Proceedings of the 19th European Young Statisticians Meeting

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Partners

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Preface

The first European Young Statisticians Meeting was organized in 1978 (Wiltshire, Great Britain), the second one in 1981 (Bressanone, Italy), and since then regularly every two years in different European countries.

From the very beginning the idea of the event is that young researchers from different countries come together and establish new research contacts at the beginning of their scientific careers.

In line with previous meetings, each of the representatives from selected European countries suggested at most two young researchers to participate in the Meeting. Also, five distinguished researchers have been invited to give plenary lectures.

We hope that you find the Meeting interesting and useful.

Welcome to the 19th EYSM 2015 in Prague. Enjoy the city, its history, its architecture and culture, and have a great time!

Local Organizing Committee
Prague, July 2015

On Calculation of the Integrals Depending on a Parameter by Monte-Carlo Method

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Abstract: In this work we use the theory $\mathbf{F}_\psi(\Omega)$ spaces in order to find the accuracy and reliability for the calculation of the improper integrals depending on a parameter t by Monte Carlo method.

Keywords: $\mathbf{F}_\psi(\Omega)$ space of random variables, condition **H**, Monte Carlo method, stochastic process

AMS subject classifications: 65C05, 60G07

1 Introduction

In this paper we developed a theory for finding the reliability and accuracy for the calculation of integrals depending on a parameter by Monte-Carlo method in $L_p(T)$ metrics.

There are many works devoted to the usage of the Monte-Carlo methods for calculation of integrals. Among them are the books by Yermakov [1] and Yermakov & Mikhailov [2].

The paper by Kurbanmuradov & Sabelfeld [7] contains the estimate for the accuracy in the space $C(T)$ and reliability for the calculation of integrals depending on a parameter if the set of integration is bounded. To obtain these results the theory of sub-Gaussian processes had been used.

The space $\mathbf{F}_\psi(\Omega)$ was introduced by Yermakov & Ostrovsky in the paper [3]. The paper [5] is devoted to studying the properties of such spaces and there had been found the conditions of fulfilling the condition **H** (see Definition 10) in this spaces. The condition **H** is necessary for finding the reliability and accuracy when we calculate integrals by Monte-Carlo method.

The choice of the space depends on particular integral and allows to find better accuracy. In this paper, the accuracy is defined via the norm in $L_p(T)$ space.

2 $\mathbf{F}_\psi(\Omega)$ – space

Definition 9. [6] Let $\psi(u) > 0$, $u \geq 1$ be monotonically increasing, continuous function for which $\psi(u) \to \infty$ as $u \to \infty$. A random variable ξ belongs to the space $\mathbf{F}_\psi(\Omega)$ if

$$\sup_{u\geq 1} \frac{\left(E\left|\xi\right|^u\right)^{1/u}}{\psi(u)} < \infty.$$

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The similar definition was formulated in the paper by S. M. Yermakov & Ye. I. Ostrovskii [3]. But there was required that $E\xi = 0$ as $\xi \in \mathbf{F}_\psi(\Omega)$. Moreover, there were considered the random variables for which $E|\xi|^u = \infty$ for some $u > 0$.

It had been proved in [3] that $\mathbf{F}_\psi(\Omega)$ is a Banach space with a norm

$$\|\xi\|_\psi = \sup_{u \geq 1} \frac{(E|\xi|^u)^{1/u}}{\psi(u)}.$$

Theorem 20. *[6] If a random variable ξ belongs to the space $\mathbf{F}_\psi(\Omega)$, then for any $\varepsilon > 0$ the following inequality holds true:*

$$P\{|\xi| > \varepsilon\} \leq \inf_{u \geq 1} \frac{\|\xi\|_\psi^u (\psi(u))^u}{\varepsilon^u}.$$

Theorem 21. *[6] If a random variable ξ belongs to the space $\mathbf{F}_\psi(\Omega)$ and $\psi(u) = u^\alpha$, where $\alpha > 0$, then for any $\varepsilon \geq e^\alpha \|\xi\|_\psi$ the following inequality is true:*

$$P\{|\xi| > \varepsilon\} \leq \exp\left\{-\frac{\alpha}{e}\left(\frac{\varepsilon}{\|\xi\|_\psi}\right)^{1/\alpha}\right\}.$$

Definition 10. [5] We say that the condition **H** for the Banach spaces $B(\Omega)$ of random variables is fulfilled if there exists such an absolute constant C_B that for any centered and independent random variables $\xi_1, \xi_2, \ldots, \xi_n$ from $B(\Omega)$ the following is true:

$$\left\|\sum_{i=1}^n \xi_i\right\|^2 \leq C_B \sum_{i=1}^n \|\xi_i\|^2.$$

The constant C_B is called a scale constant for the space $B(\Omega)$. For space $\mathbf{F}_\psi(\Omega)$ we shall denote the constants $C_{\mathbf{F}_\psi(\Omega)}$ as C_ψ.

Theorem 22. *[8] For the space $\mathbf{F}_\psi(\Omega)$, where $\psi(u) = u^\alpha$, $\alpha \geq \frac{1}{2}$ the condition **H** is fulfilled and it is true the following inequality:*

$$\left\|\sum_{i=1}^n \xi_i\right\|_\psi^2 \leq 4 \cdot 9^\alpha \sum_{i=1}^n \|\xi_i\|_\psi^2.$$

*Note, that when $\alpha < \frac{1}{2}$ then the condition **H** is not fulfilled for this space.*

3 Estimates in the norm $L_p(T)$ for the stochastic processes from the spaces $\mathbf{F}_\psi(\Omega)$

Theorem 23. *Let ν be the σ-finite measure on the compact metric space (T, ρ) and $X = \{X(t),\ t \in T\}$ be a measurable stochastic process from the space $\mathbf{F}_\psi(\Omega)$. If for some $p \geq 1$ the following condition is true*

$$\int_T \|X(t)\|_\psi^p \, d\nu(t) < \infty,$$

then

1) the integral $\int_T |X(t)|^p \, d\nu(t)$ *exists with probability one and the inequality holds true:*

$$\left\| \left(\int_T |X(t)|^p \, d\nu(t) \right)^{1/p} \right\|_\psi \leq \frac{\psi(p)}{\psi(1)} \left(\int_T \|X(t)\|_\psi^p \, d\nu(t) \right)^{1/p};$$

2) for any $\varepsilon > 0$ *the following inequality holds:*

$$P\left\{ \left(\int_T |X(t)|^p \, d\nu(t) \right)^{1/p} > \varepsilon \right\} \leq$$

$$\leq \inf_{u \geq 1} \frac{\left(\frac{\psi(p)}{\psi(1)}\right)^u \left(\int_T \|X(t)\|_\psi^p \, d\nu(t) \right)^{u/p} (\psi(u))^u}{\varepsilon^u}.$$

Example 1. *Consider the space* $\mathbf{F}_\psi(\Omega)$*, where* $\psi(u) = u^\alpha$*,* $\alpha > 0$*. It follows from the Theorems 23 and 21 that for* $\varepsilon \geq (ep)^\alpha \left(\int_T \|X(t)\|_\psi^p \, d\nu(t) \right)^{1/p}$

$$P\left\{ \left(\int_T |X(t)|^p \, d\nu(t) \right)^{1/p} > \varepsilon \right\} \leq$$

$$\leq \exp\left\{ -\frac{\alpha}{ep} \left(\frac{\varepsilon}{\left(\int_T \|X(t)\|_\psi^p \, d\nu(t) \right)^{1/p}} \right)^{1/\alpha} \right\}.$$

Theorem 24. *Let* ν *be a* σ*-finite measure on a compact metric* (T, ρ) *and* $Y = \{Y(t),\ t \in T\}$ *be the stochastic process from the space* $\mathbf{F}_\psi(\Omega)$ *and the condition* **H** *is fulfilled for this space with the constant* C_ψ*. Let* $EY(t) = m(t)$*,* $Z_n(t) = \frac{1}{n}\sum_{k=1}^{n} Y_k(t) - m(t) = \frac{1}{n}\sum_{k=1}^{n} (Y_k(t) - m(t))$*, where* $Y_k(t)$ *are the independent copies of* $Y(t)$*. Then the following inequality holds for all* $p \geq 1$

$$\left\| \left(\int_T |Z_n(t)|^p \, d\nu(t) \right)^{1/p} \right\|_\psi \leq \frac{2\sqrt{C_\psi}}{\sqrt{n}} \cdot \frac{\psi(p)}{\psi(1)} \left(\int_T \|Y(t)\|_\psi^p \, d\nu(t) \right)^{1/p}$$

and for every $\varepsilon > 0$ *the following estimate is true*

$$P\left\{\left(\int_T |Z_n(t)|^p \, d\nu(t)\right)^{1/p} > \varepsilon\right\} \leq$$

$$\leq \inf_{u\geq 1} \frac{\left(\frac{2\sqrt{C_\psi}}{\sqrt{n}} \cdot \frac{\psi(p)}{\psi(1)}\right)^u \left(\int_T \|Y(t)\|_\psi^p \, d\nu(t)\right)^{u/p} (\psi(u))^u}{\varepsilon^u}.$$

Example 2. *Let us consider the space* $\mathbf{F}_\psi(\Omega)$*, where* $\psi(u) = u^\alpha$*,* $\alpha > 0$ *then it follows from the Theorems 24 and 21 that if* $\varepsilon \geq (ep)^\alpha \frac{2\sqrt{C_\psi}}{\sqrt{n}} \left(\int_T \|Y(t)\|_\psi^p \, d\nu(t)\right)^{1/p}$

$$P\left\{\left(\int_T |Z_n(t)|^p \, d\nu(t)\right)^{1/p} > \varepsilon\right\} \leq$$

$$\leq \exp\left\{-\frac{\alpha}{ep}\left(\frac{\varepsilon}{\frac{2\sqrt{C_\psi}}{\sqrt{n}}\left(\int_T \|Y(t)\|_\psi^p \, d\nu(t)\right)^{1/p}}\right)^{1/\alpha}\right\}.$$

4 Reliability and accuracy in the space $L_p(T)$ for the calculation of integrals depending on a parameter

Let $\{\mathcal{S}, \mathcal{A}, \mu\}$ be a measurable space, μ be a σ-finite measure and $p(s) \geq 0, s \in \mathcal{S}$ be such measurable function that $\int_{\mathcal{S}} p(s)d\mu(s) = 1$. Let $m(A)$, $A \in \mathcal{A}$ be the measure $m(A) = \int_A p(s)d\mu(s)$. $m(A)$ is a probability measure and the space $\{\mathcal{S}, \mathcal{A}, m\}$ is a probability space.

Let us consider the integral $\int_{\mathcal{S}} f(s,t)p(s)d\mu(s) = I(t)$ assuming that it exists. Let the function $f(s,t)$ depend on the parameter $t \in T$, where (T, ρ) is some compact set and the function $f(s,t)$ be continuous with regard to t.

Suppose $f(s,t)$ is the stochastic process on $\{\mathcal{S}, \mathcal{A}, m\}$ and we denote it as $\xi(s,t) = \xi(t)$ and $I(t) = \int_{\mathcal{S}} f(s,t)p(s)d\mu(s) = \int_{\mathcal{S}} f(s,t)dm(s) = E\xi(t)$.

Let $\xi_i(t)$, $i = 1, 2, \ldots, n$ be the independent copies of the stochastic process $\xi(t)$, $Z_n(t) = \frac{1}{n}\sum_{i=1}^{n} \xi_i(t)$. So, according to the strong law of large numbers $Z_n(t) \to E\xi(t) = I(t)$ with probability one for any $t \in T$.

Definition 11. We say that $Z_n(t)$ approximates $I(t)$ in the space $L_p(T)$ with reliability $1-\delta>0$ and accuracy $\varepsilon>0$ if the following inequality holds true:

$$P\left\{\left(\int_T |Z_n(t)-I(t)|^p\, d\mu(t)\right)^{1/p} > \varepsilon\right\} \le \delta.$$

Theorem 25. *Let $I(t)=E\xi(t)=\int_S f(s,t)p(s)d\mu(s)$, $\xi(t)$ be the stochastic process which belongs to the space $\mathbf{F}_\psi(\Omega)$ satisfying the condition $\mathbf{H}$ with constant C_ψ, $\widetilde{Z}_n(t)=\frac{1}{n}\sum_{i=1}^{n}(\xi_i(t)-I(t))$, $\xi_i(t)$ be the independent copies of the stochastic process $\xi(t)$.*

Then, for all $p\ge 1$ the following inequality holds true

$$\left\|\left(\int_T \left|\widetilde{Z}_n(t)\right|^p d\mu(t)\right)^{1/p}\right\| \le \frac{2\sqrt{C_\psi}}{\sqrt{n}}\cdot\frac{\psi(p)}{\psi(1)}\left(\int_T \|\xi(t)\|_\psi^p\, d\mu(t)\right)^{1/p},$$

and $\widetilde{Z}_n(t)$ approximates $I(t)$ with reliability $1-\delta$ and accuracy ε in the space $L_p(T)$ for such n that

$$\inf_{u\ge 1}\frac{\left(\frac{2\sqrt{C_\psi}}{\sqrt{n}}\cdot\frac{\psi(p)}{\psi(1)}\right)^u\left(\int_T \|\xi(t)\|^p\, d\mu(t)\right)^{u/p}(\psi(u))^u}{\varepsilon^u} \le \delta. \qquad (1)$$

Example 3. *Consider the space $\mathbf{F}_\psi(\Omega)$, where $\psi(u)=u^\alpha$, $\alpha>\frac{1}{2}$. Then the Theorem 22 implies that the condition $\mathbf{H}$ is fulfilled for this space with the constant $C_\psi=4\cdot 9^\alpha$. It follows from the Example 2 and the Theorem 25 that if $\varepsilon\ge \frac{4(3pe)^\alpha\left(\int_T \|\xi(t)\|_\psi^p d\mu(t)\right)^{1/p}}{\sqrt{n}}$, then*

$$\inf_{u\ge 1}\frac{\left(\frac{2\sqrt{C_\psi}}{\sqrt{n}}\cdot\frac{\psi(p)}{\psi(1)}\right)^u\left(\int_T \|\xi(t)\|^p\, d\mu(t)\right)^{u/p}(\psi(u))^u}{\varepsilon^u} \le$$

$$\le \exp\left\{-\frac{\alpha}{e}\left(\frac{\sqrt{n}\varepsilon}{4(3pe)^\alpha\left(\int_T \|\xi(t)\|_\psi^p\, d\mu(t)\right)^{1/p}}\right)^{1/\alpha}\right\}.$$

So, the inequality (1) holds if it is true that

$$\exp\left\{-\frac{\alpha}{e}\left(\frac{\sqrt{n}\varepsilon}{4(3pe)^\alpha\left(\int_T \|\xi(t)\|_\psi^p\, d\mu(t)\right)^{1/p}}\right)^{1/\alpha}\right\} \le \delta,$$

as

$$n \geq \left(\frac{4(3pe)^{\alpha} \left(\int\limits_{T} \|\xi(t)\|_{\psi}^{p} \, d\mu(t) \right)^{1/p}}{\varepsilon} \right)^{2} \left((-\ln \delta) \frac{e}{\alpha} \right)^{2\alpha}.$$

Then

$$n \geq \left(\frac{4(3p)^{\alpha} \left(\int\limits_{T} \|\xi(t)\|_{\psi}^{p} \, d\mu(t) \right)^{1/p}}{\varepsilon} \right)^{2} \max \left(1, \left(-\frac{\ln \delta}{\alpha} \right)^{2\alpha} \right).$$

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