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A new approach to non-local boundary value problems for ordinary differential systems

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ABSTRACT

We suggest a new constructive approach for the solvability analysis and approximate solution of general non-local boundary value problems for non-linear systems of ordinary differential equations with locally Lipschitzian non-linearities. The practical application of the techniques is explained on a numerical example.

1. Introduction

The purpose of the present note is to provide a scheme for a constructive analysis of a non-local boundary value problem. More precisely, we consider the problem

$$u'(t) = f(t, u(t)), \quad t \in [a, b], \tag{1}$$

$$\phi(u) = d, \tag{2}$$

where $\phi : C([a, b], \mathbb{R}^n) \to \mathbb{R}^n$ is a vector functional (possibly non-linear), $f : [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$ is a function satisfying the Carathéodory conditions in a certain bounded set, and d is a given vector. By a solution of the problem, one means an absolutely continuous function with property (2) satisfying (1) almost everywhere on $[a, b]$.

The analysis is constructive in the sense that, when applicable, it allows one to both study the solvability of the problem and approximately construct its solutions by operating with objects that are determined explicitly in finitely many steps of computation. The topic has been addressed by many authors, see, e.g., [1,2] for related references.

It turns out that, under suitable conditions and with a certain modification, the techniques previously applied in [3,4] for periodic and two-point problems can also be used in the more general cases of problem (1) and (2) where the boundary condition may be non-local. Here, we describe this particular modification, which is based on the introduction of a suitable model problem, and outline the resulting scheme of investigation. Note that the new approach is easier to apply compared with those used earlier, e.g., in [5–7].

2. Notation and symbols

In the sequel, for any $x = \mathrm{col}(x_1, \ldots, x_n) \in \mathbb{R}^n$, the obvious notation $|x| = \mathrm{col}(|x_1|, \ldots, |x_n|)$ is used and the inequalities between vectors are understood componentwise. A similar convention is adopted implicitly for the operations 'max' and 'min'. The symbol 1_n stands for the unit matrix of dimension n and $r(K)$ denotes the spectral radius of a square matrix K.

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If $z \in \mathbb{R}^n$ and ϱ is a vector with non-negative components, $B(z, \varrho)$ stands for the componentwise ϱ-neighbourhood of z: $B(z, \varrho) := \{\xi \in \mathbb{R}^n : |\xi - z| \leqslant \varrho\}$. Similarly, given a set $\Omega \subset \mathbb{R}^n$, we define its componentwise ϱ-neighbourhood by putting

$$B(\Omega, \varrho) := \bigcup_{z \in \Omega} B(z, \varrho). \tag{3}$$

Given two sets D_0 and D_1 in $\mathbb{R}^n$, we put

$$\mathcal{B}(D_0, D_1) := \{\theta\xi + (1-\theta)\eta : \xi \in D_0, \eta \in D_1, \theta \in [0,1]\}. \tag{4}$$

For a set $\Omega \subseteq \mathbb{R}^n$ and a $n \times n$ matrix K with non-negative entries, we write $f \in \mathrm{Lip}_K(\Omega)$ if the estimate

$$|f(t, u_1) - f(t, u_2)| \leqslant K|u_1 - u_2| \tag{5}$$

holds for all u_1, u_2 from Ω and a.e. $t \in [a, b]$. Finally, we shall frequently use the notation

$$\delta_D(f) := \operatorname*{ess\,sup}_{(t,\xi)\in[a,b]\times D} f(t,\xi) - \operatorname*{ess\,inf}_{(t,\xi)\in[a,b]\times D} f(t,\xi). \tag{6}$$

3. Freezing and parametrization

The idea that we are going to employ is based on the reduction to a family of simpler auxiliary boundary problems obtained by "freezing" certain values of the solution sought for (see, e.g., [8–10]). In our case, the auxiliary problems will have two-point linear separated conditions at a and b:

$$u(a) = \xi, \quad u(b) = \eta, \tag{7}$$

where ξ and η are parameters whose values remain unknown at the moment. As will be seen from the statements below, one can then go back to the original problem by choosing the values of the introduced parameters appropriately.

Let us fix certain bounded sets $D_i \subset \mathbb{R}^n,\ i = 0, 1$, and focus on the solutions u of problem (1) and (2) with $u(a) \in D_0$ and $u(b) \in D_1$. Given an arbitrary pair $(\xi, \eta) \in D_0 \times D_1$, we set

$$u_0(t, \xi, \eta) := \left(1 - \frac{t-a}{b-a}\right)\xi + \frac{t-a}{b-a}\eta \tag{8}$$

and

$$u_{m+1}(t, \xi, \eta) = u_0(t, \xi, \eta) + \int_a^t f(s, u_m(s, \xi, \eta))ds - \frac{t-a}{b-a}\int_a^b f(\tau, u_m(\tau, \xi, \eta))d\tau \tag{9}$$

for all $t \in [a, b]$ and $m = 0, 1, \ldots$. The vectors ξ and η in (8) and (9) are treated as unknown parameters. Considering formulae (8) and (9), one arrives immediately at the following

Proposition 1. *If, for a fixed pair $(\xi, \eta) \in D_0 \times D_1$, the sequence $\{u_m(\cdot, \xi, \eta) : m \geqslant 0\}$ converges to a function $u_\infty(\cdot, \xi, \eta)$ uniformly on $[a, b]$, then:*

1. $u_\infty(b, \xi, \eta) = \eta$.
2. $u_\infty(\cdot, \xi, \eta)$ *satisfies the Cauchy problem*

$$u'(t) = f(t, u(t)) + \frac{1}{b-a}\Delta(\xi, \eta), \quad t \in [a, b], \tag{10}$$

$$u(a) = \xi, \tag{11}$$

where $\Delta : D_0 \times D_1 \to \mathbb{R}^n$ is given by formula

$$\Delta(\xi, \eta) := \eta - \xi - \int_a^b f(s, u_\infty(s, \xi, \eta))ds. \tag{12}$$

In other words, the function $u_\infty(\cdot, \xi, \eta)$, provided that it is well-defined, satisfies the equation

$$u(t) = u_0(t, \xi, \eta) + \int_a^t f(s, u(s))ds - \frac{t-a}{b-a}\int_a^b f(s, u(s))ds, \quad t \in [a, b]. \tag{13}$$

Since, clearly, the values of $u_0(\cdot, \xi, \eta)$ are convex combinations of ξ and η, we see from (13) that $u_\infty(\cdot, \xi, \eta)$ is also a solution of the two-point boundary problem (10) and (7). It turns out that this simple fact can be used to analyse the solutions of the original problem (1) and (2). In order to continue, it is however necessary to establish conditions ensuring the convergence of sequence (9) and, therefore, the fact that $u_\infty(\cdot, \xi, \eta)$ is well defined for the corresponding values of ξ and η.

4. Convergence of successive approximations

Let us put

$$\Omega := \mathcal{B}(D_0, D_1) \tag{14}$$

and $\Omega_\varrho := B(\Omega, \varrho)$ for any non-negative vector ϱ. Recall that the set $\mathcal{B}(D_0, D_1)$ is defined according to (4).

Remark 2. It is clear from (4) that $\mathcal{B}(D_0, D_1) \subset \operatorname{conv}(D_0 \cup D_1)$ but the equality is, generally speaking, not true.

Theorem 3. *Let there exist a non-negative vector ϱ satisfying the inequality*

$$\varrho \geqslant \frac{b-a}{4}\delta_{\Omega_\varrho}(f), \tag{15}$$

such that $f \in \operatorname{Lip}_K(\Omega_\varrho)$ with a matrix K for which

$$(b-a)r(K) < \frac{1}{\gamma_0}, \tag{16}$$

where

$$\gamma_0 := 3/10. \tag{17}$$

Then, for all fixed $(\xi, \eta) \in D_0 \times D_1$:

1. *The limit $\lim_{m\to\infty} u_m(t, \xi, \eta) =: u_\infty(t, \xi, \eta)$ exists uniformly in $t \in [a, b]$.*
2. *$u_\infty(\cdot, \xi, \eta)$ is the unique solution of the Cauchy problem (10) and (11).*
3. *$u_\infty(t, \xi, \eta) \in \Omega_\varrho$ for any $t \in [a, b]$.*
4. *The estimate*

$$|u_\infty(t, \xi, \eta) - u_m(t, \xi, \eta)| \leqslant \frac{5}{9}\alpha_1(t)(\gamma_0(b-a)K)^m(1_n - \gamma_0(b-a)K)^{-1}\delta_{\Omega_\varrho}(f) \tag{18}$$

holds for any $t \in [a, b]$ and $m \geqslant 0$, where

$$\alpha_1(t) = 2(t-a)\left(1 - \frac{t-a}{b-a}\right), \quad t \in [a, b]. \tag{19}$$

The proof of Theorem 3 is carried out by combining several auxiliary statements given below (see [1,11]).

Lemma 4 [1, Lemma 3.13]. *For any continuous function $u : [a, b] \to \mathbb{R}^n$, the estimate*

$$\left|\int_a^t \left(u(\tau) - \frac{1}{b-a}\int_a^b u(s)ds\right)d\tau\right| \leqslant \frac{1}{2}\alpha_1(t)\omega_{[a,b]}(u), \quad t \in [a, b], \tag{20}$$

holds, where α_1 is given by (19) and $\omega_{[a,b]}(u) := \max_{s\in[a,b]} u(s) - \min_{s\in[a,b]} u(s)$.

Let

$$\alpha_{m+1}(t) := \left(1 - \frac{t-a}{b-a}\right)\int_a^t \alpha_m(s)ds + \frac{t-a}{b-a}\int_t^b \alpha_m(s)ds, \quad t \in [a, b], \tag{21}$$

for any $m \geqslant 0$, where $\alpha_0(t) := 1, t \in [a, b]$. Clearly, formula (19) defining α_1 is obtained from (21) for $m = 0$.

Lemma 5 [1, Lemma 3.16]. *The following estimates hold:*

$$\alpha_{m+1}(t) \leqslant \gamma_0(b-a)\alpha_m(t), \quad t \in [a, b], \tag{22}$$

for $m \geqslant 2$ and

$$\alpha_{m+1}(t) \leqslant \frac{10}{9}(\gamma_0(b-a))^m\alpha_1(t), \quad t \in [a, b], \tag{23}$$

for $m \geqslant 0$, where γ_0 is given by (17).

Lemma 6. *If ϱ is a vector satisfying relation (15), then*

$$\{u_m(t, \xi, \eta) : t \in [a, b]\} \subset \Omega_\varrho \tag{24}$$

for any $m \geqslant 0$ *and* $(\xi,\eta) \in D_0 \times D_1$,

Proof. The proof is analogous to that of [4, Lemma 4] and is based on Lemma 4. Let $(\xi,\eta) \in D_0 \times D_1$ be arbitrary. In view of (14), it follows immediately from (8) that $u_0(t,\xi,\eta) \in \Omega$ for any $t \in [a,b]$, i.e., (24) holds for $m = 0$.

Let us assume that (24) holds for a certain $m = m_0$. Then, by virtue of (9), (15), and Lemma 4, we obtain

$$|u_{m_0+1}(t,\xi,\eta) - u_0(t,\xi,\eta)| \leqslant \frac{b-a}{4}\delta_\Omega(f) \leqslant \varrho \tag{25}$$

for $t \in [a,b]$. Since (24) is known to be true for $m = 0$, we see from (25) that all the values of $u_{m_0+1}(\cdot,\xi,\eta)$ are contained in $B(\Omega,\varrho)$, i.e., (24) holds with $m = m_0 + 1$. The arbitrariness of m_0 then leads us to (24) for any m. □

Proof of Theorem 3. Let $\xi \in D_0$ and $\eta \in D_1$. By Lemma 6, we have $u_m(t,\xi,\eta) \in \Omega_\varrho$ for all $t \in [a,b]$ and $m \geqslant 0$. Since, by assumption, the function f belongs to $\mathrm{Lip}_K(\Omega_\varrho)$, relation (9) yields

$$r_{m+1}(t,z,\eta) \leqslant K\left(\left(1 - \frac{t-a}{b-a}\right)\int_a^t r_m(s,z,\eta)ds + \frac{t-a}{b-a}\int_t^b r_m(s,z,\eta)ds\right), \quad t \in [a,b], \tag{26}$$

for all $m \geqslant 1$, where

$$r_m(t,\xi,\eta) := |u_m(t,\xi,\eta) - u_{m-1}(t,\xi,\eta)|, \quad t \in [a,b], \ m \geqslant 1. \tag{27}$$

On the other hand, using (9) and Lemma 4, we obtain

$$\begin{aligned} r_1(t,\xi,\eta) &= \left|\int_a^t \left(f(s,u_0(s,\xi,\eta)) - \frac{1}{b-a}\int_a^b f(s,u_0(\tau,\xi,\eta))d\tau\right)ds\right| \\ &\leqslant \frac{1}{2}\alpha_1(t)\omega_{[a,b]}(f(\cdot,u_0(\xi,\eta))) \\ &\leqslant \frac{1}{2}\alpha_1(t)\delta_{\Omega_\varrho}(f) \end{aligned} \tag{28}$$

for any $t \in [a,b]$. Putting in (26) $m = 1$ and using (21) and estimate (23) of Lemma 5, we obtain

$$\begin{aligned} r_2(t,\xi,\eta) &\leqslant \frac{1}{2}K\left(\left(1 - \frac{t-a}{b-a}\right)\int_a^t \alpha_1(s)ds + \frac{t-a}{b-a}\int_t^b \alpha_1(s)ds\right)\delta_{\Omega_\varrho}(f) \\ &\leqslant \frac{1}{2}K\alpha_2(t)\delta_{\Omega_\varrho}(f) \\ &\leqslant \frac{5\gamma_0}{9}K\alpha_1(t)\delta_{\Omega_\varrho}(f), \end{aligned} \tag{29}$$

where γ_0 is given by (17). Considering (26) and (29) and arguing by induction, we conclude that

$$r_{m+1}(t,\xi,\eta) \leqslant \frac{1}{2}K^m\alpha_{m+1}(t)\delta_{\Omega_\varrho}(f) \leqslant \frac{5}{9}(\gamma_0(b-a)K)^m\alpha_1(t)\delta_{\Omega_\varrho}(f), \quad t \in [a,b], \tag{30}$$

for any $m \geqslant 0$. Therefore, using (19) and the equality $\max_{s\in[a,b]}\alpha_1(s) = \frac{1}{2}(b-a)$, we get

$$\begin{aligned} |\,u_{m+j}(t,\xi,\eta) - u_m(t,\xi,\eta)\,| &\leqslant \sum_{i=1}^{j} r_{m+i}(t,\xi,\eta) \\ &\leqslant \frac{5}{9}\alpha_1(t)\sum_{i=1}^{j}(\gamma_0(b-a)K)^{m+i-1}\delta_{\Omega_\varrho}(f) \\ &\leqslant \frac{5(b-a)}{18}(\gamma_0(b-a)K)^m\sum_{i=0}^{j-1}(\gamma_0(b-a)K)^i\delta_{\Omega_\varrho}(f) \end{aligned} \tag{31}$$

for any $t \in [a,b], m \geqslant 0$, and $j \geqslant 1$. In view of assumption (16), the sums involved in (31) are bounded and $\lim_{m\to\infty}(\gamma_0(b-a)K)^m = 0$. Therefore, (31) implies that $\{u_m(\cdot,\xi,\eta) : m \geqslant 0\}$ is a Cauchy sequence in $C([a,b],\mathbb{R}^n)$. Passing to the limit as $j \to \infty$ in (31), one arrives at (18).

5. Properties of the function $u_\infty(\cdot,\xi,\eta)$

In terms of function $u_\infty(\cdot,\xi,\eta)$, one can characterise the solvability of the two-point problem with separated conditions (7). More precisely, apart of system (1), consider the forced system

$$u'(t) = f(t, u(t)) + \mu(b-a)^{-1}, \quad t \in [a, b], \tag{32}$$

where $\mu = \mathrm{col}(\mu_1, \ldots, \mu_n) \in \mathbb{R}^n$ is a control parameter.

Theorem 7. *Let $\xi \in D_0$ and $\eta \in D_1$ be fixed. Let there exist a non-negative vector ϱ wit property (15) such that $f \in \mathrm{Lip}_K(\Omega_\varrho)$ with a matrix K for which (16) holds. Then, for the solution of system (32) with*

$$u(a) = \xi \tag{33}$$

to have the property

$$u(b) = \eta, \tag{34}$$

it is necessary and sufficient that

$$\mu = \Delta(\xi, \eta), \tag{35}$$

where $\Delta(\xi, \eta)$ is given by (12). Moreover, in the case where (35) holds, the solution of the initial value problem (32) and (33) coincides with $u_\infty(\cdot, \xi, \eta)$.

In other words, for any given pair (ξ, η), the vector $\Delta(\xi, \eta)$ is the only value of μ in (32) for which the solution of (32) and (33) satisfies the two-point boundary conditions (7).

Proof of Theorem 7. *Sufficiency*. Assume that (35) holds. In that case, (10) coincides with (32). By virtue of Proposition 1, the function $u_\infty(\cdot, \xi, \eta)$ is the unique solution of the initial value problem (10), (11) and, moreover, $u_\infty(b, \xi, \eta) = \eta$. Thus, $u_\infty(\cdot, \xi, \eta)$ is a solution of (32) and (34).

Necessity. Let $u_\mu(\cdot, \xi)$ denote the solution of the initial value problem (32) and (33). It is obvious from (32) and (33) that

$$u_\mu(t, \xi) = \xi + \int_a^t f(s, u_\mu(s, \xi))ds + \mu \frac{t-a}{b-a}, \quad t \in [a, b]. \tag{36}$$

It follows immediately from (36) that the value of μ can be represented as

$$\mu = u_\mu(b, \xi) - \xi - \int_a^b f(s, u_\mu(s, \xi))ds \tag{37}$$

and, therefore,

$$u_\mu(t, \xi) = \xi + \int_a^t f(s, u_\mu(s, \xi))ds + \frac{t-a}{b-a}\left(u_\mu(b, \xi) - \xi - \int_a^b f(s, u_\mu(s, \xi))ds\right), \quad t \in [a, b], \tag{38}$$

for any μ. In particular, $u_{\Delta(\xi,\eta)}(\cdot, \xi)$ satisfies the equation

$$u_{\Delta(\xi,\eta)}(t, \xi) = \xi + \int_a^t f(s, u_{\Delta(\xi,\eta)}(s, \xi))ds + \frac{t-a}{b-a}\left(\eta - \xi - \int_a^b f(s, u_{\Delta(\xi,\eta)}(s, \xi))ds\right), \quad t \in [a, b], \tag{39}$$

since, in view of Proposition 1, $u_{\Delta(\xi,\eta)}(\cdot, \xi)$ coincides with $u_\infty(\cdot, \xi, \eta)$ and the latter function has the property $u_\infty(b, \xi, \eta) = \eta$. Assuming now that

$$u_\mu(b, \xi) = \eta, \tag{40}$$

we immediately find from (38) and (39) that each of the functions $u_\mu(\cdot, \xi)$ and $u_{\Delta(\xi,\eta)}(\cdot, \xi)$ satisfies Eq. (13), where $u_0(\cdot, \xi, \eta)$ is given by (8). By Theorem 3, the function $u_\infty(\cdot, \xi, \eta)$, which is the uniform limit of the successive approximations (9), is the only solution of (13). Therefore, under assumption (40), $u_\mu(\cdot, \xi)$ coincides with $u_\infty(\cdot, \xi, \eta)$. Recalling (37), we conclude that μ necessarily has form (35) in that case.

Theorem 7 leads one immediately to the following

Proposition 8. *Under the assumptions of Theorem 3, the function $u_\infty(\cdot, \xi, \eta)$ is a solution of the boundary value problem (1) and (2) if and only if the pair (ξ, η) satisfies the system of $2n$ equations*

$$\Delta(\xi, \eta) = 0, \tag{41}$$

$$\phi(u_\infty(\cdot, \xi, \eta)) = d, \tag{42}$$

where $\Delta : D_0 \times D_1 \to \mathbb{R}^n$ is given by (12).

Proof. It suffices to apply Theorem 7 and notice that the differential Eq. (10) coincides with (1) if and only if (ξ, η) satisfies (41). □

Equations of the type appearing in the last proposition are usually referred to as a *determining equations* and, indeed, as the following statement shows, the system of Eqs. (41) and (42) determines all possible solutions of the original boundary value problem (1), (2) with graphs lying in Ω_ϱ.

Theorem 9. *Let there exist a non-negative vector ϱ with property (15) such that $f \in \mathrm{Lip}_K(\Omega_\varrho)$ with a matrix K for which (16) holds.*

1. *If there exists a pair $(\xi, \eta) \in D_0 \times D_1$ satisfying (41) and (42), then the non-local boundary value problem (1) and (2) has a solution $u(\cdot)$ such that*
$$\{u(t) : t \in [a, b]\} \subset \Omega_\varrho \tag{43}$$
and $u(a) = \xi, u(b) = \eta$.
2. *If the boundary value problem (1) and (2) has a solution $u(\cdot)$ such that (43) holds, then the pair $(u(a), u(b))$ is a solution of system (41) and (42).*

Proof. The first assertion is an immediate consequence of Theorem 3 and Propositions 1 and 8 since $u_\infty(\cdot, \xi, \eta)$ is the required solution in that case. To prove the second one, assume that problem (1) and (2) has a solution u with property (43). Then u is a solution of the Cauchy problem (32) and (33) with $\mu = 0$ and $\xi = u(a)$ and, therefore, by Theorem 3,
$$u = u_\infty(\cdot, u(a), u(b)). \tag{44}$$
In view of Theorem 7, we obtain
$$\Delta(\xi, u(b)) = 0, \tag{45}$$
which means that (41) holds with $\eta = u(b)$. Finally, equality (42) is an immediate consequence of (44) and the assumption that $\phi(u) = d$. □

6. Approximation of a solution

The last theorem suggests an approach to the study of the non-local problem (1) and (2) by looking for its solution among those of the family of equations (13), which are, as Theorem 7 shows, motivated by auxiliary problems with separated two-point conditions (7). The study of the problem then consists of two parts, namely, the analytic part, when the integral Eq. (13) is dealt with by using the method of successive approximations (9), and the numerical one, which consists in finding a values of the $2n$ unknown parameters from the system of Eqs. (41) and (42). This closely correlates with the idea of the Lyapunov–Schmidt reduction (see, e.g., [12]). The solvability of the determining system (41) and (42), in turn, can be established in a rigorous manner by studying some its approximate versions
$$\Delta_m(\xi, \eta) = 0, \tag{46}$$
$$\phi(u_m(\cdot, \xi, \eta)) = d, \tag{47}$$
where m is fixed and $\Delta_m : D_0 \times D_1 \to \mathbb{R}^n$ is given by the relation
$$\Delta_m(\xi, \eta) := \eta - \xi - \int_a^b f(s, u_m(s, \xi, \eta))ds \tag{48}$$
for all $(\xi, \eta) \in D_0 \times D_1$. The solvability analysis based on properties of equations (46) and (47), which can be carried out by analogy to [3,4,13], is not treated here.

In practice, one constructs analytically the function $u_{m_0}(\cdot, \xi, \eta)$ for a certain m_0 keeping ξ and η as parameters, then finds numerically a root $(\tilde{\xi}, \tilde{\eta})$ of the approximate determining system (46) and (47) with $m = m_0$, and forms the function
$$U_{m_0}(t) := u_{m_0}(t, \tilde{\xi}, \tilde{\eta}), \quad t \in [a, b], \tag{49}$$
which is natural to be interpreted as the m_0th approximation of a solution of the original problem (1) and (2) the values of which at a and b lie in a neighbourhood of $\tilde{\xi}$ and $\tilde{\eta}$ respectively. Possible multiple roots of system (46) and (47), under appropriate assumptions, correspond to multiple solutions of the exact determining system (41), (42) and, thus, determine distinct solutions of the given problem.

The above-mentioned property of U_{m_0} is justified by the estimate
$$|u_\infty(t, \tilde{\xi}, \tilde{\eta}) - U_{m_0}(t)| \leqslant \frac{5}{9}\alpha_1(t)(\gamma_0(b-a)K)^{m_0}(1_n - (\gamma_0(b-a)K))^{-1}\delta_{\Omega_\varrho}(f), \quad t \in [a, b], \tag{50}$$
which is a direct consequence of inequality (18) of Theorem 3. In (50), ϱ is the vector appearing in Theorem 3, whereas $\delta_{\Omega_\varrho}(f)$ and γ_0 are given by (6) and (17) respectively. Note that, by Theorem 9, a solution of problem (1) and (2), when it exists,

necessarily has the form $u_\infty(\cdot,\xi_*,\eta_*)$, where (ξ_*,η_*) satisfies (41) and (42). The pair $(\tilde{\xi},\tilde{\eta})$ involved in (50) is, in a sense, an approximation of the explicitly unknown (ξ_*,η_*). A rigorous proof of the existence of the solution in question would involve an analysis of the approximate determining Eqs. (46) and (47) in the spirit of [4,13].

The most difficult part of the scheme is, of course, the construction of the function $u_{m_0}(\cdot,\xi,\eta)$. Quite often systems of symbolic computation can be used for this purpose, which facilitates greatly the operations with functions depending on multiple parameters. Otherwise, if the explicit integration in the (9) if impossible or difficult, one employs suitable modifications of the formulae which, at the expense of a certain loss in accuracy, lead one to schemes better suited for practical realisation. We mention two natural modifications of this kind which make the scheme more constructive.

Version 1 ("Frozen" parameters) Instead of $\{u_m : m \geqslant 0\}$ defined by (9), one uses the sequence $\{v_m : m \geqslant 0\}$ defined by the equalities

$$v_0(t,\xi,\eta) := u_0(t,\xi,\eta), \quad t \in [a,b], \tag{51}$$

and

$$v_{m+1}(t,\xi,\eta) := u_0(t,\xi,\eta) + \int_a^t f(s, v_m(s,\xi_m,\eta_m))ds - \frac{t-a}{b-a}\int_a^b f(\tau, v_m(\tau,\xi_m,\eta_m))d\tau, \quad t \in [a,b], \tag{52}$$

for any $m = 0,1,\ldots$, where $u_0(\cdot,\xi,\eta)$ is given by (8) and (ξ_m,η_m) is a root of the system

$$\begin{aligned} \eta - \xi &= \int_a^b f(s, v_m(s,\xi,\eta))ds, \\ \phi(v_m(\cdot,\xi,\eta)) &= d. \end{aligned} \tag{53}$$

Then one defines the function U_{m_0}, which is to be treated as the m_0th approximation of a solution u with $(u(a),u(b))$ lying in a neighbourhood of (ξ_{m_0},η_{m_0}), as

$$U_{m_0}(t) := v_{m_0}(t,\xi_{m_0},\eta_{m_0}), \quad t \in [a,b]. \tag{54}$$

Note that, as follows from (51) and (52), the mapping $(\xi,\eta) \mapsto v_m(t,\xi,\eta)$ is linear for any $t \in [a,b]$ and, moreover, the dependence on the parameters in (52) is localised to the first summand outside the integration sign. This facilitates greatly the construction of iterations compared to formula (9). For the same reason, system (53), which has to be solved numerically, is considerably simpler than (46) and (47).

System (53) should be solved in a domain where the values $(u(a),u(b))$ of a solution are expected to lie. A natural starting point for that is a root (ξ_0,η_0) of the zeroth approximate determining system ((46) and (47) with $m = 0$):

$$\begin{aligned} \eta - \xi &= \int_a^b f(s, u_0(s,\xi,\eta))ds, \\ \phi(u_0(\cdot,\xi,\eta)) &= d, \end{aligned} \tag{55}$$

where u_0 is given by (8).

Version 2 (Polynomial interpolation) Formula (52) is modified so that the polynomial approximations of the integrands are used, i.e., instead of (9), one uses the formula

$$v_{m+1}(t,\xi,\eta) := u_0(t,\xi,\eta) + \int_a^t p_l f(\cdot, v_m(\cdot,\xi_m,\eta_m))(s)ds - \frac{t-a}{b-a}\int_a^b p_l f(\cdot, v_m(\cdot,\xi_m,\eta_m))(\tau)d\tau, \quad t \in [a,b],$$

where l is fixed and $p_l y$ stands for the polynomial of degree l interpolating the function y at l suitably chosen nodes. The substantiation is similar to other similar cases (see, e.g., [14] where Dirichlet problems for systems of two equations are considered). In this case, one assumes that f satisfies the Dini condition in the time variable [15].

Combining Versions 1 and 2 and using computer algebra systems to facilitate the computation, one arrives at a scheme which is quite efficient and easy to be programmed.

7. A numerical example

Let us apply the numerical-analytic approach described above to the system of differential equations

$$\begin{aligned} u_1'(t) &= u_2^2(t) - \frac{t}{5}u_1(t) + \frac{t^3}{100} - \frac{t^2}{25}, \\ u_2'(t) &= \frac{t^2}{10}u_2(t) + \frac{t}{8}u_1(t) - \frac{21}{800}t^3 + \frac{1}{16}t + \frac{1}{5}, \quad t \in [0,1/2], \end{aligned} \tag{56}$$

considered under the non-linear boundary conditions of integral type

$$\int_0^{\frac{1}{2}} s u_1(s)u_2(s)ds = -\frac{197}{48000}, \quad \int_0^{\frac{1}{2}} s^2 u_2^2(s)ds = \frac{1}{4000}. \tag{57}$$

Problem (56) and (57) has form (1) and (2) with $a = 0, b = 1/2$,

$$u \mapsto \phi(u) := \begin{pmatrix} \int_0^{\frac{1}{2}} s u_1(s) u_2(s) ds \\ \int_0^{\frac{1}{2}} s^2 u_2^2(s) ds \end{pmatrix},$$

and the obvious definitions of the function $f : [0, 1/2] \times \mathbb{R}^2 \to \mathbb{R}^2$ and vector d.

We need to choose some domains where the values of a solution at 0 and 1/2 should belong. Let us put, e.g.,

$$D_0 \ := \ \{(u_1, u_2) : -0.55 \leqslant u_1 \leq 0.45, -0.2 \leqslant u_2 \leq 0.15\}, \quad D_1 \ := \ D_0. \tag{58}$$

It is clear from (4) that $\mathcal{B}(D_0, D_0) = D_0$ and, therefore, according to (14), we have $\Omega = D_0$ in this case. Putting

$$\varrho := \text{col}(0.2, 0.2), \tag{59}$$

we find that the componentwise ϱ-neighbourhood of the set Ω has the form

$$\Omega_\varrho = \{(u_1, u_2) : -0.75 \leqslant u_1 \leq 0.65, -0.4 \leqslant u_2 \leq 0.35\} \tag{60}$$

and, according to (6), one gets that $\delta_{\Omega_\varrho}(f) = \text{col}(0.3, 0.10625)$.

Therefore,

$$\frac{b-a}{4} \delta_{\Omega_\varrho}(f) = \begin{pmatrix} 0.0375 \\ 0.01328125 \end{pmatrix} \leqslant \begin{pmatrix} 0.2 \\ 0.2 \end{pmatrix} = \varrho, \tag{61}$$

which means that the value of ϱ given by (59) satisfies inequality (15) of Theorem 3. A direct computation also shows that $f \in \text{Lip}_K(\Omega_\varrho)$ with

$$K := \begin{pmatrix} 1/10 & 9/10 \\ 1/16 & 1/40 \end{pmatrix}$$

and, therefore,

$$\frac{3}{20} r(K) = 0.045 < 1, \tag{62}$$

which means that (16) holds. We see that all the conditions of Theorem 3 are satisfied. The sequence of functions (9) is thus convergent and one can continue to the construction of approximations.

According to Theorem 9, the number of roots of the determining system (41) and (42) in $D_0 \times D_1$ coincides with the number of solutions u of problem (56) and (57) with $\{u(0), u(1/2)\}$ lying in the set (60). The approximate determining systems (46) and (47) are regarded as approximations to (41), (42) and, thus, their roots may serve as approximations to those of (41) and (42). Let us consider several approximations of a concrete solution.

We start from the zeroth approximation, in which case no iteration is carried out at all. Formula (8) in this example gives

$$u_{0i}(t, \xi, \eta) = (1 - 2t)\xi_i + 2t\eta_i \tag{63}$$

for $i = 1, 2$, where $u_m = \text{col}(u_{m1}, u_{m2}), m \geqslant 0$. Substituting (63) into (48), we find that the zeroth approximate determining system $\Delta_0(\xi, \eta) = 0$ in this case has the form

$$\begin{aligned} -\frac{119}{60}\xi_1 + \frac{61}{30}\eta_1 - \frac{1}{6}(\eta_2 - \xi_2)^2 - \xi_2(\eta_2 - \xi_2) - \xi_2^2 + \frac{29}{9600} = 0, \\ -\frac{961}{480}\xi_2 + \frac{319}{160}\eta_2 - \frac{1}{96}\xi_1 - \frac{1}{48}\eta_1 - \frac{5499}{25600} = 0. \end{aligned} \tag{64}$$

It is easy to verify that the pair of functions

$$u_1^*(t) \ = \ \frac{t^2}{20} - \frac{1}{2}, \quad u_2^*(t) \ = \ \frac{t}{5}, \quad t \in [0, 1/2], \tag{65}$$

is a solution of problem (56) and (57). Obviously, $(u_1^*(0), u_2^*(0)) = (\xi_1^*, \xi_2^*)$ and $(u_1^*(1/2), u_2^*(1/2)) = (\eta_1^*, \eta_2^*)$ with

$$\begin{aligned} \xi_1^* = -0.5, \quad \xi_2^* = 0, \\ \eta_1^* = -0.4875, \quad \eta_2^* = 0.1. \end{aligned} \tag{66}$$

Solving the system of Eqs. (64) in a neighbourhood of the point $(-0.5, 0, 0.4875, 0.1)$, we find its root $(\xi_{01}, \xi_{02}, \eta_{01}, \eta_{02})$:

$$\begin{aligned} \xi_{01} \ \approx \ -0.5018743329, \quad \xi_{02} \ \approx \ -0.2568969557 \cdot 10^{-5} \\ \eta_{01} \ \approx \ -0.4893794933, \quad \eta_{02} \ \approx \ 0.1000006422 \end{aligned} \tag{67}$$

and, after the substitution of (67) into (63), obtain the corresponding zeroth approximation $U_0 = u_0(\cdot, \xi_0, \eta_0)$ of solution (65):

$$\begin{aligned} U_{01}(t) &\approx -0.5018743329 + 0.0249896794t, \\ U_{02}(t) &\approx -0.000002568969557 + 0.2000064223t, \quad t \in [0, 1/2], \end{aligned} \tag{68}$$

shown on Fig. 1. Here and below, we use the notation $U_m = \mathrm{col}(U_{m1}, U_{m2})$, $\xi_m = \mathrm{col}(\xi_{m1}, \xi_{m2})$, $\eta_m = \mathrm{col}(\eta_{m1}, \eta_{m2})$ for any m.

According to (8) and (49), the zeroth approximation is always a linear function and, therefore, one cannot expect a satisfactory degree of accuracy at the very beginning of computation (see the graphs of u_1^* and U_{01} at Fig. 1(a)). However, values (67) can already serve as approximations of (66) and, thus, even the zeroth approximate determining system (64) helps us to obtain a certain space localisation of the corresponding roots of the approximate determining systems at further steps. Indeed, let us construct the first approximation. Using (9) and carrying out computations in *Maple*, at the first iteration ($m = 1$), we obtain

$$\begin{aligned} u_{11}(t, \xi, \eta) = {} & \xi_1 + \frac{t^4}{400} + \frac{t^3}{3}\left(4(-\xi_2 + \eta_2)^2 + \frac{2}{5}(\xi_1 - \eta_1) - \frac{1}{25}\right) \\ & + \frac{t^2}{2}\left(4\xi_2(-\xi_2 + \eta_2) - \frac{1}{5}\xi_1\right) + \xi_2^2 t \\ & - 2t\left(-\frac{29}{19200} + \frac{1}{6}(-\xi_2 + \eta_2)^2 - \frac{1}{120}\xi_1 - \frac{1}{60}\eta_1 + \frac{1}{2}\xi_2(-\xi_2 + \eta_2) + \frac{1}{2}\xi_2^2\right) \\ & + 2t(\eta_1 - \xi_1), \\ u_{12}(t, \xi, \eta) = {} & \xi_2 + \frac{t}{5} + \frac{t^4}{20}\left(-\xi_2 + \eta_2 - \frac{21}{160}\right) + \frac{t^3}{6}\left(-\frac{1}{2}\xi_1 + \frac{1}{2}\eta_1 + \frac{1}{5}\xi_2\right) + \frac{t^2}{16}\left(\xi_1 + \frac{1}{2}\right) \\ & - \frac{t}{16}\left(\frac{5499}{1600} + \frac{1}{30}\xi_2 + \frac{1}{10}\eta_2 + \frac{1}{6}\xi_1 + \frac{1}{3}\eta_1\right) + 2t(\eta_2 - \xi_2) \end{aligned} \tag{69}$$

for any $t \in [0, 1/2]$ and $\{\xi, \eta\} \subset D_0$. Solving numerically the approximate determining system (46) and (47) for $m = 1$ in a neighbourhood of $(\xi_{01}, \xi_{02}, \eta_{01}, \eta_{02})$, we find its root $(\xi_{11}, \xi_{12}, \eta_{11}, \eta_{12})$:

$$\begin{aligned} \xi_{11} &\approx -0.5000145056, \quad \xi_{12} \approx 5.750026703 \cdot 10^{-7}, \\ \eta_{11} &\approx -0.4875143149, \quad \eta_{12} \approx 0.1000004007. \end{aligned} \tag{70}$$

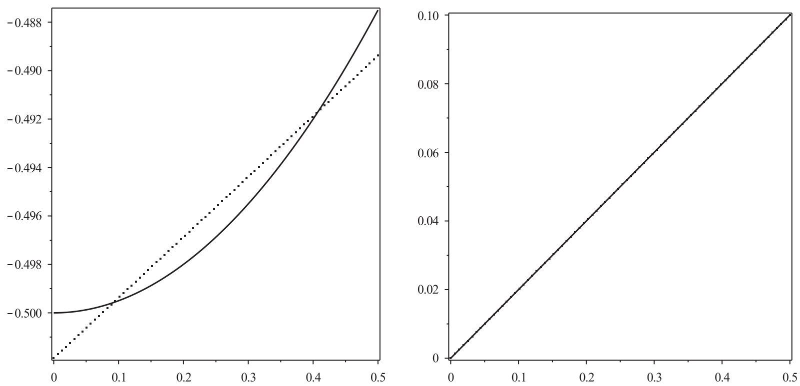
Recall that $(\xi_{01}, \xi_{02}, \eta_{01}, \eta_{02})$ is the root (67) of system (64). Using (49) and substituting the values (70) into (69), we obtain the first and second components of the first approximation $U_1 = \mathrm{col}(U_{11}, U_{12})$ of the solution of problem (56) and (57):

$$\begin{aligned} U_{11}(t) &= -0.5000145056 + \frac{t^4}{400} - 0.001666738533t^3 + 0.05000156555t^2 + 0.00010378326t, \\ U_{12}(t) &= 5.750026703 \cdot 10^{-7} + 0.1999349926t - 0.001562508715t^4 + 0.001041701733t^3 - 9.066 \cdot 10^{-7}t^2 \end{aligned} \tag{71}$$

for $t \in [0, 1/2]$. Comparing (71) with (65), we find that the error of the first approximation is estimated as

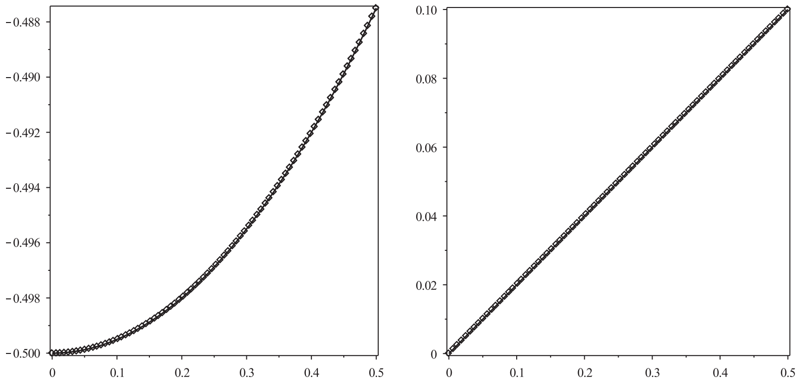
$$\max_{t \in [0,1/2]} | u_1^*(t) - U_{11}(t) | \leqslant 2 \cdot 10^{-5}, \quad \max_{t \in [0,1/2]} | u_2^*(t) - U_{12}(t) | \leqslant 6 \cdot 10^{-6}. \tag{72}$$

The graphs of the solution (65) and its first approximation are shown on Fig. 2. Considering estimates (72), we see that, in fact, there is no need to draw the graphs of any higher approximations.



(a) First component (b) Second component

Fig. 1. The exact solution u^* (solid line) and its zeroth approximation U_0 (dots).



(a) First component (b) Second component

Fig. 2. Solution (65) and its first approximation (71). The graphs of the components of U_1 (the symbol '$\diamond$') visually coincide with those of u^* drawn with the solid line.

In case a better accuracy is needed, higher approximations can be constructed in a similar manner. For example, solving the second approximate determining system (46) and (47) ($m = 2$), we obtain the roots

$$\begin{aligned} \xi_{21} &\approx -0.4999999582, \quad \xi_{22} \approx -1.145685349 \cdot 10^{-8}, \\ \eta_{21} &\approx -0.4874999580, \quad \eta_{22} \approx 0.09999998851 \end{aligned} \tag{73}$$

and the corresponding second approximation $U_2 = \mathrm{col}(U_{21}, U_{22})$ of the form

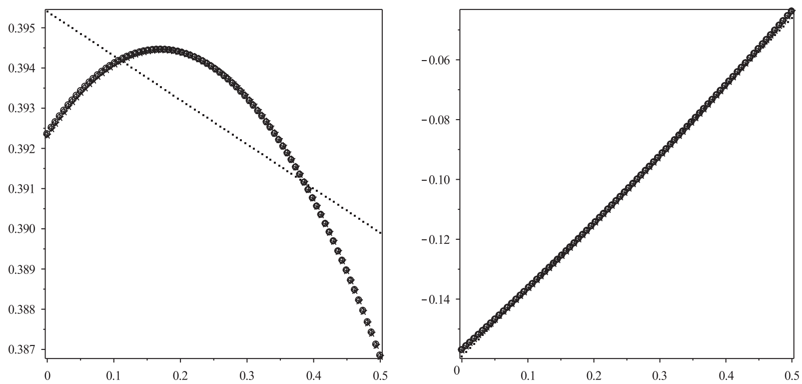
$$\begin{aligned} U_{21}(t) = {} & -0.4999999582 + 2.712673614 \cdot 10^{-7} t^9 - 4.06900898 \cdot 10^{-7} t^8 \\ & + 1.550086457 \cdot 10^{-7} t^7 - 0.00018746609 t^6 + 0.0001499728534 t^5 + 5.7900 \cdot 10^{-10} t^4 \\ & - 0.00001562404305 t^3 + 0.04999999353 t^2 + 3.9424 \cdot 10^{-7} t, \\ U_{22}(t) = {} & -1.145685349 \cdot 10^{-8} + 0.1999998999 t - 0.00002232142859 t^7 \\ & + 0.0000694444383 t^6 - 0.00004166661640 t^5 - 0.000001627838355 t^4 \\ & + 0.00000434008107 t^3 + 2.61 \cdot 10^{-9} t^2 \end{aligned} \tag{74}$$

for all $t \in [0, 1/2]$. We see that (73) as an approximation of (66) is more accurate than (70). A further computation leads to the uniform estimates

$$\max_{t \in [0,1/2]} | u_1^*(t) - U_{21}(t) | \leqslant 6 \cdot 10^{-8}, \quad \max_{t \in [0,1/2]} | u_2^*(t) - U_{22}(t) | \leq 1.5 \cdot 10^{-8},$$

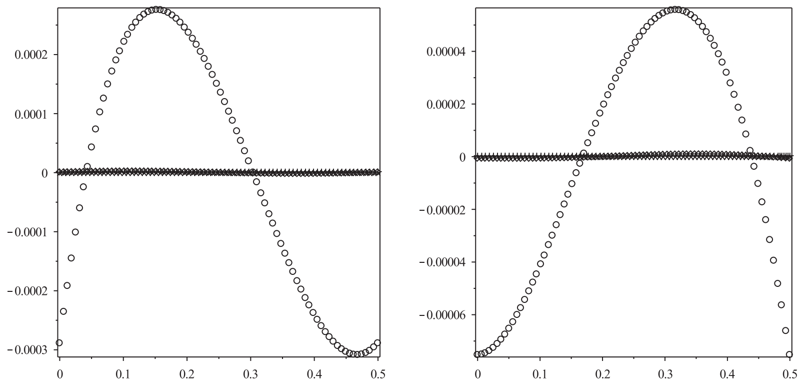
which are significantly better than (72) for U_1. The third approximation U_3, not given here explicitly, provides still better accuracy:

$$\max_{t \in [0,1/2]} | u_1^*(t) - U_{31}(t) | \leq 8 \cdot 10^{-10}, \quad \max_{t \in [0,1/2]} | u_2^*(t) - U_{32}(t) | \leq 1.5 \cdot 10^{-10}.$$



(a) First component (b) Second component

Fig. 3. The zeroth ('$\cdot$'), first ('$\circ$'), second ('$\diamond$') and third ('$*$') approximations to $\tilde{u}$.



(a) First component (b) Second component

Fig. 4. The residual functions of the first ('∘'), second ('◇') and third ('∗') approximations to $\tilde{u}$.

Let us now note that the considerations shown above have been related to the approximation of solution (65), which is known explicitly in this particular example. A computation shows however that, along with (67), the zeroth approximate determining system (64) has another root

$$\begin{aligned} \tilde{\xi}_{01} &\approx 0.3954059502, \quad \tilde{\xi}_{02} \approx -0.1592025648, \\ \tilde{\eta}_{01} &\approx 0.389899163, \quad \tilde{\eta}_{02} \approx -0.0459889168 \end{aligned} \tag{75}$$

and, likewise, the first approximate determining system, along with its root (70), has the root

$$\begin{aligned} \tilde{\xi}_{11} &\approx 0.3923536713, \quad \tilde{\xi}_{12} \approx -0.1570525052, \\ \tilde{\eta}_{11} &\approx 0.386849396, \quad \tilde{\eta}_{12} \approx -0.04383992217. \end{aligned} \tag{76}$$

This indicates a possible existence of a solution $\tilde{u} = \mathrm{col}(\tilde{u}_1, \tilde{u}_2)$ of the boundary value problem (56) and (57) which is different from (65) and has the initial data $(\tilde{u}_1(0), \tilde{u}_2(0), \tilde{u}_1(1/2), \tilde{u}_2(1/2))$ in a neighbourhood of the corresponding values (76). The rigorous analysis confirming the existence of $\tilde{u}$, which consists in the verification of suitable sufficient conditions similar to [3], is omitted here, and we focus on the construction of approximations only. In this case, arguing as shown above and substituting the values from (76) into (69), we obtain the following expression for the first approximation to $\tilde{u}$:

$$\begin{aligned} \widetilde{U}_{11}(t) &= 0.3923536713 + 0.0025\,t^4 + 0.004490021967\,t^3 - 0.07479600670\,t^2 + 0.02495444725\,t, \\ \widetilde{U}_{12}(t) &= -0.1570525052 + 0.200075291\,t - 0.0009018708475\,t^4 - 0.005693773113\,t^3 + 0.05577210445\,t^2 \end{aligned}$$

for $t \in [0, 1/2]$. Solving the second determining system in a neighbourhood of $(\tilde{\xi}_{11}, \tilde{\xi}_{12}, \tilde{\eta}_{11}, \tilde{\eta}_{12})$, we find

$$\begin{aligned} \tilde{\xi}_{21} &\approx 0.3923271761, \quad \tilde{\xi}_{22} \approx -0.1570509845, \\ \tilde{\eta}_{21} &\approx 0.3868231849, \quad \tilde{\eta}_{22} \approx -0.04383842534, \end{aligned} \tag{77}$$

and obtain the second approximation $\widetilde{U}_2$ of $\tilde{u}$:

$$\begin{aligned} \widetilde{U}_{21}(t) &= 0.3923271761 + 9.037479779 \cdot 10^{-8} t^9 + 0.000001283746929\,t^8 \\ &\quad - 0.000009739630169\,t^7 - 0.0002493277103\,t^6 + 0.0000434563574\,t^5 + 0.01226591636\,t^4 \\ &\quad - 0.00749262869\,t^3 - 0.07065485865\,t^2 + 0.02466458183\,t, \\ \widetilde{U}_{22}(t) &= -0.1570509845 + 0.2000007654\,t - 0.00001288388631\,t^7 - 0.00004281164578\,t^6 \\ &\quad + 0.001227658392t^5 - 0.0038978801\,t^4 - 0.004195302573\,t^3 + 0.0557704485\,t^2 \end{aligned} \tag{78}$$

for $t \in [0, 1/2]$. Similarly, one finds the root of the third approximate determining system

$$\begin{aligned} \tilde{\xi}_{31} &\approx 0.3923269706, \quad \tilde{\xi}_{32} \approx -0.1570509371, \\ \tilde{\eta}_{31} &\approx 0.3868229824, \quad \tilde{\eta}_{32} \approx -0.04383837836 \end{aligned} \tag{79}$$

and constructs the third approximation $\widetilde{U}_3$:

$$\begin{aligned}\widetilde{U}_{31}(t) = {} & 0.3923269706 + 1.106630226 \cdot 10^{-11} t^{15} + 7.879714253 \cdot 10^{-11} t^{14} \\ & - 2.29239859 \cdot 10^{-9} t^{13} - 3.896994250 \cdot 10^{-10} t^{12} + 1.755383703 \cdot 10^{-7} t^{11} \\ & - 0.00000109051396\, t^{10} - 3.431360411 \cdot 10^{-7} t^{9} + 0.00002580341034\, t^{8} \\ & + 0.00001123554886\, t^{7} - 0.0008109829273\, t^{6} + 0.0008310140856\, t^{5} \\ & - 0.0003399644806\, t^{5} + 0.00634405846\, t^{4} - 0.00149463484\, t^{3} - 0.0694077454\, t^{2} \\ & + 0.02241968655\, t + 0.01193924453\, t^{4} - 0.007483401567\, t^{3} - 0.07064300545\, t^{2} \\ & + 0.02466500215\, t, \\ \widetilde{U}_{32}(t) = {} & -0.1570509371 + 0.200000005\, t + 1.026986386 \cdot 10^{-9} t^{11} - 1.12792034 \cdot 10^{-7} t^{10} \\ & - 6.109572563 \cdot 10^{-7} t^{9} + 0.00001144998193\, t^{8} - 0.00005490798177\, t^{7} + 0.0001856181722\, t^{6} \\ & + 0.000928093109\, t^{5} - 0.00377044416\, t^{4} - 0.004207340707\, t^{3} + 0.05577043565\, t^{2}\end{aligned} \tag{80}$$

for $t \in [0, 1/2]$.

The graphs of the functions $U_{mi}, m = 1,2,3, i = 1,2$, presented on Fig. 3, show a clear tendency of convergence to $\tilde{u}_i, i = 1,2$. Substituting the third approximation (80) into the differential system (56), one obtains a residual such that

$$\max_{t\in[0,1/2]} \left| \widetilde{U}'_{31}(t) - \widetilde{U}^2_{32}(t) + \frac{t}{5}\widetilde{U}_{31}(t) - \frac{t^3}{100} + \frac{t^2}{25} \right| \approx 1.530806 \cdot 10^{-9},$$

$$\max_{t\in[0,1/2]} \left| \widetilde{U}'_{32}(t) - \frac{t^2}{10}\widetilde{U}_{32}(t) + \frac{t^3}{50} - \frac{1}{5} \right| \approx 9.9868 \cdot 10^{-11},$$

whereas the residual of the first approximation U_1 does not exceed 0.0004 (see also Fig. 4).

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References

[1] A. Rontó, M. Rontó, Successive approximation techniques in non-linear boundary value problems for ordinary differential equations, in: Handbook of Differential Equations: Ordinary Differential Equations, Handb. Differ. Equ., vol. IV, Elsevier/North-Holland, Amsterdam, 2008, pp. 441–592.
[2] A. Rontó, M. Rontó, On constructive investigation of a class of non-linear boundary value problems for functional differential equations, Carpathian J. Math. 29 (1) (2013) 91–108.
[3] A. Rontó, M. Rontó, N. Shchobak, Constructive analysis of periodic solutions with interval halving, Bound. Value Probl. (2013), http://dx.doi.org/10.1186/1687-2770-2013-57. 2013:57, 34.
[4] A. Rontó, M. Rontó, N. Shchobak, Notes on interval halving procedure for periodic and two-point problems, Bound. Value Probl., to appear.
[5] M. Rontó, K. Marynets, On the parametrization of boundary-value problems with three-point non-linear restrictions, Miskolc Math. Notes 13 (1) (2012) 91–106.
[6] T. Jankowski, Numerical-analytic method for implicit differential equations, Math. Notes (Miskolc) 2 (2) (2001) 137–144.
[7] M. Rontó, K. Marynets, Parametrization for non-linear problems with integral boundary conditions, Electron. J. Qual. Theory Differ. Equ. (2012) No. 99, 2.
[8] A.N. Ronto, M. Ronto, N.M. Shchobak, On the parametrization of three-point nonlinear boundary value problems, Nonlinear Oscil. 7 (3) (2004) 384–402.
[9] M. Ronto, N. Shchobak, On parametrized problems with non-linear boundary conditions, in: Proceedings of the 7th Colloquium on the Qualitative Theory of Differential Equations, vol. 7 of Proc. Colloq. Qual. Theory Differ. Equ., Electron. J. Qual. Theory Differ. Equ., Szeged, 2004, pp. 20, 25 pp. (electronic).
[10] M. Rontó, N. Shchobak, On the numerical-analytic investigation of parametrized problems with nonlinear boundary conditions, Nonlinear Oscil. 6 (4) (2003) 469–496, http://dx.doi.org/10.1023/B:NONO.0000028586.11256.d7 (N.Y.).
[11] M. Rontó, J. Mészáros, Some remarks on the convergence of the numerical-analytical method of successive approximations, Ukr. Math. J. 48 (1) (1996) 101–107, http://dx.doi.org/10.1007/BF02390987.
[12] R.E. Gaines, J.L. Mawhin, Coincidence degree, and nonlinear differential equations, Lecture Notes in Mathematics, vol. 568, Springer-Verlag, Berlin, 1977.
[13] A. Rontó, M. Rontó, Existence results for three-point boundary value problems for systems of linear functional differential equations, Carpathian J. Math. 28 (1) (2012) 163–182.
[14] A. Rontó, M. Rontó, G. Holubová, P. Nečesal, Numerical-analytic technique for investigation of solutions of some nonlinear equations with Dirichlet conditions, Bound. Value Probl. 58 (2011) 1–20, http://dx.doi.org/10.1186/1687-2770-2011-58.
[15] V.L. Goncharov, The theory of interpolation and approximation of functions, second ed., GITTL, Moscow, 1954, in Russian.