection methods for nonstationary time series*, Technometrics 50, 40-52, (2008)

24. Coates D.S. and Diggle P., *Tests for comparing two estimated spectral densities*, Journal of Time Series Analysis, **7**, 7-20, (1986)

25. Cramer H. and Lidbetter M. *Stationary and Related Stochastic Process.* - J.Willey, New York, (1967).

26. Cressie N.A.C. *Statistics for Spatial Data.* John Wiley, New York, revised edition (1993)

27. Dychovychnyj A.A. On the estimation of the correlation function of homogeneous and isotropic Gaussian field // Theory of probab. and math. stat. - 1983. - Issue 29, - P. 37–40. (in Russian)

28. Djenkins G., Vatts D. *Spectral analysis and its applications.* "Mir Moscow (1971) (in Russian)

29. Dovgaj B.V., Kozachenko Yu.V., Rozora I.V. *Simulation of random processes in physical systems.* "Zadruga Kyiv (2010) (in Ukrainian)

30. Dudley R.M. *Sample functions of the Gaussian processes.* Ann. of Probab. **Vol. 1** , № 1, 3–68 (1973)

31. Ermakov S.M., Mikhajlov G.A. *Statistical simulation.* "Nauka Moscow (1982) (in Russian)

32. Fan J. and Zhang W., *Generalized likelihood ratio tests for spectral density*, Biometrika 91, 195-209, (2004)

33. Fan J., Zhang C. and Zhang J., *Generalised likelihood ratio statistics and Wilks phenomenon*, Ann. Statist. 29, 153–93, (2001)

34. Fedoryanych T.V., *One estimate of the correlation function for Gaussian stochastic process*, Bulletin of Taras Shevchenko National University of Kyiv **2**, 72-76, (2004)

35. Fernique X. *Régularité des trajectoires des fonctions aléatoires gaussiennes.* Lecture Notes in Mathematics. **Vol. 489**, 2–187 (1975)

36. Feller V. *Introduction in probabiblity theory and mathematical statistics.* Vol 2. "Mir Moscow (1984) (in Russian)

37. Fuentes M., *A formal test for nonstationarity of spatial stochastic processes*, Journal of Multivariate Analysis 96, 30-54, (2005)

38. Fuentes M., *Testing for separability of spatial-temporal covariance functions*, Journal of Statistical Planning and Inference 136, 447-466, (2006)

39. Gikhman I.I., Skorohod A.V. *The theory of stochastic pprocesses I- II.* Springer-Verlag, Berlin Heidelberg (2004)

40. Gikhman I.I., Skorohod A.V., Yadrenko M.I. *Probability theory and mathematical statistics.* "Vyscha shkola Kyiv (1988) (in Russian)

41. Hardi G.H. Rogosinski W.W. *Fourier Series.* Dover Publ. (1999).

42. Hunt G.A. *Random Fourie transforms.*- Trans. Amer. Math. Soc. 71.- 1951.- P.38-69.

43. Ianevych T. O., Kozachenko Yu. V., Troshki V. B. *Goodness-of-fit tests for random sequences incorporating several components*, Random Oper. Stoch. Equ., **25**, № 1, 1–10 (2017).

44. Ibragimov I.A. *About condotions of smoothness of trajectory of random funktion.* Theory probability and its applicatons. **Vol. 28** , № 2, 229-250 (1973) (in Russian)

45. Ibragimov I.A., Linnik Yu. V. *Independance and inhomogeneous related variables.* "Nauka Moscow (1973) (in Russian)

46. Ibragimov I.A., Rozanov Yu.A. *Random gaussian processes.* "Nauka Moscow (1970) (in Russian)

47. Ivanov A., *A limit theorem for the evaluation of the correlation function*, Theor. Probability and Math. Statist **19**, 76-81, (1978)

48. Jain N.C., Marcus M.B. *Integrability of infinite sums of independent vector-valued random variables.* Trans. Amer. Math. Soc. **212**, № 1, 1–36 (1975)

49. Kahane J.P. *Propriétés locales des fonctions à series de Fouries aléatories.* Studia Math. **19**, № 1, 1–25 (1960)

50. Kampé de Feriet. *Fonctions de la physique mathématique.* Centre National de la Recherche Scientifique, Paris (1957)

51. Kolmogorov A.N, Fomin S.V. *Elements of the theory of functions and functional analysis.* M.: Nauka, (1976). (in Russian)

52. Kozachenko Yu.V. *Pre-Gaussian random processes* Functional Analysis V. – 1998. – Vol 1. – P.35-59.

53. Yu.V. Kozachenko, O.E. Kamenshchikova, textit Approximation $SSub_{varphi}$ *Omega* of random processes in the space $L_p(\mathbb{T})$, Theory Probably and mat. mute. textbf 79 (2008), 73–78. (in Ukranian)

54. Kozachenko Yu.V. and Kozachenko L.F., *On the accuracy of modelling in $L^2(0,T)$ of Gaussian random processes.*, Vichisl. Prikl. Mat. (Kiev) **74**, 88-93 (1991). (in Russian)

55. Kozachenko Yu.V. and Kozachenko L.F., *On the accuracy of modelling in $L^2(0,T)$ of stationary Gaussian random processes.*, Vichisl. Prikl. Mat. (Kiev) **75**, 108-115 (1991). (in Russian)

56. Kozachenko Yu.V. and Kozachenko L.F., *On the accuracy of modelling of stationary processes with absolutely continuous spectrum*, Theor. Probab. and Math. Statist. (Kiev) **47**, 47-54 (1992). (in Russian)

57. Kozachenko Yu.V., Kozachenko L.F. Criterion for checking hypotheses about covariance function of Gaussian random processes // Computational and Applied Mathematics. - №77. - P.61–74 (1993). (in Ukranian)

58. Kozachenko Yu.V., Kuchinka K. J., Slyvka-Tylyshchak A. I. *Random processes in problems of mathematical physics*, Monography, Uzhhorod, 256, (2017). (in Ukranian)

59. Kozachenko, Yu. V. and Moklyachuk, O. M. *Large deviation probabilities for square-Gaussian stochastic processes.* Extremes, Vol. **2** (3),269–293 (1999).

60. Kozachenko, Yu. V. and Moklyachuk, O. M. *Square-Gaussian stochastic processes.* Theory of Stoch. Processes, **Vol. 6(22)**, N 3-4, 98-121(2000)

61. Kozachenko Yu., Oleshko T., *Analytic properties of certain classes of pre-Gaussian stochastic processes*, Theor. Probability and Math. Statist **48**, 37-51, (1993)

62. Kozachenko Yu. V. and Ostrovskij E.I. *Banach spaces of random variables of type subGaussian.*, Theor. Probab. and Math. Statist. (Kiev) **32**, 42-53 (1985). (in Russian)

63. Kozachenko Yu. V. and Pashko A.O. *Simulation of stochastic Processes.*, "Kyiv University Kyiv (1999) (in Ukrainian)

64. Kozachenko Y. V., Pashko A. O. *Accuracy of simulation of stochastic processes in norms of Orlicz spaces. I* , Theory of Probability and Mathematical Statistics. №. 58, 51-66. (1999)

65. Kozachenko Yu. V., Pashko A.O. and Rozora I.V. *Simulation of stochastic Processes and fields.*, "Zadruga Kyiv (2007) (in Ukrainian)

66. Kozachenko Yu. V. and Pashko A.O. *The accuracy of the simulation of random processes in norms Orlicz spaces I.* Theor. Probab. and Math. Statist. (Kiev) **58**, 45-60 (1988). (in Ukrainian)

67. Kozachenko Yu. V., Pashko A.O., Rozora I.V. *Simulation of random processes and fields.* "Zadruga Kyiv (2007). (in Ukrainian)

68. Kozachenko Yu. V., Pogoriliak O.O. *Simulation of Cox processes driven by random field.* Reports of national sience academia, № 10, 20-23 (2006). (in Ukrainian)

69. Kozachenko Yu. V., Pogoriliak O.O. *Simulation of log Gaussian Cox processes with given reliability and accuracy.* Theor. Probab. and Math. Statist., № 76, 58-71 (2007). (in Ukrainian)

70. Kozachenko Yu. V., Pogoriliak O.O. *About one of the method of simulation log Gaussian Cox processes.* Theor. Probab. and Math. Statist., № 77, 82-95 (2007). (in Ukrainian)

71. Kozachenko Yu. V., Pogoriliak O.O., Tegza A.M. *Simulation of Gaussian random processes and Cox processes.* "Karpaty Uzhgorod (2012). (in Ukrainian)

72. Kozachenko Yu. , Pogorilyak O., Rozora I., Tegza A., *Simulation of Stochastic Processes with Given Accuracy and Reliability*, ISTE Press Ltd London. Elsevier Ltd, Oxford, (2016)

73. Kozachenko, Yu., Rozora, I. *Simulation of Gaussian stochastic processes.* Random Operators Stoch. Equations, **11**, № 3, 275–296(2003)

74. Kozachenko Yu., Rozora I. *Simulation of Gaussian Stochastic Fields.* Theory of Stoch. Processes. **Vol.10 (26)**, N.1-2, 48-60 (2004)

75. Kozachenko Yu., Rozora I. *On simulation of stochastic processes from the space* $Sub_{\varphi}(\Omega)$. Applied Statist. and Financial Math, № 1, 72–79 (2004)

76. Kozachenko Yu., Rozora I. *Application of the theory of Square-Gaussian Processes to simulation of Stochastic Processes.* Theory of Stoch. Processes. **12(28)**, № 3-4, 43-54 (2006)

77. Kozachenko Yu.V, Rozora I.V. and Turchyn Ye.V. *On an expansion of random processes in series.* Random operators Stoch. Equations, **15**,15-33 (2007)

78. Kozachenko Yu.V., Sidorenko O.O. About one criteria for testing the hypothesis regarding the type of correlation functions of Gaussian stationary stochastic process // Proceedings of the Second Ukrainian-Hungarian Conference on New Directions of Probability Theory and Mathematical Statistics. - Mukachevo, 1992. - P. 179-186. (in Russian)

79. Yu. Kozachenko, A. Slyvka-Tylyshchak *The Cauchy Problem for the Heat Equation with a Random Right Part from the Space* $Sub_{\varphi}(\Omega)$, Applied Mathematics., Vol. 5, 2318–2333, (2014).

80. Kozachenko Yu., Stadnik A., *On the convergence of some functionals of Gaussian vectors in Orlicz spaces*, Theor. Probability and Math. Statist **44**, 80-87, (1991).

81. Kozachenko Yu. V., Stus O. V. *Square-Gaussian random processes and estimators of covariance functions.* Math.Communications, **Vol. 3**, № 1, 83–94 (1998)

82. Kozachenko Yu., Sottinen T. and Vasylyk O. *Simulation of weakly self-similar stationary increment $Sub_{\varphi}(\Omega)$-processes: a serias expansion approach.* Methodology and Computing in Applied Probability. **Vol. 7**, № 3, 379–400 (2005)

83. Kozachenko Yu., Troshki V., *A criterion for testing hypotheses about the covariance function of a stationary Gaussian stochastic process*, Modern Stochastics: Theory and Applications **2**, 1-11, (2015)

84. Kozachenko Yu. V., Troshki V. B. *Construction of a criterion for testing hypothesis about covariance function of a stationary Gaussian stochastic process with unknown mean*, Communications in Statistics - Theory and Methods, 2017, DOI: 10.1080/03610926.2017.1377253.

85. Kozachenko Yu.V., Troshki N.V. *Accuracy and reliability of a model of Gaussian random process in $C(\mathbb{T})$ space*, International Journal of Statistics and Management System, **10**, (2015), № 1-2, 1–15.

86. Kozachenko Yu., Fedoryanych T., *A criterion for testing hypotheses about the covarians function of a Gaussian stationary process*, Theor. Probability and Math. Statist **69**, 85-94, (2005)

87. Kozachenko Yu.V., Yadrenko M.Y. *Local properties of sample functions of random fields.* Teor. Veroyat. i Mat.Stat., **Vol. 15**, 82–98 (1976)

88. Krasnosel'skii M.A., Rutitsky Ya.B. *Convex functions and Orlicz space.* - M .: Fizmatgiz, - 271p., (1958) (in Russian)

89. Landau H.J., Shepp L.A. *On the supremum of Gaussian processes.* Sankhya, Ser. A. **Vol. 32**, № 4, 369–378 (1970)

90. Leonenko N., Ivanov A., *Statistical Analysis of Random Fields*, Kluwer, Dordrecht, (1989).

91. Leonov V.P. and Shiryaev A. N. *Sur le calcul des semi-invariants.* Theor. Probab. Appl, 319-329 (1960)

92. Li B., Genton M.G. and Sherman M., *Testing the covariance structure of multivariate random fields*, Biometrika, 95, 813-829, (2008)

93. Loeve M., *Probability Theory.*, D. van Nostrand, Princeton, NJ (1960)

94. Lucacz E., *Characteristic functions.*, 2nd ed., Griffin, London (1970)

95. Lund R., Bassily H. and Vidakovic B., *Testing equality of stationary autocovariances*, Journal of Time Series Analysis, 30, 332-348, (2009)

96. Mikhailov G.A. Numerical construction of a random field with a given spectral density, Dokl. AN SSSR. 1978. P. 238. N 4. P. 793-795.

97. Mikhailov G. A., *Simulation of random processes and fields on the basis of point flows Palma*, Dokl. USSR Academy of Sciences, **262**, No. 3, 531-535 (1982) (in Russian)

98. Mikhailov G.A., *Some questions of Monte Carlo theory and methods*, Science, Novosibirsk (1974) (in Russian)

99. Mikhailov G. A., Sabelfeld K.K. *On numerical modeling of impurity diffusion in stochastic velocity fields*, Izv. Academy of Sciences of the USSR, Ser. FAO. T.16, No. 3, , 229-235 (1980) (in Russian)

100. Mikhailov G.A., *Approximate models of random processes and fields*, Zh. Calculated. Mathematics and mathematics. Physics, Vol. 23, No. 3, 558-566 (1983) (in Russian)

101. Mihaylov G.A., Voitishek A. V. *Numerical statistical modelling. Monte-Carlo methods, textbook for university students.*, "Academiya Moscow (2006) (in Russian)

102. Macak I.K., Plychko A.N. *Some inequalities for sums of independent random variables in Banach spaces.* Theor. Probab. and Math. Statist. (Kiev) **38**, 81-86 (1988). (in Russian)

103. Moller J., Syversveen A.R., Waagepetersen R.P. *Log gaussian Cox Processes.* Scandinavian Journal of Statistics, – Vol. 25, № 4, P. 451-482. (1998)

104. Nanopoulos C. and Nobelis P. *Regularité et propriétés limites des fonctions aléatoires.* Lecture Notes in Math.,**vol. 49**, 567–690 (1976)

105. Nikolskiy S.M. *Approximation of multivariable functions.*, "Nauka Moscow (1977)

106. Ogorodnikov V.A., Prigarin S.M. *Numerical Modeling of Random Processes and Fields: Algorithms and Applications.* Utrecht: VSP (1996)

107. Ostrovskyy E.I. *Exponential estimates of maximum distribution of non-Gaussian random field.* Probab. Theory and its Appl., **vol. 35(3)**, 482–493 (1990)

108. Park M. S., Fuentes M., *Testing lack of symmetry in spatial-temporal processes*, Journal of Statistical Planning and Inference 138, 2847-2866, (2008)

109. Pisier G. *Conditions déntropie assurant la continuité de certains processus et applications a ĺanalyse harmonique.* Semin. Anal. Fonct, 13-24 (1979-1980)

110. Pogoriliak O.O. *Simulation of square Gaussian Cox processes when intensity are generated of inhomogeneous field.* Scientific Bulletin of the Uzhgorod University. Series mathematica and informatica. № 14, 95-102 (2007) (in Ukrainian)

111. Pogoriliak O.O. *Simulation of random Cox processes driven by square gaussian random field.* Scientific Bulletin of the Uzhgorod University. Series mathematica and informatica. № 18, 113-120 (2009) (in Ukrainian)

112. Pogoriliak O.O., Tegza A.M. *On the simulation method of square gaussian Cox processes.* Scientific Bulletin of the Uzhgorod University. Series mathematica and informatica. № 22, 131-137 (2011) (in Ukrainian)

113. Prigarin S. M. *Convergence of numerical models of random fields in Monte Carlo methods.* Russian Journal of Numerical Analysis and Math.Modelling. **vol. 7(5)**, 441–456 (1992)

114. Prigarin S. M. *Numerical models of random processes and fields and some applications in Monte Carlo methods.* Russian Journal of Numerical Analysis and Math.Modelling. **vol. 9(2)**, 147–177 (1994)

115. Prigarin S. M. *Spectral models of random fields and their applications.Advences in modelling and analysis.* **vol. A 29(2)**, 39–51(1995)

116. Rakhimov G.M. On the estimation of the correlation function of homogeneous and isotropic field on a sphere. I, II // Theory Probab. and Math. Statistics. - No. 28.(in Russian) - 1983. - P. 112-1-115.

117. Revenko A.O. Construction of the criterion for checking hypotheses relative to the covariance function of homogeneous and isotropic Gaussian random field. // Theory Probab. and Math. Statistics. - 2000. -No. 62. - P. 124-133.(in Ukrainian)

118. Ripley B.D. *Stochastic Simulation.* Wiley, New York (1987)

119. Ross, Sheldom M. *Simulation.* Elsevier Inc. (2006)

120. Scaccia L., Martin R. J., *Testing axial symmetry and separability of lattice processes*, Journal of Statistical Planning and Inference 131, 19-39, (2005)

121. Shinuzuky M. *Simulation of Multivariate Multidimensional Random Processes.* J.Acoust.Soc.Am., 357–367 (1971)

122. Shumway R.H., *Time-frequency clustering and discriminant analysis*, Statistics and Probability Letters, 24, 307-314, (2006)

123. Taheriyoun A.R., *Testing the covariance function of stationary Gaussian random fields*, Statistics and Probability Letters 82, 606-613, (2012)

124. Tegza A.M. *On the accuracy and reliability of some models of Gaussian processes with a limited range.* Scientific Bulletin of the Uzhgorod University. **Vol. 6**, 125-131 (2001) (in Ukrainian)

125. Tegza A.M. *Finding the accuracy and reliability of models Gaussian processes with continuous spectrum.* Bulletin of the University of Kiev, Series: Physics & Mathem., 4, p. 38-43, 2002 (in Ukrainian)

126. Tegza A.M. *Simulating of Gaus-sian stationary process with given accuracy in C([0,T]).* Theory of Stochastic Processes., Vol. 10(26) p.172-177, 2004

127. Tegza A.M. *Models Gaussian stationary random process in space* $L_p([0,T])$, $p > 1$. Bulletin of the University of Kiev, Series: Physics & Mathem., 4, p.17-21, 2004

128. Tegza A.M. *Some estimates for the construction of a generalized model of Gaussian stationary processes.* Scientific Bulletin of the Uzhgorod University. Series math. and inform., 21, p.137-145, 2010

129. Teugels Jozef L., Sundt Bjorn. *Poisson Processes. Encyclopedia of actuarial science.* Vol. 3. Wiley, (2004)

130. Trikomi H.F. *Integral equations.* Moscow, (1960).

131. Troshki N. *Construction models of Gaussian random processes with a given accuracy and reliability in* $L_p(\mathbb{T}), p \geqslant 1$, J. Classical Anal., **3**, № 2, 157-165, (2013)

132. Troshki N. V textit Construction of models of Gaussian random fields with given reliability and accuracy in $L_p(mathbbT), p geq 1$, Applied statistics. Actuarial and Financial Mathematics, No. 1-2, 149-156 (2013). (in Ukranian)

133. Troshki N. *Accuracy and reliability of a model for a Gaussian homogeneous and isotropic random field in the space* $L_p(\mathbb{T}), p \geqslant 1$ Theory of Probability and Mathematical Statistics, V. 90, 183-200 (2015)

134. Troshki N. *Upper bounds for supremums of the norms of the deviation between a homogeneous isotropic random fields and its model,* Theory of Probability and Mathematical Statistics, V. 94, 159-184 (2017).

135. Vasylyk O.I., Kozachenko Yu.V., Yamnenko R.E. *φ-sub-Gaussian random processes.* "Kyivskyi Universytet Kyiv (2008)

136. Vladimirov V. *Mathematical physics equations.* "Nauka Mascow (1967) (in Russian)

137. Vojtyshek A.V. *Weak convergence investigation of models of gaussian random field with given correlation function spectral decomposition.* Simulation on the calculated systems. 119-129 (1982)

138. Vojtyshek A.V. *Randomization numerical spectral model of homogeneous random function.* Mathematical and imitation models of systems. 17-25 (1983)

139. Z. O. Vyzhva *On Aprocsimation of 3-D Isotropic Random Fields on the Sphere and Statistical Simulation*, Theory of Stochastic Processes, **3**, (1997), №3-4, 463-467.

140. Vyzhva Z.O. *About Approximation of 3-D Random Fields and Statistical Simulation.* Random Operators Stoch. Equations. **Vol. 4**, № 3,255–266 (2003)

141. M. Yadrenko, *Spectral theory of random field*, Kyiv, "Vyscha shkola"(1980)

142. Yadrenko M. Y., Rakhimov G. K., *Statistic of the model of one-dimensional and anisotropic field on a plane*, Theory of functions and mathematical statistics. V. 49, 245-251 (1993) (in Russian)

143. Yadrenko M.Y., Yadrenko O.M., Grikh Z.O. *About Approximation and Statistical Simulation of Izotropic Fields.* Random Operators Stoch. Equations,**Vol. 1**, № 1, 37–45 (1993)

144. Yaglom A.M. Correlation theory of stationary random fields. - L .: Gidrometeoizdat, 1981. - 280 p. (in Russian)

145. Zelepugina I.I., Kozachenko Yu.V. *About estimates oof accuracy of simulation of random fields in the sapce L_p.* Operations research and ACS, N 32 (1988) (in Russian)

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