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AN APPLICATION OF THE THEORY OF SPACES $\mathbf{F}_\psi(\Omega)$ FOR EVALUATING MULTIPLE INTEGRALS BY USING THE MONTE CARLO METHOD

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Abstract. The reliability and accuracy in the space $C(T)$ of the Monte Carlo method for evaluating multiple integrals are established.

1. Introduction

This paper is a continuation of [1] and [2]. It is shown in the paper [2] how general results on the spaces $\mathbf{F}_\psi(\Omega)$ can be applied for finding the reliability and accuracy in the uniform metric for the evaluation of integrals that depend on a parameter with the help of the Monte Carlo method. In contrast to the paper [1] where the theory of Orlicz spaces has been used, we apply the theory of the $\mathbf{F}_\psi(\Omega)$ spaces in the current paper and estimate the accuracy in terms of the norm in the space $C(T)$.

The space $\mathbf{F}_\psi(\Omega)$ was introduced by Ermakov and Ostrovskiĭ in [3]. Properties of these spaces were studied in [2]. In particular, sufficient conditions are found in [2] under which the so-called condition $\mathbf{H}$ holds for these spaces. Recall that condition $\mathbf{H}$ is used for finding the accuracy and reliability of the evaluation of integrals with the help of the Monte Carlo method.

This paper contains three main sections. Necessary definitions and results of the theory of the spaces $\mathbf{F}_\psi(\Omega)$ are given in Section 2. Some bounds for the distribution of supremums of stochastic processes belonging to the spaces $\mathbf{F}_\psi(\Omega)$ are considered in Section 3. Section 4 contains some estimates of the reliability and accuracy in $C(T)$ for the evaluation of integrals depending on a parameter with the help of the Monte Carlo method. Some examples are also considered in Section 4.

2. $\mathbf{F}_\psi(\Omega)$-spaces

Definition 2.1 ([2]). Let $\psi(u) > 0$, $u \ge 1$, be an increasing continuous function such that $\psi(u) \to \infty$ as $u \to \infty$. We say that a random variable ξ belongs to the space $\mathbf{F}_\psi(\Omega)$ if

$$\sup_{u\ge 1} \frac{\left(\mathsf{E}\,|\xi|^u\right)^{1/u}}{\psi(u)} < \infty.$$

A similar definition was introduced by Ermakov and Ostrovskiĭ in [3] with an extra assumption that $\mathsf{E}\,\xi = 0$. The random variables such that $\mathsf{E}\,|\xi|^u = \infty$ for some $u > 0$ are also considered in [3].

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The proof of the property that $\mathbf{F}_\psi(\Omega)$ is a Banach space for the norm

$$\|\xi\|_\psi = \sup_{u\ge 1} \frac{\left(\mathsf{E}\,|\xi|^u\right)^{1/u}}{\psi(u)}$$

is the same in our case as that in the paper [3].

Some examples of random variables belonging to the corresponding spaces $\mathbf{F}_\psi(\Omega)$ are provided below.

Example 2.1. According to Definition 2.1, a random variable ξ such that $|\xi| < C$ with probability one, where $C > 0$ is a nonrandom constant, belongs to the space $\mathbf{F}_\psi(\Omega)$ generated by any function ψ, since

$$\|\xi\|_\psi = \sup_{u\ge 1} \frac{\left(\mathsf{E}\,|\xi|^u\right)^{1/u}}{\psi(u)} \le \sup_{u\ge 1} \frac{\left(C^u\right)^{1/u}}{\psi(u)} = \sup_{u\ge 1} \frac{C}{\psi(u)} = \frac{C}{\psi(1)}.$$

Example 2.2. A random variable with the Laplace distribution (a random variable with the probability density $p(x) = \frac{1}{2}e^{-|x|}$) belongs to the space $\mathbf{F}_\psi(\Omega)$, where $\psi(u) = u$, since

$$\sqrt[k]{\mathsf{E}\,|\xi|^k} = \sqrt[k]{k!} \sim k \quad \text{as } k \to \infty.$$

Example 2.3. A normal random variable $\xi = N(0,1)$ belongs to the space $\mathbf{F}_\psi(\Omega)$, where $\psi(u) = u^{1/2}$, since

$$\sqrt[2l]{\mathsf{E}\,|\xi|^{2l}} = \sqrt[2l]{\frac{(2l)!}{2^l l!}} \sim l^{1/2} \quad \text{as } l \to \infty.$$

Theorem 2.1 ([2]). *Let a random variable ξ belong to the space $\mathbf{F}_\psi(\Omega)$. Then*

$$\mathsf{P}\left\{|\xi| > \varepsilon\right\} \le \inf_{u\ge 1} \frac{\|\xi\|_\psi^u (\psi(u))^u}{\varepsilon^u}$$

for all $\varepsilon > 0$.

Theorem 2.2. *Let a random variable ξ belong to the space $\mathbf{F}_\psi(\Omega)$ and $\psi(u) = u^\alpha$ for $\alpha > 0$. Then*

$$\mathsf{P}\left\{|\xi| > \varepsilon\right\} \le \exp\left\{ -\frac{\alpha}{e}\left(\frac{\varepsilon}{\|\xi\|_\psi}\right)^{1/\alpha}\right\}$$

for all $\varepsilon \ge e^\alpha \|\xi\|_\psi$.

Proof. Using Theorem 2.1 we obtain

$$\mathsf{P}\left\{|\xi| > \varepsilon\right\} \le \inf_{u\ge 1} \frac{\|\xi\|_\psi^u u^{\alpha u}}{\varepsilon^u}. \tag{1}$$

Put $\|\xi\|_\psi/\varepsilon = b$. Then the equalities

$$\left(\ln\left(b^u u^{\alpha u}\right)\right)' = (u\ln b + \alpha u \ln u)' = \ln b + \alpha \ln u + \alpha = 0,$$
$$\ln u = -\frac{\ln b + \alpha}{\alpha}$$

imply that the infimum is attained at the point $u = \frac{1}{e}b^{-1/\alpha}$. Since $u \ge 1$, we get $\varepsilon \ge e^\alpha \|\xi\|_\psi$. Substituting this value for u in inequality (1) yields

$$\mathsf{P}\left\{|\xi| > \varepsilon\right\} \le b^{\frac{1}{e}b^{-1/\alpha}} \left(\frac{1}{e}b^{-\frac{1}{\alpha}}\right)^{\alpha \frac{1}{e} b^{-1/\alpha}} = \exp\left\{-\frac{\alpha}{e}\left(\frac{1}{b}\right)^{1/\alpha}\right\},$$

whence Theorem 2.2 follows. ∎

Theorem 2.3. *Let a random variable ξ belong to the space $\mathbf{F}_\psi(\Omega)$ and $\psi(u) = e^{au^\beta}$, where $a > 0$ and $\beta > 0$. Then*

$$\mathsf{P}\{|\xi| > \varepsilon\} \leq \exp\left\{-\frac{\beta}{a^{1/\beta}}\left(\frac{\ln\frac{\varepsilon}{\|\xi\|_\psi}}{\beta+1}\right)^{\frac{\beta+1}{\beta}}\right\}$$

for all $\varepsilon \geq e^{a(\beta+1)}\|\xi\|_\psi$.

Proof. Theorem 2.1 implies

$$\mathsf{P}\{|\xi| > \varepsilon\} \leq \inf_{u\geq 1}\frac{\|\xi\|_\psi^u e^{au^{\beta+1}}}{\varepsilon^u}. \tag{2}$$

We have $\|\xi\|_\psi/\varepsilon = b$. Then

$$\left(\ln\left(b^u e^{au^{\beta+1}}\right)\right)' = \left(u\ln b + au^{\beta+1}\right)' = \ln b + a(\beta+1)u^\beta = 0,$$

$$u^\beta = -\frac{\ln b}{a(\beta+1)}$$

implies that the infimum is attained at the point $u = \left(-\frac{\ln b}{a(\beta+1)}\right)^{1/\beta}$. Since $u \geq 1$, we have $\varepsilon \geq e^{a(\beta+1)}\|\xi\|_\psi$. Substituting this value for u in inequality (2) yields

$$\mathsf{P}\{|\xi| > \varepsilon\} \leq b^{\left(-\frac{\ln b}{a(\beta+1)}\right)^{1/\beta}} e^{a\left(-\frac{\ln b}{a(\beta+1)}\right)^{\frac{\beta+1}{\beta}}} = \exp\left\{-\frac{\beta}{a^{1/\beta}}\left(\frac{\ln\frac{1}{b}}{\beta+1}\right)^{\frac{\beta+1}{\beta}}\right\},$$

which has to be proved. □

Theorem 2.4. *Let a random variable ξ belong to the space $\mathbf{F}_\psi(\Omega)$ and*

$$\psi(u) = (\ln(u+1))^\lambda, \quad \text{where } \lambda > 0.$$

Then

$$\mathsf{P}\{|\xi| > \varepsilon\} \leq e^\lambda \exp\left\{-\lambda\exp\left\{\left(\frac{\varepsilon}{\|\xi\|_\psi}\right)^{1/\lambda}\frac{1}{e}\right\}\right\}$$

for all $\varepsilon \geq (e\ln 2)^\lambda\|\xi\|_\psi$.

Proof. Theorem 2.1 implies

$$\mathsf{P}\{|\xi| > \varepsilon\} \leq \inf_{u\geq 1}\frac{\|\xi\|_\psi^u(\ln(u+1))^{\lambda u}}{\varepsilon^u}. \tag{3}$$

We have

$$u+1 = \exp\left\{\left(\frac{\varepsilon}{\|\xi\|_\psi}\right)^{1/\lambda}\frac{1}{z}\right\},$$

where $z > 0$. Substituting this expression in inequality (3) yields

$$\begin{aligned}\left(\frac{\|\xi\|_\psi(\ln(u+1))^\lambda}{\varepsilon}\right)^u &= \frac{1}{z^{\lambda u}} = \exp\{-\lambda u\ln z\}\\ &= \exp\left\{-\lambda(\ln z)\left(\exp\left\{\left(\frac{\varepsilon}{\|\xi\|_\psi}\right)^{1/\lambda}\frac{1}{z}\right\}-1\right)\right\}\\ &= z^\lambda\exp\left\{-\lambda(\ln z)\exp\left\{\left(\frac{\varepsilon}{\|\xi\|_\psi}\right)^{1/\lambda}\frac{1}{z}\right\}\right\}.\end{aligned}$$

Consider the latter equality for $z = e$. Since $u \geq 1$, Theorem 2.4 follows in view of

$$\mathsf{P}\{|\xi| > \varepsilon\} \leq e^{\lambda} \exp\left\{-\lambda \exp\left\{\left(\frac{\varepsilon}{\|\xi\|_{\psi}}\right)^{1/\lambda} \frac{1}{e}\right\}\right\}. \qquad \square$$

Definition 2.2 ([2]). A nondecreasing sequence $(\varkappa(n) > 0, n \geq 1)$ is called an M-characteristic (majorant characteristic) of the space $\mathbf{F}_{\psi}(\Omega)$ if

$$\left\|\max_{1 \leq i \leq n} |\xi_i|\right\|_{\psi} \leq \varkappa(n) \max_{1 \leq i \leq n} \|\xi_i\|_{\psi}$$

for all random variables ξ_i, $i = 1, 2, \ldots, n$, belonging to this space.

Theorem 2.5 ([2]). *The sequence*

$$\varkappa(n) = \sup_{u \geq 1} \inf_{v > 0} n^{\frac{1}{u+v}} \frac{\psi(u+v)}{\psi(u)}$$

is a majorant characteristic of the space $\mathbf{F}_{\psi}(\Omega)$.

Majorant characteristics are found in the papers [2] and [4] for the space $\mathbf{F}_{\psi}(\Omega)$ generated by the functions $\psi(u) = u^{\alpha}$, $\psi(u) = e^{au^{\beta}}$, and $\psi(u) = (\ln(u+1))^{\lambda}$. In particular, the sequence

$$\varkappa(n) = \left(\frac{e}{\alpha}\right)^{\alpha} (\ln n)^{\alpha}, \qquad n > 1,$$

is a majorant characteristic of the space $\mathbf{F}_{\psi}(\Omega)$ where $\psi(u) = u^{\alpha}$, $\alpha > 0$.

Definition 2.3 ([2]). We say that condition $\mathbf{H}$ holds for the space of random variables $\mathbf{F}_{\psi}(\Omega)$ if there exists an absolute constant C_{ψ} such that

$$\left\|\sum_{i=1}^{n} \xi_i\right\|^2 \leq C_{\psi} \sum_{i=1}^{n} \|\xi_i\|^2$$

for all jointly independent centered random variables $\xi_1, \xi_2, \ldots, \xi_n$ belonging to the space $\mathbf{F}_{\psi}(\Omega)$.

The constant C_{ψ} in Definition 2.3 is called the scale constant of the space $\mathbf{F}_{\psi}(\Omega)$.

3. Stochastic processes belonging to the spaces $\mathbf{F}_{\psi}(\Omega)$ that are defined on a compact set

Definition 3.1 ([5]). We say that a stochastic process $X = \{X(t), t \in T\}$, where T is a certain set of parameters, belongs to the space $\mathbf{F}_{\psi}(\Omega)$ if the random variable $X(t)$ belongs to the space $\mathbf{F}_{\psi}(\Omega)$ whenever $t \in T$.

Examples of stochastic processes of the space $\mathbf{F}_{\psi}(\Omega)$ can be found in the paper [6].

Definition 3.2 ([5]). The minimal number of closed balls whose radii do not exceed a number u and that cover a set T is called the metric massiveness of the metric space (T, ρ) and is denoted by $N(u)$.

Theorem 3.1. *Let (T, ρ) be a compact metric space, $Y = \{Y(t), t \in T\}$ a stochastic process that belongs to the space $\mathbf{F}_{\psi}(\Omega)$ and for which condition $\mathbf{H}$ holds with a constant C_{ψ}, and let Y be a separable process in (T, ρ). Moreover, let there exist an increasing continuous function $\sigma(h)$ such that $\sigma(0) = 0$ and*

$$\sup_{\rho(t,s) \leq h} \|Y(t) - Y(s)\|_{\psi} \leq \sigma(h).$$

Assume further that

$$\int_0^z \varkappa\left(N\left(\sigma^{(-1)}(u)\right)\right) du < \infty$$

for all $z > 0$*, where* $\varkappa(n)$ *is a majorant characteristic of the space* $\mathbf{F}_\psi(\Omega)$*.*

Let $X(t) = Y(t) - m(t)$*, where* $m(t) = \mathsf{E}\, X(t)$ *and* $X_k(t)$ *are independent copies of the process* $X(t)$*,* $\sigma_1(t) = 2\sqrt{C_\psi}\sigma(t)$*, and*

$$S_n(t) = \frac{1}{n}\sum_{k=1}^n X_k(t).$$

Then

$$\left\|\sup_{t\in T}|S_n(t)|\right\|_\psi \le \frac{1}{\sqrt{n}}B(p)$$

for all $0 < p < 1$*, where*

$$B(p) = 2\sqrt{C_\psi}\inf_{t\in T}\|Y(t)\|_\psi + \frac{1}{p(1-p)}\int_0^{\gamma p}\varkappa\left(N\left(\sigma_1^{(-1)}(u)\right)\right) du,$$
$$\gamma = \sigma_1\left(\sup_{t,s\in T}\rho(t,s)\right).$$

Moreover,

$$\mathsf{P}\left\{\sup_{t\in T}|S_n(t)| > \varepsilon\right\} \le \inf_{u\ge 1}\left(\frac{B(p)\psi(u)}{\varepsilon\sqrt{n}}\right)^u \tag{4}$$

for all $\varepsilon > 0$*.*

Proof. Theorem 3.1 follows from Theorem 4.1 and Corollary 4.1 of the paper [2]. □

Example 3.1. Consider the space $\mathbf{F}_\psi(\Omega)$ for $\psi(u) = u^\alpha$, $\alpha > 0$. Then Theorems 3.1 and 2.2 with $\varepsilon \ge \frac{e^\alpha B(p)}{\sqrt{n}}$ imply

$$\mathsf{P}\left\{\sup_{t\in T}|S_n(t)| > \varepsilon\right\} \le \exp\left\{-\frac{\alpha}{e}\left(\frac{\varepsilon\sqrt{n}}{B(p)}\right)^{1/\alpha}\right\}.$$

Example 3.2. Consider the space $\mathbf{F}_\psi(\Omega)$ with $\psi(u) = e^{au^\beta}$ for $a > 0$ and $\beta > 0$. Then Theorems 3.1 and 2.3 with $\varepsilon \ge e^{a(\beta+1)}B(p)/\sqrt{n}$ imply

$$\mathsf{P}\left\{\sup_{t\in T}|S_n(t)| > \varepsilon\right\} \le \exp\left\{-\frac{\beta}{a^{1/\beta}}\left(\frac{\ln\frac{\varepsilon\sqrt{n}}{B(p)}}{\beta+1}\right)^{\frac{\beta+1}{\beta}}\right\}.$$

Example 3.3. Consider the space $\mathbf{F}_\psi(\Omega)$ with $\psi(u) = (\ln(u+1))^\lambda$, $\lambda > 0$. Then Theorems 3.1 and 2.4 with $\varepsilon \ge (e\ln 2)^\lambda B(p)/\sqrt{n}$ imply

$$\mathsf{P}\left\{\sup_{t\in T}|S_n(t)| > \varepsilon\right\} \le e^\lambda \exp\left\{-\lambda\exp\left\{\left(\frac{\varepsilon\sqrt{n}}{B(p)}\right)^{1/\lambda}\frac{1}{e}\right\}\right\}.$$

4. Reliability and accuracy in the space $C(T)$ for the evaluation of integrals depending on parameters by using the Monte Carlo method

Let $\{\mathcal{S}, \mathcal{A}, \mu\}$ be a measurable space, μ a σ-finite measure, $p(s) \geq 0$, $s \in \mathcal{S}$, a function such that $\int_{\mathcal{S}} p(s)\, d\mu(s) = 1$, and $P(A)$, $A \in \mathcal{A}$, a measure defined by

$$P(A) = \int_A p(s)\, d\mu(s).$$

Since $P(A)$ is a probability measure, the triple $\{\mathcal{S}, \mathcal{A}, P\}$ is a probability space.

Let a function $f(s,t)$ depend on a parameter $t \in T$, where (T, ρ) is a compact metric space. Assume that $f(s,t)$ is continuous with respect to t. Assume that the integral

$$\int_{\mathcal{S}} f(s,t) p(s)\, d\mu(s) = I(t)$$

is well defined

We view $f(s,t)$ as a stochastic process in $\{\mathcal{S}, \mathcal{A}, P\}$ and denote it by $\xi(s,t) = \xi(t)$. Then

$$I(t) = \int_{\mathcal{S}} f(s,t) p(s)\, d\mu(s) = \int_{\mathcal{S}} f(s,t)\, dP(s) = \mathsf{E}\,\xi(t).$$

Let $\xi_i(t)$, $i = 1, 2, \ldots, n$, be independent copies of the stochastic process $\xi(t)$ and let

$$Z_n(t) = \frac{1}{n} \sum_{i=1}^{n} \xi_i(t).$$

Then the strong law of large numbers implies that $Z_n(t) \to \mathsf{E}\,\xi(t) = I(t)$ with probability one for all $t \in T$.

Definition 4.1. We say that $Z_n(t)$ approximates $I(t)$ in the space $C(T)$ with reliability $1 - \delta$, $\delta > 0$, and accuracy $\varepsilon > 0$ if

$$\mathsf{P}\left\{\sup_{t \in T} |Z_n(t) - I(t)| > \varepsilon\right\} \leq \delta.$$

Theorem 4.1. *Let a stochastic process $\xi(t)$, $t \in T$, belong to the space $\mathbf{F}_\psi(\Omega)$ and let condition $\mathbf{H}$ hold with a constant C_ψ. Let $Z_n(t) - I(t) = \frac{1}{n}\sum_{i=1}^{n}(\xi_i(t) - I(t))$, where $\xi_i(t)$ are independent copies of the stochastic process $\xi(t)$.*

Assume that there exists a continuous increasing function $\sigma(h)$ such that $\sigma(0) = 0$ and

$$\sup_{\rho(t,s) \leq h} \left\|\xi(t) - \xi(s)\right\|_\psi \leq \sigma(h).$$

Further, assume that

$$\int_0^z \varkappa\left(N\left(\sigma^{(-1)}(u)\right)\right) du < \infty$$

for all $z > 0$, where $\varkappa(n)$ is a majorant characteristic, $N(u)$ is the metric massiveness of the space $\mathbf{F}_\psi(\Omega)$, and $\sigma^{(-1)}(u)$ is the inverse function to $\sigma(u)$. Then $Z_n(t)$ approximates $I(t)$ with reliability $1 - \delta$, $\delta > 0$, and accuracy ε in the space $C(T)$ if n is such that

$$\inf_{u \geq 1} \left(\frac{B(p)\psi(u)}{\varepsilon\sqrt{n}}\right)^u \leq \delta \tag{5}$$

for all $0 < p < 1$, where

$$B(p) = 2\sqrt{C_\psi}\inf_{t \in T}\|\xi(t)\|_\psi + \frac{1}{p(1-p)}\int_0^{\gamma p} \varkappa\left(N\left(\sigma_1^{(-1)}(u)\right)\right) du,$$

$\sigma_1(u) = 2\sqrt{C_\psi}\sigma(u)$, and $\gamma = \sigma_1\left(\sup_{t,s \in T}\rho(t,s)\right)$.

Proof. According to Theorem 3.1, we have

$$\|Z_n(t) - I(t)\|_\psi \le \frac{1}{\sqrt{n}} B(p).$$

Then inequality (4) implies

$$\mathsf{P}\left\{\sup_{t\in T} |Z_n(t) - I(t)| > \varepsilon\right\} \le \inf_{u\ge 1} \left(\frac{B(p)\psi(u)}{\varepsilon\sqrt{n}}\right)^u.$$

The latter inequality completes the proof of Theorem 4.1. □

Theorem 4.2. *Let $\xi(\vec{t})$ be a separable stochastic process defined in the space (T,ρ), where $T = \{a_j \le t_j \le b_j, j = \overline{1,d}\}$ and $\rho(\vec{t},\vec{s}) = \max_{1\le j\le d} |t_j - s_j|$ for $\vec{t} = (t_1,\dots,t_d)$ and $\vec{s} = (s_1,\dots,s_d)$. Moreover let $\xi(\vec{t})$ belong to the space $\mathbf{F}_\psi(\Omega)$ with $\psi(u) = u^\alpha$, $\alpha \ge \frac{1}{2}$. Assume that*

$$\sup_{\rho(\vec{t},\vec{s})\le h} \left\|\xi(\vec{t}) - \xi(\vec{s})\right\|_\psi \le C\,|h|^\beta,$$

where $C > 0$ and $0 < \beta \le 1$. Finally let $Z_n(\vec{t}) = \frac{1}{n}\sum_{i=1}^n \xi_i(\vec{t})$, where $\xi_i(\vec{t})$ are independent copies of the stochastic process $\xi(\vec{t})$, and $I(\vec{t}) = \int_S f(\vec{s},\vec{t})p(\vec{s})\,d\mu(\vec{s}) = \mathsf{E}\,\xi(\vec{t})$.

Then $Z_n(\vec{t})$ approximates $I(\vec{t})$ with reliability $1-\delta$, $\delta > 0$, and accuracy ε in the space $C(T)$ if

$$n \ge \left(\frac{e^\alpha B(\beta)}{\varepsilon}\right)^2 \max\left(1, \left(-\frac{\ln\delta}{\alpha}\right)^{2\alpha}\right), \tag{6}$$

where

$$B(\beta) = 4\cdot 3^\alpha \inf_{t\in T}\left\|\xi(\vec{t})\right\|_\psi + \left(\frac{e}{\alpha}\right)^\alpha \cdot \frac{4Cd3^{\alpha+\frac{3}{2}}R^\beta}{2^{\frac{\beta}{2}-1}\beta}.$$

Proof. The result follows from Theorem 4.1. Note that $\sigma(h) = C\,|h|^\beta$ in the case under consideration. It is shown in [5] that $C_\psi = 4\cdot 9^\alpha$ if $\psi(u) = u^\alpha$ and $\alpha \ge \frac{1}{2}$.

Thus $\sigma_1(u) = 4\cdot 3^\alpha C\,|u|^\beta$ and $\sigma_1^{(-1)}(u) = \left(\frac{u}{4\cdot 3^\alpha C}\right)^{1/\beta}$. The majorant characteristic for the space $\mathbf{F}_\psi(\Omega)$ for $\psi(u) = u^\alpha$ is found in the paper [2]. This characteristic is given by

$$\varkappa(n) = \left(\frac{e}{\alpha}\right)^\alpha (\ln n)^\alpha.$$

It is easy to see that

$$N\left(\sigma_1^{(-1)}(u)\right) \le \prod_{j=1}^d \left(\frac{b_j - a_j}{2\sigma_1^{(-1)}(u)} + 1\right) \le \left(\frac{R}{2\sigma_1^{(-1)}(u)} + 1\right)^d,$$

where $R = \max_{1\le j\le d}(b_j - a_j)$. In the case under consideration,

$$N\left(\sigma_1^{(-1)}(u)\right) \le \left(\frac{R(4\cdot 3^\alpha C)^{1/\beta}}{2u^{1/\beta}} + 1\right)^d.$$

Hence

$$B(p) \le 4\cdot 3^\alpha \inf_{t\in T}\left\|\xi(\vec{t})\right\|_\psi + \frac{1}{p(1-p)}\int_0^{\tilde{\gamma}p} \left(\frac{e}{\alpha}\right)^\alpha \ln\left(\frac{R(4\cdot 3^\alpha C)^{1/\beta}}{2u^{1/\beta}} + 1\right)^d du, \tag{7}$$

where $\tilde{\gamma} = \sigma_1(R) = 4\cdot 3^\alpha C R^\beta$. Since

$$\ln(1+x) \le \frac{x^\theta}{\theta}$$

for all $0 < \theta \le 1$, inequality (7) implies

$$\begin{aligned} B(p) &\le 4 \cdot 3^\alpha \inf_{t \in T} \left\| \xi(\vec{t}) \right\|_\psi + \frac{1}{p(1-p)} \int_0^{\tilde{\gamma} p} \left(\frac{e}{\alpha} \right)^\alpha \frac{d}{\theta} \left(\frac{R(4 \cdot 3^\alpha C)^{1/\beta}}{2u^{1/\beta}} \right)^\theta du \\ &= 4 \cdot 3^\alpha \inf_{t \in T} \left\| \xi(\vec{t}) \right\|_\psi + \frac{1}{p(1-p)} \left(\frac{e}{\alpha} \right)^\alpha \frac{d}{\theta 2^\theta} \left(R^\beta 4 \cdot 3^\alpha C \right)^{\theta/\beta} \cdot \frac{1}{\left(1 - \frac{\theta}{\beta}\right)} (\tilde{\gamma} p)^{1-\theta/\beta} \\ &= 4 \cdot 3^\alpha \inf_{t \in T} \left\| \xi(\vec{t}) \right\|_\psi + \left(\frac{e}{\alpha} \right)^\alpha \frac{d}{\theta 2^\theta} \cdot \frac{4 \cdot 3^\alpha R^\beta C}{\left(1 - \frac{\theta}{\beta}\right)} \cdot \frac{1}{p^{\theta/\beta}(1-p)} \end{aligned}$$

for $\theta < \beta$.

Example 3.1 yields

$$\inf_{u \ge 1} \left(\frac{B(p)\psi(u)}{\varepsilon \sqrt{n}} \right)^u = \exp\left\{ -\frac{\alpha \varepsilon^{1/\alpha} n^{1/2\alpha}}{e(B(p))^{1/\alpha}} \right\}$$

for $\varepsilon \ge e^\alpha B(p)/\sqrt{n}$.

Now it is clear that restriction (5) follows from the inequality

$$n \ge \left(\frac{e^\alpha B(p)}{\varepsilon} \right)^2 \max\left(1, \left(-\frac{\ln \delta}{\alpha} \right)^{2\alpha} \right). \tag{8}$$

Since inequality (8) holds for all $0 < p < 1$, one can minimize the right hand side of the latter inequality with respect to θ and p. Then

$$\begin{aligned} B(p) &\le 4 \cdot 3^\alpha \inf_{t \in T} \left\| \xi(\vec{t}) \right\|_\psi + \left(\frac{e}{\alpha} \right)^\alpha \frac{d}{\theta 2^\theta} \cdot \frac{4 \cdot 3^\alpha R^\beta C}{\left(1 - \frac{\theta}{\beta}\right)} \cdot \frac{1}{p^{\frac{\theta}{\beta}}(1-p)} \\ &= 4 \cdot 3^\alpha \inf_{t \in T} \left\| \xi(\vec{t}) \right\|_\psi + \left(\frac{e}{\alpha} \right)^\alpha \cdot \frac{4Cd 3^{\alpha + \frac{3}{2}} R^\beta}{2^{\frac{\beta}{2} - 1} \beta} = B(\beta) \end{aligned}$$

for $\theta = \frac{\beta}{2}$ and $p = \frac{1}{3}$. The latter inequality together with (8) implies inequality (6). $\square$

Using Theorems 2.3 and 2.4 and Examples 3.2 and 3.3, one can prove the results similar to Theorem 4.2 for the spaces $\mathbf{F}_\psi(\Omega)$ generated by the functions $\psi(u) = e^{au^\beta}$ and $\psi(u) = (\ln(u+1))^\lambda$.

Example 4.1. Consider the integral

$$I(t) = rq \int_0^{+\infty} \int_0^{+\infty} e^{-rx - qy} \sin\left(2\sqrt{txy}\right) dx\, dy,$$

where $0 \le t \le T$, $r > 0$, and $q > 0$.

Let ξ and η be independent random variables with exponential distribution with parameters r and q, respectively. This means that

$$\mathsf{P}\{\xi < x\} = \begin{cases} 1 - e^{-rx}, & x > 0; \\ 0, & x < 0, \end{cases}$$

$$\mathsf{P}\{\eta < y\} = \begin{cases} 1 - e^{-qx}, & y > 0; \\ 0, & y < 0. \end{cases}$$

Therefore

$$I(t) = \int_0^{+\infty} \int_0^{+\infty} re^{-rx} qe^{-qy} \sin\left(2\sqrt{txy}\right) dx\, dy = \mathsf{E} \sin\left(2\sqrt{t\xi\eta}\right).$$

Consider the space $\mathbf{F}_\psi(\Omega)$ for $\psi(u) = u^2$. Let $\xi(t) = \sin\left(2\sqrt{t\xi\eta}\right)$ and let $\xi_i(t)$, $i = 1,\dots,n$, be independent copies of the stochastic process $\xi(t)$. Then $Z_n(t) = \frac{1}{n}\sum_{i=1}^n \xi_i(t)$ approximates $I(t)$. Now we are going to estimate the norm of ξ and that of the increments of this process. We have $\|\xi(t)\|_\psi = \left\|\sin\left(2\sqrt{t\xi\eta}\right)\right\|_\psi$ and thus $\inf_{0\le t\le T}\|\xi(t)\|_\psi = 0$. An estimate for the norm of increments is given by

$$\begin{aligned}\|\xi(t)-\xi(s)\|_\psi &= \left\|\sin\left(2\sqrt{t\xi\eta}\right) - \sin\left(2\sqrt{s\xi\eta}\right)\right\|_\psi \\ &\le 2\left\|\sin\left(\sqrt{\xi\eta}\left(\sqrt{t}-\sqrt{s}\right)\right)\right\|_\psi \le 2\left\|\sqrt{\xi\eta}\right\|_\psi \left|\sqrt{t}-\sqrt{s}\right| \le 2\sqrt{|t-s|}\left\|\sqrt{\xi\eta}\right\|_\psi .\end{aligned} \tag{9}$$

It follows from the definition of the norm in the space $\mathbf{F}_\psi(\Omega)$ that

$$\left\|(\xi\eta)^{1/2}\right\|_\psi = \sup_{u\ge 1}\frac{\left(\mathsf{E}(\xi\eta)^{u/2}\right)^{1/u}}{u^2} = \sup_{u\ge1}\frac{\left(\mathsf{E}\,\xi^{u/2}\,\mathsf{E}\,\eta^{u/2}\right)^{1/u}}{u^2},$$

whence

$$\mathsf{E}\,\xi^{u/2} = \int_0^{+\infty} x^{u/2} r e^{-rx}\,dx = \frac{1}{r^{u/2}}\Gamma\left(\frac{u}{2}+1\right).$$

Since $\Gamma(z) \le e^{-z} z^{z-\frac12} C_z$ for $C_z = \sqrt{2\pi}e^{1/(12z)}$, we get

$$\Gamma\left(\frac{u}{2}+1\right) \le e^{-\left(\frac{u}{2}+1\right)}\left(\frac{u}{2}+1\right)^{\frac{u}{2}+\frac12}\sqrt{2\pi}e^{\frac{1}{12\left(\frac{u}{2}+1\right)}} \le e^{-\left(\frac{u}{2}+1\right)}\left(\frac{u}{2}+1\right)^{\frac{u}{2}+\frac12}\sqrt{2\pi}e^{1/18}.$$

Thus

$$\left(\mathsf{E}\,\xi^{u/2}\right)^{\frac1u} \le \frac{1}{\sqrt{r}}e^{-\frac12}e^{-\frac1u}\left(\frac{u}{2}+1\right)^{\frac12+\frac{1}{2u}}\sqrt{2\pi}e^{1/18} \le \frac{1}{\sqrt{r}}\left(\frac{u}{2}+1\right)\sqrt{2\pi}e^{-8/18}$$

for $u \ge 1$.

A similar reasoning allows one to prove the following inequality:

$$\left(\mathsf{E}\,\eta^{u/2}\right)^{\frac1u} \le \frac{1}{\sqrt{q}}\left(\frac{u}{2}+1\right)\sqrt{2\pi}e^{-8/18}.$$

Hence

$$\sup_{u\ge1}\frac{\left(\mathsf{E}\,\xi^{u/2}\,\mathsf{E}\,\eta^{u/2}\right)^{1/u}}{u^2} = \sup_{u\ge1}\frac{1}{\sqrt{rq}}\left(\frac{u}{2}+1\right)^2 2\pi e^{-8/9}\frac{1}{u^2} = \frac{1}{\sqrt{rq}}\cdot\frac{9\pi}{2e^{8/9}}.$$

Then inequality (9) is rewritten as

$$\|\xi(t)-\xi(s)\|_\psi \le \sqrt{|t-s|}\frac{1}{\sqrt{rq}}\cdot\frac{9\pi}{e^{8/9}}.$$

This means that

$$\sigma(h) = Ch^{\frac12},$$

where $C = \frac{1}{\sqrt{rq}}\cdot\frac{9\pi}{e^{\frac89}}$.

Moreover, $Z_n(t)$ approximates $I(t)$ in the space $C([0,1])$ with reliability $1-\delta$ and accuracy ε if

$$n \ge 60665$$

and $\varepsilon = 0.03B(\beta)$ in inequality (6) with $\delta = 0.01$.

Example 4.2. Consider the integral

$$I(t) = rq\int_0^{+\infty}\int_0^{+\infty}\frac{1}{\sqrt{xy}}e^{-rx-qy}\sin\left(\sqrt{txy}\right)\,dx\,dy,$$

where $0 \le t \le T$, $r > 0$, and $q > 0$.

Let ξ and η be independent random variables with exponential distribution with parameters r and q, respectively. Therefore

$$I(t)=\int_0^{+\infty}\int_0^{+\infty}\frac{1}{\sqrt{xy}}re^{-rx}qe^{-qy}\sin\left(\sqrt{txy}\right)\,dx\,dy=\mathsf{E}\left(\frac{\sin\left(\sqrt{t\xi\eta}\right)}{\sqrt{\xi\eta}}\right).$$

Consider the space $\mathbf{F}_\psi(\Omega)$ with $\psi(u)=u^{\frac{1}{2}}$. Let

$$\xi(t)=\frac{\sin\left(\sqrt{t\xi\eta}\right)}{\sqrt{\xi\eta}}$$

and let $\xi_i(t)$, $i=1,\dots,n$, be independent copies of the stochastic process $\xi(t)$. Then $Z_n(t)=\frac{1}{n}\sum_{i=1}^n\xi_i(t)$ approximates the integral $I(t)$.

Next we estimate the norm of the process and the norm of its increments. First,

$$\|\xi(t)\|_\psi=\left\|\frac{\sin\left(\sqrt{t\xi\eta}\right)}{\sqrt{\xi\eta}}\right\|_\psi\le\sqrt{t},$$

and thus $\inf_{0\le t\le T}\|\xi(t)\|_\psi=0$. An estimate for the norm of increments of this process is given by

$$\begin{aligned}\|\xi(t)-\xi(s)\|_\psi&=\left\|\frac{\sin\left(\sqrt{t\xi\eta}\right)}{\sqrt{\xi\eta}}-\frac{\sin\left(\sqrt{s\xi\eta}\right)}{\sqrt{\xi\eta}}\right\|_\psi\\&\le 2\left\|\frac{\sin\left(\sqrt{\xi\eta}\left(\sqrt{t}-\sqrt{s}\right)\right)}{2\sqrt{\xi\eta}}\right\|_\psi\le\left\|\sqrt{t}-\sqrt{s}\right\|_\psi\le|t-s|^{\frac{1}{2}}.\end{aligned}$$

Therefore

$$\sigma(h)=Ch^{\frac{1}{2}}$$

for $C=1$.

Moreover, $Z_n(t)$ approximates $I(t)$ in the space $C([0,1])$ with reliability $1-\delta$ and accuracy ε if

$$n\ge 3021$$

and $\varepsilon=0.03B(\beta)$ in inequality (6) with $\delta=0.01$.

5. Concluding remarks

The paper contains some definitions and results of the theory of the spaces $\mathbf{F}_\psi(\Omega)$. Some bounds for the distribution of the supremum of stochastic processes defined on a compact set and belonging to the space $\mathbf{F}_\psi(\Omega)$ are considered. Some sufficient conditions are found under which multiple integrals can be evaluated with a given reliability and accuracy in the space $C(T)$ by using the Monte Carlo method. Some methods of the theory of stochastic processes belonging to the spaces $\mathbf{F}_\psi(\Omega)$ are used in the proofs of this paper.

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Bibliography

1. Yu. V. Kozachenko and Yu. Yu. Mlavets, *Probability of large deviations of sums of random processes from Orlicz space*, Monte Carlo Methods Appl. **17** (2011), 155–168. MR2819705
2. Yu. V. Kozachenko and Yu. Yu. Mlavets, *The Banach space* $\mathbf{F}_\psi(\Omega)$ *of random variables*, Teor. Ĭmovir. Mat. Stat. **86** (2012), 92–107; English transl. in Theor. Probability and Math. Statist. **86** (2013), 105–121. MR2986453
3. S. V. Ermakov and E. I. Ostrovskiĭ, *Conditions for the continuity, exponential bounds, and central limit theorem for random fields*, Dep. VINITI no. 3752-B.86.0, 1986. (Russian)
4. Yu. Yu. Mlavets, $\mathbf{F}_\psi(\Omega)$*-spaces of random variables with exponential function* ψ, Visnyk Kyiv Taras Shevchenko National University Phys. Math. **2** (2012), 19–22.
5. Yu. Kozachenko and Yu. Mlavets, *Stochastic processes of* $\mathbf{F}_\psi(\Omega)$ *spaces*, Contemporary Math. Stat. **2** (2014), no. 1, 55–75.
6. Yu. Yu. Mlavets, *Conditions for the uniform convergence of random series of functions belonging to the spaces* $\mathbf{F}_\psi(\Omega)$, Appl. Stat. Actuar. Finance Math. **1** (2014), 97–103. (Ukrainian)

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