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PARTIALLY SOLVED DIFFERENTIAL SYSTEMS WITH TWO-POINT NON-LINEAR BOUNDARY CONDITIONS

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Abstract. We suggest a new constructive approach for the solvability analysis and approximate solution of certain types of partially solved Lipschitzian differential systems with two-point non-linear boundary conditions. The practical application of the suggested technique is shown on a numerical example.

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1. INTRODUCTION AND SUBSIDIARY STATEMENTS

The solvability analysis and approximate construction of solutions of various types of regular and singular boundary value problems were successfully done mainly in case of an explicit form of differential systems

$$\frac{dx(t)}{dt} = f(t, x(t)).$$

There is a large gap in the study of solutions of boundary value problems given for systems of differential equations of implicit form, in particular partially resolved with respect to the derivative. This work in a certain form fills this shortcoming.

We study the following boundary value problem on a compact interval

$$\frac{dx(t)}{dt} = f\left(t, x(t), \frac{dx(t)}{dt}\right), t \in [a,b], \tag{1.1}$$

$$g(x(a), x(b)) = d. \tag{1.2}$$

Here we suppose that $f : [a,b] \times D \times D_1 \to \mathbb{R}^n$ and $g : D \times D \to \mathbb{R}^n$ are continuous functions defined on a bounded sets $D \subset \mathbb{R}^n$ and $D^1 \subset \mathbb{R}^n$ (domain D will be concretized later, see (1.8)), D_1 is given), and the function f is Lipschitzian with respect to the second and third variables in the following form:

$$\left|\frac{du}{dt}-\frac{dv}{dt}\right| = \left|f\left(t,u,\frac{du}{dt}\right)-f\left(t,v,\frac{dv}{dt}\right)\right| \leq K_1\left|u-v\right|+K_2\left|\frac{du}{dt}-\frac{dv}{dt}\right| \quad (1.3)$$

for any $t \in [a,b]$ fixed and all $\{u,v\} \subset D$, $\left\{\frac{du}{dt},\frac{dv}{dt}\right\} \subset D_1$, where K_1, K_2 are a non-negative constant matrix of dimension $n \times n$.

Here and below, the absolute value sign and inequalities between vectors are understood componentwise. A similar convention is adopted for the operations "max", "min". The symbol 1_n stands for the unit matrix of dimension n, $r(K)$ denotes a spectral radius of a square matrix K.

If the maximal in modulus eigenvalue of matrix K_2 is less then one

$$r(K_2) < 1,$$

then from (1.3), if $u \neq v$, we obtain

$$[1_n - K_2]\left|\frac{du}{dt}-\frac{dv}{dt}\right| \leq K_1\left|u-v\right|,$$

or

$$\left|f\left(t,u,\frac{du}{dt}\right)-f\left(t,v,\frac{dv}{dt}\right)\right| \leq K\left|u-v\right|, \quad (1.4)$$

where

$$K = [1_n - K_2]^{-1} K_1 = K_1 + K_2 [1_n - K_2]^{-1} K_1.$$

Moreover, we suppose that for the maximal in modulus eigenvalue of matrix

$$Q = \frac{3(b-a)}{10} K \quad (1.5)$$

holds

$$r(Q) < 1. \quad (1.6)$$

If $z \in \mathbb{R}^n$ and ρ is a vector with non-negative components, $B(z,\rho)$ stands for the componentwise ρ-neighbourhood of z:

$$B(z,\rho) := \{\xi \in \mathbb{R}^n : |\xi - z| \leq \rho\}.$$

Similarly, for the given bounded connected set $\Omega \subset \mathbb{R}^n$, we define its componentwise $\rho-$neighbourhood by putting

$$B(\Omega,\rho) := \underset{\xi\in\Omega}{\cup} B(\xi,\rho).$$

Let us fix certain closed bounded sets $D_a \subset \mathbb{R}^n$ and $D_b \subset \mathbb{R}^n$ and focus on the continuously differentiable solutions $x : [a,b] \to D$, $x^{'} : [a,b] \to D_1$ of problem (1.1)-(1.2) with values $x(a) \in D_a$ and $x(b) \in D_b$. For given two bounded connected sets $D_a \subset \mathbb{R}^n$ and $D_b \subset \mathbb{R}^n$, introduce the set

$$D_{a,b} := (1-\theta)z + \theta\eta,\ z \in D_a, \eta \in D_b, \theta \in [0,1] \quad (1.7)$$

and its componentwise $\rho-$neighbourhood

$$D := B(D_{a,b}, \rho)\ . \tag{1.8}$$

It is important to emphasize that D and D_1 are supposed to be bounded and, thus, the Lipschitz condition for f is not assumed globally. The boundary conditions (1.2), generally speaking, non-separated and non-linear.

With the function f involved in equation (1.1), we associate the vector

$$\delta_{[a,b],D,D_1}(f) := \frac{\max\limits_{(t,x,\frac{dx}{dt})\in[a,b]\times D\times D_1} f\left(t,x,\frac{dx}{dt}\right) - \min\limits_{(t,x,\frac{dx}{dt})\in[a,b]\times D\times D_1} f\left(t,x,\frac{dx}{dt}\right)}{2}. \tag{1.9}$$

We recall some subsidiary statements which are needed below.

Lemma 1 ([1], Lemma.3.13)**.** *Let $f : [\tau, \tau + I] \to \mathbb{R}^n$ be a continuous function. Then, for an arbitrary $t \in [\tau, \tau + I]$, the inequality*

$$\left| \int\limits_{\tau}^{t} \left[f(\tau) - \frac{1}{T} \int\limits_{\tau}^{\tau+I} f(s)ds \right] d\tau \right| \le \frac{1}{2}\alpha_1(t,\tau,I) \left[\max_{t\in[\tau,\tau+I]} f(t) - \min_{t\in[\tau,\tau+I]} f(t) \right] \tag{1.10}$$

holds, where

$$\alpha_1(t,\tau,I) = 2(t-\tau)\left(1 - \frac{t-\tau}{I}\right),\ |\alpha_1(t,\tau,I)| \le \frac{I}{2},\ t \in [\tau,\tau+I]. \tag{1.11}$$

Lemma 2 ([1], Lemma 3.16)**.** *Let the sequence of continuous functions $\{\alpha_m(t,\tau,I)\}_{m=0}^{\infty}$, for $t \in [\tau,\tau+I]$, $m = 0,1,2,...$ be defined by the recurrence relation*

$$\alpha_{m+1}(t,\tau,I) = \left(1 - \frac{t-\tau}{I}\right) \int\limits_{\tau}^{t} \alpha_m(s,\tau,I)ds + \frac{t-\tau}{I} \int\limits_{t}^{\tau+I} \alpha_m(s,\tau,I)ds, \tag{1.12}$$

$\alpha_0(t,\tau,I) = 1.$

Then the following estimates hold for $t \in [\tau,\tau+I]$:

$$\alpha_{m+1}(t,\tau,I) \le \frac{10}{9}\left(\frac{3I}{10}\right)^m \alpha_1(t,\tau,I),\ m \geqslant 0, \tag{1.13}$$

$$\alpha_{m+1}(t,\tau,I) \le \frac{3I}{10}\alpha_m(t,\tau,I), m \geqslant 2,$$

2. Parametrization and convergence of successive approximations

The idea that we are going to employ is based on the reduction to a family of simple auxiliary boundary value problems [5]. This approach was used also in [2–4, 6, 7]. Namely, we introduce the vectors of parameters

$$z = col(z_1, z_2, ..., z_n),\ \eta = col(\eta_1, \eta_2, ..., \eta_n) \tag{2.1}$$

by formally putting

$$z = x(a),\, \eta = x(b). \tag{2.2}$$

Instead of boundary value problem (1.1)-(1.2) we will consider the following "model" problem with two-point linear separated parametrized conditions at a and b:

$$\frac{dx}{dt} = f\left(t, x, \frac{dx}{dt}\right), t \in [a,b], \tag{2.3}$$

$$x(a) = z,\ x(b) = \eta. \tag{2.4}$$

As will be seen from statements below, one can then go back to the original problem by choosing the values of the introduced parameters appropriately.

Let us connect with the two-point parametrized boundary value problem (2.3)-(2.4) the sequence of functions

$$x_{m+1}(t,z,\eta) = z + \int_a^t f\left(s, x_m(s,z,,\eta), \frac{dx_m(s,z,\eta)}{ds}\right) ds - \tag{2.5}$$

$$-\frac{t-a}{b-a}\int_a^b f\left(s, x_m(s,z,\eta), \frac{dx_m(s,z,\eta)}{ds}\right) ds + \frac{t-a}{b-a}[\eta - z],\ t \in [a,b],$$

$m = 1, 2, ...$, satisfying (2.4) for arbitrary $z, \eta \in \mathbb{R}^n$, where

$$x_0(t,z,\eta) = z + \frac{t-a}{b-a}[\eta - z] = \left(1 - \frac{t-a}{b-a}\right) z + \frac{t-a}{b-a}\eta,\, t \in [a,b]. \tag{2.6}$$

It is easy to see from (2.6) that $x_0(t,z,\eta)$ is a linear combination of vectors z and η, when $z \in D_a,\ \eta \in D_b$.

Theorem 1. *Assume that*

$$\exists \text{ non negative vector } \rho : \ \rho \geq \frac{b-a}{2}\delta_{[a,b],D,D_1}(f), \tag{2.7}$$

where D is the $\rho-$neighhourhood of the set $D_{a,b}$ defined according to (1.7), (1.8) and $\delta_{[a,b],D,D_1}(f)$ is given as in (1.9):

$$\delta_{[a,b],D,D_1}(f) := \frac{\max\limits_{(t,x,\frac{dx}{dt})\in[a,b]\times D\times D_1} f(t,x,\frac{dx}{dt}) - \min\limits_{(t,x,\frac{dx}{dt})\in[a,b]\times D\times D_1} f(t,x,\frac{dx}{dt})}{2}, \tag{2.8}$$

the function $f \in C([a,b]\times D\times D_1, \mathbb{R}^n)$ is Lipschitzian with respect to the second and third variables according to condition (1.3) and for the matrix Q of form (1.5) holds an inequality (1.6).

Then, for all fixed $z \in D_a$, and $\eta \in D_b$:

1. The functions of the sequence (2.5) belonging to the domain D are continuously differentiable on the interval $[a,b]$, and satisfy the two-point separated boundary conditions (2.4).

2. The sequence of functions (2.5) for $t \in [a,b]$ converges as $m \to \infty$ to the limit function uniformly

$$x_\infty(t,z,\eta) = \lim_{m\to\infty} x_m(t,z,\eta). \tag{2.9}$$

3. The limit function satisfies the two-point separated boundary conditions (2.4).

4. The limit function $x_\infty(t,z,\eta)$ for all $t \in [a,b]$ is a unique continuously differentiable solution of the integral equation

$$x(t) = z + \int_a^t f(s,x(s),\frac{dx(s)}{ds})ds - \frac{t-a}{b-a}\int_a^b f(s,x(s),\frac{dx(s)}{ds})ds \tag{2.10}$$
$$+\frac{t-a}{b-a}[\eta - z],$$

i.e. it is the solution of the Cauchy problem for the modified system of integro-differential equations:

$$\frac{dx}{dt} = f\left(t,x,\frac{dx(t)}{dt}\right) + \frac{1}{b-a}\Delta(z,\eta),\ x(a) = z \tag{2.11}$$

where $\Delta(z,\eta)) : D_a \times D_b \to \mathbb{R}^n$ is a mapping given by formula

$$\Delta(z,\eta) := [\eta - z] - \int_a^b f\left(s, x_\infty(s,z,\eta), \frac{dx_\infty(s,z,\eta)}{ds}\right)ds. \tag{2.12}$$

5. The following error estimation holds:

$$|x_\infty(\cdot,z,\eta)-x_m(\cdot,z,\eta)| \leqslant$$

$$\leqslant \frac{10}{9}\alpha_1(t,a,b-a)Q^m\left(1_n-Q\right)^{-1}\delta_{[a,b],D,D_1}(f),\ t\in[a,b],m\geq 0. \tag{2.13}$$

Proof. We will prove that for $z\in D_a$, $\eta\in D_b$ and $t\in[a,b]$ the values of the functions (2.5) belong to the domain D and it is a Cauchy sequence in the Banach space $C\left([a,b],\mathbb{R}^n\right)$. Indeed, using the estimation (1.10) of Lemma 1, relation (2.5) for $m=0, t\in[a,b]$ implies that

$$|x_1(t,z,\eta)-x_0(t,z,\eta)| \leq$$

$$\leq \alpha_1(t,a,b-a)\left[\frac{\max\limits_{(t,x,\frac{dx}{dt})\in[a,b]\times D\times D_1} f(t,x,\frac{dx}{dt}) - \min\limits_{(t,x,\frac{dx}{dt})\in[a,b]\times D\times D_1} f(t,x,\frac{dx}{dt})}{2})\right] \tag{2.14}$$

$$\leq \alpha_1(t,a,b-a)\ \delta_{[a,b],D,D_1}(f) \leq \frac{b-a}{2}\ \delta_{[a,b],D,D_1}(f),$$

which means according to (2.7), that $x_1(t,z,\eta)\in D$, whenever $(t,z,\eta)\in[a,b]\times D_a\times D_b$.

Using this and arguing by induction according to Lemma 1 we can easily establish that

$$|x_m(t,z,\eta)-x_0(t,z,\eta)| \leq \alpha_1(t,a,b-a)\ \delta_{[a,b],D,D_1}(f) \leq \frac{b-a}{2}\ \delta_{[a,b],D,D_1}(f),$$

$m=2,3,...$, which means that all functions (2.5) are also contained in the domain D, for all $m=1,2,3,..$ and $(t,z,\eta)\in[a,b]\times D_a\times D_b$.

Now, consider the difference of functions

$$x_{m+1}(t,z,\eta)-x_m(t,z,\eta)=$$

$$=\int_a^t\left[f\left(s,x_m(s,z,\eta),\frac{dx_m(s,z,\eta)}{ds}\right)-f\left(s,x_{m-1}(s,z,\eta),\frac{dx_{m-1}(s,z,\eta)}{ds}\right)\right]ds \tag{2.15}$$

$$-\frac{t-a}{b-a}\int_a^b\left[f\left(s,x_m(s,z,\eta),\frac{dx_m(s,z,\eta)}{ds}\right)-f\left(s,x_{m-1}(s,z,\eta),\frac{dx_{m-1}(s,z,\eta)}{ds}\right)\right]ds,$$

$m=1,2,...$ and introduce the notation

$$r_m(t,z,\eta)=|x_m(t,z,\eta)-x_{m-1}(t,z,\eta)|, m=1,2,.... \tag{2.16}$$

By virtue of the Lipschitz condition (1.4), for $m = 1$ from (2.15), we have

$$r_2(t,z,\eta) \leq K\left[\left(1-\frac{t-a}{b-a}\right)\int\limits_a^t r_1(s,z,\eta)ds + \frac{t-a}{b-a}\int\limits_t^b r_1(s,z,\eta)ds\right]. \tag{2.17}$$

According to the recurrence relation (1.12) and estimation (1.13) from (2.14) and (2.17) follows that

$$r_2(t,z,\eta) \leq K\left(1-\frac{t-a}{b-a}\right)\int\limits_a^t \alpha_1(s,a,b-a)\,\delta_{[a,b],D,D_1}(f)ds+$$
$$+\frac{t-a}{b-a}\int\limits_t^b \alpha_1(s,a,b-a)\,\delta_{[a,b],D,D_1}(f)ds \leq$$
$$\leq K\alpha_2(t,a,b-a)\,\delta_{[a,b],D,D_1}(f).$$

By induction using estimation (1.13), we can easily establish that

$$r_{m+1}(t,z,\eta) \leq K^m\alpha_{m+1}(t,a,b-a)\,\delta_{[a,b],D,D_1}(f) \leq \frac{10}{9}Q^m\alpha_1(t,a,b-a)\,\delta_{[a,b],D,D_1}(f) \tag{2.18}$$

Therefore, in view of (2.18)

$$\left|x_{m+j}(t,z,\eta) - x_m(t,z,\eta)\right| \leq$$
$$\leq \left|x_{m+j}(t,z,\eta) - x_{m+j-1}(t,z,\eta)\right| + \left|x_{m+j-1}(t,z,\eta) - x_{m+j-2}(t,z,\eta)\right| + \ldots$$
$$+\left|x_{m+1}(t,z,\eta) - x_m(t,z,\eta)\right| = \sum_{i=1}^{j} r_{m+i}(t,z,\eta) \leq$$
$$\leq \frac{10}{9}\alpha_1(t,a,b-a)\sum_{i=1}^{j} Q^{m+i-1}\delta_{[a,b],D,D_1}(f) =$$
$$= \frac{10}{9}\alpha_1(t,a,b-a)Q^m\sum_{i=0}^{j-1} Q^i\,\delta_{[a,b],D,D_1}(f), \tag{2.19}$$

where $\delta_{[a,b],D,D_1}(f)$ is given by (2.8). Since, due to (1.6), the maximum eigenvalue of the matrix (1.5) does not exceed the unity, we have

$$\sum_{i=0}^{j-1} Q^i \leq (1_n - Q)^{-1}, \ \lim_{m\to\infty} Q^m = 0_n. \tag{2.20}$$

Therefore, we conclude from (2.19) that, according to Cauchy criterium, the sequence $\{x_m(t,z,\eta)\}_{m=0}^{\infty}$ of the form (2.5) uniformly converges in the domain $(t,z,\eta) \in [a,b]\times D_a \times D_b$ to the limit function $x_\infty(t,z,\eta)$. Since all functions of the sequence

(2.5) satisfy the linear separated parametrized conditions (2.4) for all values of the introduced parameter $z \in D_a$, $\eta \in D_b$ the limit function $x_\infty(t,z,\eta)$ also satisfies these conditions. Passing to the limit as $m \to \infty$ in equality (2.5) we show that the limit function satisfies both the integral equation (2.10) and the Cauchy problem (2.11), where $\Delta(z,\eta)$ is given by (2.12). Passing to the limit as $j \to \infty$ in (2.19) we get the estimation (2.13). □

3. Connection of the limit function $x_\infty(t,z,\eta)$ to the solution of the original problem

Theorem 2. *Under the assumptions of Theorem 1, the limit function*

$$x_\infty(t,z^*,\eta^*) = \lim_{m\to\infty} x_m(t,z^*,\eta^*) \tag{3.1}$$

of the sequence (2.5) is a solution of the non-linear boundary value problem (1.1)-(1.2) if and only if the pair of parameters (z^*,η^*) from (2.2) satisfies the system of $2n$ algebraic or transcendental equations

$$\Delta(z,\eta) := [\eta - z] - \int_a^b f\left(s, x_\infty(s,z,\eta), \frac{dx_\infty(s,z,\eta)}{ds}\right) ds = 0,$$
$$\Lambda(z,\eta) := g\left(x_\infty(a,z,\eta), x_\infty(b,z,\eta)\right) - d = 0. \tag{3.2}$$

Proof. The proof can be carried out similarly as in Theorems 2 and 3 from [8]. □

Remark 1. The system of equations (3.2) is usually referred to as a *determining equations*. In such a manner, the original infinite-dimensional problem (1.1)- (1.2) is reduced to a system of $2n$ equations numerical equations.

The method thus consists of two parts, namely, the analytic part, when the integral equation (2.10) is dealt with by using the method of successive approximations (2.5), and the numerical one, which consists in finding values of the $2n$ unknown parameters from equations (3.2).

The next statement proves that the system of determining equations (3.2) defines all possible solutions of the original non-linear boundary value problem (1.1)-(1.2).

Theorem 3. *Let the assumptions of Theorem 1 hold. Furthermore, assume there exist some pair of vectors $\left(z^0,\eta^0\right) \in D_a \times D_b$ that satisfy the system of determining equations (3.2).*

Then:

1. The non-linear boundary value problem (1.1)-(1.2) has a solution $x^0(\cdot)$ such that

$$x^0(a) = z^0,\ x^0(b) = \eta^0. \tag{3.3}$$

Moreover, this solution is given by the limit function of the sequence (2.5):

$$x^0(\cdot) = x_\infty\left(\cdot, z^0, \eta^0\right) = \lim_{m\to\infty} x_m(\cdot, z^0, \eta^0),\ t \in [a,b]. \tag{3.4}$$

2. If the non-linear boundary value problem (1.1)-(1.2) has a solution $x^0(\cdot)$, then the system of determining equations (3.2) is satisfied with

$$z = x^0(a), \eta = x^0(b). \tag{3.5}$$

Proof. The proof can be carried out similarly as in Theorem 4 from [8]. □

4. Solvability analysis based on the approximate determining system

Although Theorem 2 provides a theoretical answer to the question on the construction of a solution of the original non-linear boundary value problem (1.1)-(1.2), its application faces certain difficulties due to the fact that the explicit form of the limit function $x_\infty(\cdot, z, \eta)$ and consequently the explicit form of the functions

$$\Delta : D_a \times D_b \to \mathbb{R}^n, \ \Lambda : D_a \times D_b \to \mathbb{R}^n$$

in (3.2) is usually unknown. This complication can be overcome by using the so-called *approximate determining equations*

$$\Delta_m(z,\eta) := [\eta - z] - \int_a^b f(s, x_m(s,z,\eta), \frac{dx_m(s,z,\eta)}{ds})ds = 0,$$

$$\Lambda_m(z,\eta) := g(x_m(a,z,\eta), x_m(b,z,\eta))) - d = 0 \tag{4.1}$$

for a fixed m.

Lemma 3. *Under the assumptions of Theorem 1 and if the function $g : D_a \times D_b \to \mathbb{R}^n$ in the boundary restrictions (1.2) satisfy the Lipschitz condition*

$$|g(u_1,u_2) - g(v_1,v_2)| \leq K_{g1}|u_1 - u_2| + K_{g2}|v_1 - v_2|, \tag{4.2}$$

for all $\{u_1,u_2\} \subset D_a \times D_b$ and $\{v_1,v_2\} \subset D_a \times D_b$, where K_{g1}, K_{g2} are a non-negative constant matrix of dimension $n \times n$, then for the exact and approximate determining functions defined by (3.2) and (4.1) for any $(z,\eta) \in D_a \times D_b$ and $m \geq 1$ hold the following estimates:

$$|\Delta(z,\eta) - \Delta_m(z,\eta)| \leq \frac{10}{27} K Q^m (1_n - Q)^{-1} \delta_{[a,b],D,D_1}(f),$$

$$|\Lambda(z,\eta) - \Lambda_m(z,\eta)| \leq \left[K_{g1} + K_{g2}\right] \frac{5}{9}(b-a) Q^m (1_n - Q)^{-1} \delta_{[a,b],D,D_1}(f), \tag{4.3}$$

where the matrix Q and the vector $\delta_{[a,b],D,D_1}(f)$ are given respectively in (1.5) and (2.8).

Proof. Let us fix an arbitrary $(z,\eta) \in D_a \times D_b$. Using the Lipschitz condition (1.4), estimate (2.13) and the equality

$$\int_a^b \alpha_1(t,a,b-a)dt = \frac{(b-a)^2}{3}, \tag{4.4}$$

we have

$$\begin{aligned}
&|\Delta(z,\eta) - \Delta_m(z,\eta)| = \\
&= \left| \int_a^b f(s, x_\infty(s,z,\eta))ds - \int_a^b f(s, x_m(s,z,\eta))ds \right| \\
&\leq K \int_a^b \frac{10}{9} \alpha_1(s,a,b) Q^m (1_n - Q)^{-1} \delta_{[a,b],D,D_1}(f) ds = \\
&= \frac{10}{27} K Q^m (1_n - Q)^{-1} \delta_{[a,b],D,D_1}(f),
\end{aligned}$$

which proves the first inequality in (4.3).

From (3.2) and (4.1) using the Lipschitz condition (4.2) and (2.13), we obtain

$$\begin{aligned}
|\Lambda(z,\eta) - \Lambda_m(z,\eta)| &= |g(x_\infty(a,z,\eta), x_\infty(b,z,\eta)) - g(x_m(a,z,\eta), x_m(b,z,\eta))| \\
&\leq K_{g1} |x_\infty(a,z,\eta) - x_m(a,z,\eta)| + K_{g2} |x_\infty(a,z,\eta) - x_m(a,z,\eta)| \\
&\leq \left[K_{g1} + K_{g2}\right] \frac{5}{9}(b-a) Q^m (1_n - Q)^{-1} \delta_{[a,b],D,D_1}(f),
\end{aligned}$$

i.e. the second estimate in (4.3) holds also. □

Based on both exact an approximate determining systems (3.2) and (4.1) let us introduce the mappings $H : D_a \times D_b \to \mathbb{R}^{2n}$ and $H_m : D_a \times D_b \to \mathbb{R}^{2n}$ by setting

$$H(z,\eta) := \begin{bmatrix} [\eta - z] - \int_a^b f(s, x_\infty(s,z,\eta), \frac{dx_\infty(s,z,\eta)}{ds})ds, & z \in D_a \times D_b \\ g(x_\infty(a,z,\eta), x_\infty(b,z,\eta)) - d & \end{bmatrix}, \tag{4.5}$$

$$H_m(z,\eta) := \begin{bmatrix} [\eta - z] - \int_a^b f(s, x_m(s,z,\eta), \frac{dx_m(s,z,\eta)}{ds})ds, & z \in D_a \times D_b \\ g(x_m(a,z,\eta), x_m(b,z,\eta)) - d & \end{bmatrix}, \tag{4.6}$$

We see from Theorem 2 that the critical points of the vector field H of the form (4.5) determine solutions of the non-linear boundary value problem (1.1)-(1.2). The next statement establishes a similar result based upon properties of vector field H_m explicitly known from (4.6).

Theorem 4. *Assume that the conditions of Lemma 3 hold. Moreover, one can specify an* $m \geq 1$ *and a set*

$$\Omega := D_1 \times D_2 \subset \mathbb{R}^{2n},$$

where $D_1 \subset D_a, D_2 \subset D_b$ *are certain bounded open sets such that the mapping* H_m *satisfies the relation*

$$|H_m(z,\eta)| \rhd_{\partial\Gamma} \begin{bmatrix} \frac{10}{27} K Q^m (1_n - Q)^{-1} \delta_{[a,b],D,D_1}(f) \\ [K_{g1} + K_{g2}] \frac{5}{9}(b-a) Q^m (1_n - Q)^{-1} \delta_{[a,b],D,D_1}(f) \end{bmatrix} \tag{4.7}$$

on the boundary $\partial\Omega$ *of the set* Ω. *If, in addition*

$$\deg(H_m, \Omega, 0) \neq 0, \tag{4.8}$$

then there exists a pair $(z^*, \eta^*) \in D_1 \times D_2$ *for which the function*

$$x^*(\cdot) := x_\infty(\cdot, z^*, \eta^*)$$

is a solution of the non-linear boundary value problem (1.1)-(1.2).

In (4.7) the binary relation $\rhd_{\partial\Gamma}$ is defined in [1] as a kind of strict inequality for vector functions and it means that at every point on the boundary $\partial\Omega$ at least one of the components of the vector $|H_m(z,\eta)|$ is greater than the corresponding component of the vector in the right-hand side. The degree in (4.8) is the Brouwer degree because all the vectors fields are finite-dimensional. Likewise, all the terms in the right-hand side of (4.7) are computed explicitly e.g. by using computer algebra system.

Proof. The proof can be carried out similarly as in Theorem 4 from [6]. □

5. Example

Let us apply the approach described above to the system of differential equations

$$\begin{cases} \frac{dx_1(t)}{dt} = x_1(t)\, x_2(t) - \frac{dx_2(t)}{dt} + x_2^2(t) = f_1\left(t, x_1(t), x_2(t), \frac{dx_1(t)}{dt}, \frac{dx_2(t)}{dt}\right) \\ \frac{dx_2(t)}{dt} = \frac{dx_1(t)}{dt}\frac{dx_2(t)}{dt} + \frac{1}{2}\, x_2(t) + \frac{t}{4} = f_2\left(t, x_1(t), x_2(t), \frac{dx_1(t)}{dt}, \frac{dx_2(t)}{dt}\right) \end{cases}, \tag{5.1}$$

$t \in [a,b] = \left[0, \frac{1}{2}\right]$, considered under the two- point non-linear boundary conditions

$$\begin{cases} x_1^2(a) - x_2(b) = -\frac{1}{32} \\ x_2^2(a) - x_1(b) = \frac{1}{32} \end{cases} \tag{5.2}$$

Following (2.1), (2.2), introduce the parameters $z = col(z_1, z_2)$, $\eta = col(\eta_1, \eta_2)$.

Let us consider the following choice of subsets D_a, D_b and D_1, where one looks for the values $x(a)$, $x(b)$ and the values of the derivatives $\frac{dx_1(t)}{dt}, \frac{dx_2(t)}{dt}$:

$$D_a = D_b = \{(x_1, x_2) : \ -0.2 \le x_1 \le 0.2, \ -0.2 \le x_1 \le 0.2\}, \tag{5.3}$$

$$D_1 = \left\{(x_1, x_2) : \ -0.2 \le \frac{dx_1}{dt} \le 0.2 \ , \ -0{,}2 \le \frac{dx_2}{dt} \le 0.2\right\}. \tag{5.4}$$

This choice of the sets D_a and D_b is motivated by the fact that the zero-th approximate determining system (i. e., (4.1) with m = 0) has roots lying in these sets (5.3), see the second line in Table 1. Recall that, in order to obtain it, only function (2.6) are used, and no iteration is yet carried out. We see that this piecewise linear function provides quite reasonable approximate values of the parameters. In this case, according to (1.7), we have

$$D_{a,b} = D_a = D_b. \tag{5.5}$$

For ρ involved in (2.7), we choose the vector

$$\rho := col(0.3, 0.3). \tag{5.6}$$

Then, in view of (5.3), (5.5), (5.6), the set (1.8) takes the form

$$D = \{(x_1, x_2) : \ -0.5 \le x_1 \le 0.5, \ -0.5 \le x_1 \le 0.5\}. \tag{5.7}$$

A direct computation shows that the Lipschitz condition (1.4) for f given by (5.1) on D and D_1 of forms (5.7) and (5.4) holds with matrices

$$K_1 = \begin{pmatrix} 0.5 & 1 \\ 0 & 0.5 \end{pmatrix}, \ K_2 = \begin{pmatrix} 0 & 1 \\ 0.2 & 0.2 \end{pmatrix},$$

$$[1_n - K_2]^{-1} = \begin{pmatrix} 1.333333333 & 1.666666667 \\ 0.333333333 & 1.666666667 \end{pmatrix},$$

$$K = [1_n - K_2]^{-1} K_1 = \begin{pmatrix} 0.6666666665 & 2.166666666 \\ 0.1666666666 & 1.166666667 \end{pmatrix}.$$

Therefore, by (1.5)

$$Q = \begin{pmatrix} 0.09999999998 & 0.3249999999 \\ 0.02499999999 & 0.1750000000 \end{pmatrix}$$

and $r(Q) = 0.235128120913226 < 1$.

Furthermore, in view of (1.9)

$$\delta_{[a,b],D,D_1}(f) := \frac{\max\limits_{(t,x,\frac{dx}{dt})\in[a,b]\times D\times D_1} f(t,x,\frac{dx}{dt}) - \min\limits_{(t,x,\frac{dx}{dt})\in[a,b]\times D\times D_1} f(t,x,\frac{dx}{dt})}{2} =$$

$$= \begin{pmatrix} 0.4812500000 \\ 0.3525000000 \end{pmatrix}$$

and by (5.6) we have

$$\frac{b-a}{2}\delta_{[a,b],D,D_1}(f) = \begin{pmatrix} 0.1203125000 \\ 0.0881250000 \end{pmatrix} \le \rho.$$

TABLE 1.

m	z_1	z_2	η_1	η_2
Exact	0	0	$-\frac{1}{32}=-0.03125$	$\frac{1}{32}=0.03125$
0	0.001561487459	-0.001556662026	-0.0312475768	0.03125243824
1	0.0004122967488	-0.0003873893367	-0.03124984993	0.03125016999
2	-0.000157576675	0.0001562942819	-0.03124997556	0.03125002482
3	$-0.8640511032 \cdot 10^{-5}$	$0.6489136242 \cdot 10^{-5}$	-0.03124999997	0.03125000006
4	$0.1264993624 \cdot 10^{-4}$	$-0.1236038310 \cdot 10^{-4}$	-0.03124999985	0.03125000015
5	$-4.892901202 \cdot 10^{-7}$	$6.586000019 \cdot 10^{-7}$	-0.03125000000	0.03125000001
6	$-0.1073874769 \cdot 10^{-5}$	$0.1030957108 \cdot 10^{-5}$	-0.03125000001	0.03125000000
7	$1.587193848 \cdot 10^{-7}$	$-1.712004563 \cdot 10^{-7}$	-0.03124999999	0.03124999999
8	$8.595697086 \cdot 10^{-8}$	$-8.040876912 \cdot 10-8$	-0.03125000000	0.03125000000
9	$-2.502073667 \cdot 10^{-8}$	$2.574184204 \cdot 10^{-8}$	-0.03124999999	0.03125000001

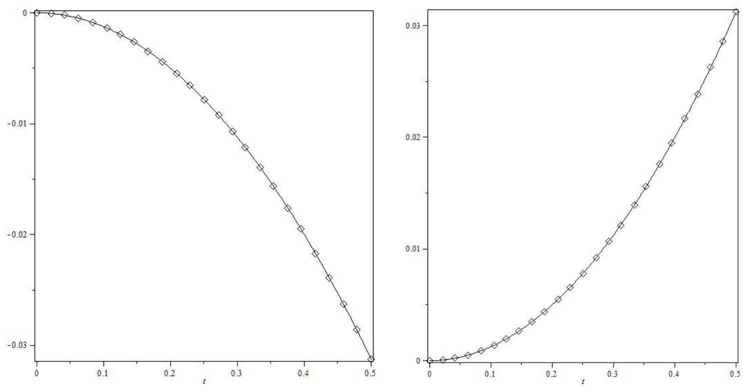
We thus see that all the conditions of Theorem 1 are fulfilled, and the sequence of functions (2.5) for this example is convergent.

It is easy to verify that the pair of functions

$$x_1^*(t) = -\frac{t^2}{8},\ x_2^*(t) = \frac{t^2}{8}$$

is a solution of the given boundary value problem (5.1)-(5.2).

Using (2.5) and applying Maple 13 for different values of m to implement the approximations $x_m(t,z,\eta) = col(x_{m1}(t,z,\eta), x_{m2}(t,z,\eta))$ and solving the approximate determining system (4.1), we find the following values of introduced parameters, which are presented in Table 1. The graphs of the exact and approximate solution for $m=9$ for the first and second components are shown on the Fig. 1.



(a) First component (b) Second component

FIGURE 1. The exact solution $(x_1^*(t), x_2^*(t))$ (solid line) and its nineth approximation (dots)

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