Miskolc Mathematical Notes
Vol. 19 (2018), No. 2, pp. 1233–1241
HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2018.2738

ON INVESTIGATION OF SOME NON-LINEAR INTEGRAL BOUNDARY VALUE PROBLEM

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Received 07 November, 2018

Abstract. We suggest a constructive approach for the solvability analysis and approximate solution of certain types of partially solved Lipschitzian differential systems with mixed two-point and integral non-linear boundary conditions. The practical application of the suggested technique is shown on a numerical example.

2010 *Mathematics Subject Classification:* 34B15

Keywords: implicit differential systems, non-linear integral boundary conditions, parametrization technique, successive approximations

1. PROBLEM SETTING

This article uses the approach proposed in [2], [5], [4] in the case of the following non-linear boundary value problem with mixed two-point and integral restrictions

$$\frac{dx(t)}{dt} = f\left(t, x(t), \frac{dx(t)}{dt}\right), t \in [a,b], \tag{1.1}$$

$$g\left(x(a), x(b), \int_a^b h(s, x(s))ds\right) = d. \tag{1.2}$$

We suppose that $f : [a,b] \times D \times D_1 \to \mathbb{R}^n$ is a continuous function defined on a bounded sets $D \subset \mathbb{R}^n$, $D^1 \subset \mathbb{R}^n$ (domain $D := D_\rho$ will be concretized later, see (1.8)), D^1 is given) and $d \in \mathbb{R}^n$ is a given vector. Moreover f, g: $D \times D \times D_2 \to \mathbb{R}^n$ and $h : [a,b] \times D \to \mathbb{R}^n$ are Lipschitzian in the following form

$$|f(t,u,v) - f(t,\widetilde{u},\widetilde{v})| \le K_1 |u - \widetilde{u}| + K_2 |v - \widetilde{v}|, \tag{1.3}$$

$$|g(u,w,p) - g(\widetilde{u},\widetilde{w},\widetilde{p})| \le K_3 |u - \widetilde{u}| + K_4 |w - \widetilde{w}| + K_5 |p - \widetilde{p}| \tag{1.4}$$

$$|h(t,u) - h(t,\widetilde{u})| \le K_6 |u - \widetilde{u}| \tag{1.5}$$

for any $t \in [a,b]$ fixed, all $\{u,\widetilde{u}\} \subset D$, $\{v,\widetilde{v}\} \subset D^1$, $\{w,\widetilde{w}\} \subset D$, $\{p,\widetilde{p}\} \subset D_2$, where

$$D_2 := \left\{ \int_a^b h(t,x(t))dt : t \in [a,b],\ x \in D \right\}$$

and $K_1 - K_5$ are non-negative square matrices of dimension n. The inequalities between vectors are understood componentwise. A similar convention is adopted for the "absolute value", "max", "min" operations. The symbol I_n stands for the unit matrix of dimension n, $r(K)$ denotes a spectral radius of a square matrix K.

By the solution of the problem (1.1), (1.2) we understand a continuously differentiable function with property (1.2) satisfying (1.1) on $[a,b]$.

We fix certain bounded sets $D_a \subset \mathbb{R}^n$ and $D_b \subset \mathbb{R}^n$ and focus on the solutions x of the given problem with property $x(a) \in D_a$ and $x(b) \in D_b$. Instead of the non-local boundary value problem (1.1), (1.2), we consider the parameterized family of two-point "model -type " problems with simple separated conditions

$$\frac{dx(t)}{dt} = f\left(t, x(t), \frac{dx(t)}{dt}\right),\ t \in [a,b], \tag{1.6}$$

$$x(a) = z,\ x(b) = \eta, \tag{1.7}$$

where $z = (z_1, z_2, ..., z_n),\ \eta = (\eta_1, \eta_2, ..., \eta_n)$ are considered as parameters.

If $z \in \mathbb{R}^n$ and ρ is a vector with non-negative components, $O(z,\rho) := \{\xi \in \mathbb{R}^n : |\xi - z| \leq \rho\}$ stands for the componentwise ρ -neighbourhood of z. For given two bounded connected sets $D_a \subset \mathbb{R}^n$ and $D_b \subset \mathbb{R}^n$, introduce the set

$$D_{a,b} := (1-\theta)z + \theta\eta,\ z \in D_a,\ \eta \in D_b,\ \theta \in [0,1]$$

and its componentwise $\rho-$neighbourhood by putting

$$D = D_\rho := O(D_{a,b}, \rho) = \bigcup_{\xi \in D_{a,b}} O(\xi, \rho) \tag{1.8}$$

We suppose that

$$r(K_2) < 1,\ r(Q) < 1, \tag{1.9}$$

where

$$Q := \frac{3(b-a)}{10} K, \tag{1.10}$$

$$K = K_1 + K_2\ [I_n - K_2]^{-1} K_1 = [I_n - K_2]^{-1} K_1.$$

On the base of function $f : [a,b] \times D \times D^1 \to \mathbb{R}^n$ we introduce the vector

$$\delta_{[a,b],D,D^1}(f) := \frac{1}{2}\left[\max_{(t,x,y)\in[a,b]\times D\times D^1} f(t,x,y) - \min_{(t,x,y)\in[a,b]\times D\times D^1} f(t,x,y)\right] \tag{1.11}$$

and suppose that the $\rho-$neighbourhood in (1.8) is such that

$$\rho \geq \frac{b-a}{2}\delta_{[a,b],D,D^1}(f). \tag{1.12}$$

2. Main statements

The investigation of the solutions of parameterized problem (1.6) and (1.7) is connected with the properties of the following special sequence of functions well posed on the interval $t \in [a,b]$

$$x_0(t,z,\eta) = z + \frac{t-a}{b-a}[\eta - z] = \left[1 - \frac{t-a}{b-a}\right] z + \frac{t-a}{b-a}\eta,\ t \in [a,b], \tag{2.1}$$

$$x_{m+1}(t,z,\eta) = z + \int_a^t f\left(s, x_m(s,z,\eta), \frac{dx_m(s,z,\eta)}{ds}\right) ds -$$

$$- \frac{t-a}{b-a}\int_a^b f\left(s, x_m(s,z,\eta), \frac{dx_m(s,z,\eta)}{ds}\right) ds + \frac{t-a}{b-a}[\eta - z],\ t \in [a,b], \tag{2.2}$$

$m = 0,1,2,...,$

Theorem 1. *Let assumptions (1.3)-(1.5) and (1.9) hold. Then, for all fixed* $(z,\eta) \in D_a \times D_b$ *:*

1. The functions of the sequence (2.2) are continuously differentiable functions on the interval $t \in [a,b]$*, have values in the domain* $D = D_\rho$ *and satisfy the two-point separated boundary conditions (1.7).*

2. The sequence of functions (2.2) in $t \in [a,b]$ *converges uniformly as* $m \to \infty$ *to the limit function*

$$x_\infty(t,z,\eta) = \lim_{m\to\infty} x_m(t,z,\eta), \tag{2.3}$$

satisfying the two-point separated boundary conditions (1.7).

3. The limit function $x_\infty(t,z,\eta)$ *is the unique continuously differentiable solution of the integral equation*

$$x(t) = z + \int_a^t f\left(s, x(s), \frac{dx(s)}{ds}\right) ds - \frac{t-a}{b-a}\int_a^b f\left(s, x(s), \frac{dx(s)}{ds}\right) ds +$$

$$+ \frac{t-a}{b-a}[\eta - z], \tag{2.4}$$

i.e. it is the solution of the Cauchy problem for the modified system of integro-differential equations :

$$\frac{dx}{dt} = f\left(t, x, \frac{dx(t)}{dt}\right) + \frac{1}{b-a}\Delta(z,\eta),\ x(a) = z \tag{2.5}$$

where $\varDelta(z,\eta): D_a \times D_b \to \mathbb{R}^n$ *is a mapping given by formula*

$$\varDelta(z,\eta) := [\eta - z] - \int_a^b f\left(s, x_\infty(s,z,\eta), \frac{dx_\infty(s,z,\eta)}{ds}\right) ds. \tag{2.6}$$

4.The following error estimate holds:

$$|x_\infty(t,z,\eta) - x_m(t,z,\eta)| \leqslant$$
$$\leqslant \frac{10}{9}\alpha_1(t,a,b-a) Q^m (1_n - Q)^{-1} \delta_{[a,b],D,D^1}(f), \tag{2.7}$$

for any $t \in [a,b]$ *and* $m \geq 0$, *where* $\delta_{[a,b],D,D^1}(f)$ *is given in (1.11) and*

$$\alpha_1(t,a,b-a) = 2(t-a)\left(1 - \frac{t-a}{b-a}\right), \ \alpha_1(t,a,b-a) \leq \frac{b-a}{2}. \tag{2.8}$$

Proof. The validity of this statement can be established similarly to Theorem 1 in [4]. □

Theorem 2. *Under the assumption of Theorem 1, the limit function* $x_\infty(t,z,\eta)$: $[a,b] \times D_a \times D_b \to \mathbb{R}^n$ *defined by (2.3) is a continuously differentiable solution of the original BVP (1.1), (1.2) if and only if the pair of vectors* (z,η) *satisfies the system of* $2n$ *determining algebraic equations*

$$\begin{cases} \varDelta(z,\eta) = \eta - z - \int_a^b f\left(s, x_\infty(s,z,\eta), \frac{dx_\infty(s,z,\eta)}{ds}\right) ds = 0, \\ \varLambda(z,\eta) = g\left(x_\infty(a,z,\eta), x_\infty(b,z,\eta), \int_a^b h(s, x_\infty(s,z,\eta)) ds\right) - d = 0. \end{cases} \tag{2.9}$$

Note, that similarly as in [3], the solvability of the determining system (2.9) on the base of (1.3)-(1.5) and (1.9) can be established by studying its $m-$th approximate versions:

$$\begin{cases} \varDelta_m(z,\eta) = \eta - z - \int_a^b f\left(s, x_m(s,z,\eta), \frac{dx_m(s,z,\eta)}{ds}\right) ds = 0, \\ \varLambda_m(z,\eta) = g\left(x_m(a,z,\eta), x_m(b,z,\eta), \int_a^b h(s, x_m(s,z,\eta)) ds\right) - d = 0. \end{cases} \tag{2.10}$$

where m is fixed.

Lemma 1. *Under the assumptions of Theorem 1, for the exact and approximate determining functions defined by (2.9) and (2.10) for any* $(z,\eta) \in D_a \times D_b$ *and* $m \geq 1$

hold the following estimates:

$$|\Delta(z,\eta)-\Delta_m(z,\eta)| \le \frac{10(b-a)^2}{27} K Q^m (1_n-Q)^{-1}\delta_{[a,b],D,D_1}(f), \tag{2.11}$$

$$|\Lambda(z,\eta)-\Lambda_m(z,\eta)| \le \frac{5(b-a)}{9}[K_3+K_4+ \\ +(b-a)K_5K_6]\; Q^m(1_n-Q)^{-1}\delta_{[a,b],D,D_1}(f), \tag{2.12}$$

where the matrix Q and the vector $\delta_{[a,b],D,D_1}(f)$ are given respectively in (1.10) and (1.11).

Proof. Let us fix an arbitrary $(z,\eta)\in D_a\times D_b$. Direct computation gives that

$$\int_a^b \alpha_1(t,a,b-a)dt = \frac{(b-a)^2}{3}.$$

On the base of (1.1) and (1.3), when $u\neq\widetilde{u}$, we have

$$|f(t,u,v)-f(t,\widetilde{u},\widetilde{v})| \le K|u-\widetilde{u}|,$$

where matrix K is given in (1.10). Taking into account (2.7) we obtain

$$|\Delta(z,\eta)-\Delta_m(z,\eta)| =$$

$$=\left|\int_a^b f\left(s,x_\infty(s,z,\eta),\frac{dx_\infty(s,z,\eta)}{dt}\right)ds-\int_a^b f\left(s,x_m(s,z,\eta),\frac{dx_m(s,z,\eta)}{dt}\right)ds\right|\le$$

$$\le K\int_a^b \frac{10}{9}\alpha_1(s,a,b)Q^m(1_n-Q)^{-1}\delta_{[a,b],D,D_1}(f)ds =$$

$$=\frac{10(b-a)^2}{27}KQ^m(1_n-Q)^{-1}\delta_{[a,b],D,D_1}(f),$$

which proves (2.11).

From (2.9) and (2.10) using the Lipschitz conditions (1.4), (1.5) and estimates (2.7), (2.8), we obtain

$$|\Lambda(z,\eta)-\Lambda_m(z,\eta)| = \left|g\left(x_\infty(a,z,\eta),x_\infty(b,z,\eta),\int_a^b h(s,x_\infty(b,z,\eta))ds\right)-\right.$$

$$\left.-\,g\left(x_m(a,z,\eta),x_m(b,z,\eta),\int_a^b h(s,x_m(s,z,\eta))ds\right)\right|\le$$

$$\le K_3|x_\infty(a,z,\eta)-x_m(a,z,\eta)|+K_4|x_\infty(b,z,\eta)-x_m(b,z,\eta)|+$$

$$+(b-a)K_5K_6\left|x_\infty(t,z,\eta)-x_m(t,z,\eta)\right|\le$$
$$\le\frac{5(b-a)}{9}\left[K_3+K_4+(b-a)K_5K_6\right]\,Q^m(1_n-Q)^{-1}\delta_{[a,b],D,D_1}(f),$$

i.e. (2.12) holds also. □

Based on both exact and approximate determining systems (2.9) and (2.10) let us introduce the mappings $H: D_a\times D_b\to\mathbb{R}^{2n}$ and $H_m: D_a\times D_b\to\mathbb{R}^{2n}$ by setting

$$H(z,\eta)=\begin{bmatrix}[\eta-z]-\int_a^b f\left(s,x_\infty(a,z,\eta),x_\infty(b,z,\eta),\int_a^b h(s,x_\infty(s,z,\eta))\right)ds,\\ g\left(x_\infty(a,z,\eta),x_\infty(b,z,\eta),\int_a^b h(s,x_\infty(b,z,\eta))ds\right)-d,\end{bmatrix}\tag{2.13}$$

$$H_m(z,\eta)=\begin{bmatrix}[\eta-z]-\int_a^b f\left(s,x_m(a,z,\eta),x_m(b,z,\eta),\int_a^b h(s,x_m(s,z,\eta))\right)ds,\\ g\left(x_m(a,z,\eta),x_m(b,z,\eta),\int_a^b h(s,x_m(b,z,\eta))ds\right)-d,\end{bmatrix}\tag{2.14}$$

$(z,\eta)\in D_a\times D_b$. We see from Theorem 2 that the critical points of the vector field H of the form (2.13) determine solutions of the non-linear boundary value problem (1.1)-(1.2). The next statement establishes a similar result based upon properties of vector field H_m explicity known from (2.14).

Theorem 3. *Assume that the conditions of Lemma 1 hold. Moreover, one can specify an $m\ge 1$ and a set*

$$\Gamma:=D_1\times D_2\subset\mathbb{R}^{2n},$$

where $D_1\subset D_a, D_2\subset D_b$ are certain bounded open sets such that the mapping H_m satisfies the relation

$$|H_m(z,\eta)|\rhd_{\partial\Gamma}\begin{bmatrix}\frac{10(b-a)^2}{27}KQ^m(1_n-Q)^{-1}\delta_{[a,b],D,D_1}(f)\\ \frac{5(b-a)}{9}\left[K_3+K_4+(b-a)K_5K_6\right]\,Q^m(1_n-Q)^{-1}\delta_{[a,b],D,D_1}(f)\end{bmatrix}\tag{2.15}$$

on the boundary $\partial\Gamma$ of the set Ω. If, in addition

$$\deg(H_m,\Omega,0)\ne 0,\tag{2.16}$$

then there exists a pair $(z^*,\eta^*)\in D_1\times D_2$ for which the function

$$x^*(\cdot):=x_\infty(\cdot,z^*,\eta^*)$$

is a solution of the non-linear boundary value problem (1.1)-(1.2).

In (2.15) the binary relation $\rhd_{\partial \Gamma}$ is defined in [1] as a kind of strict inequality for vector functions and it means that at every point on the boundary $\partial \Gamma$ at least one of the components of the vector $|H_m(z,\eta)|$ is greater than the corresponding component of the vector on the right-hand side . The degree in (2.16) is the Brouwer degree because all the vectors fields are finite-dimensional. Likewise, all the terms on the right-hand side of (2.15) are computed explicitly e.g. by using computer algebra system.

Proof. The proof can be carried out similarly as in Theorem 4 from [3]. □

3. Example

Let us apply the approach described above to the system of differential equations

$$\begin{cases} \frac{dx_1(t)}{dt} = \frac{1}{2}x_2^2(t) - t\frac{dx_2(t)}{dt}x_1(t) + \frac{1}{32}t^3 - \frac{1}{32}t^2 + \frac{9}{40}t \\ \frac{dx_2(t)}{dt} = \frac{1}{2}\frac{dx_1(t)}{dt}x_1(t) - t^2x_2(t) + \frac{15}{64}t^3 + \frac{1}{80}t + \frac{1}{4} \end{cases}, \ t \in [0,1],$$

considered with non-linear boundary conditions

$$x_1(0)\,x_2(1) + \left[\int_0^1 x_1(s)ds\right]^2 = -\frac{311}{14400},$$

$$x_1(1)x_2(0) - \int_0^1 x_2(s)ds = -\frac{1}{8}.$$

Introduce the vector of parameters $z = col(z_1,z_2)$, $\eta = col(\eta_1,\eta_2)$. Let us consider the following choice of the subsets D_a, D_b and D^1 :

$$D_a = D_b = \{(x_1,x_2) : -0.1 \le x_1 \le 0.2,\ -0.2 \le x_2 \le 0.3\},$$

$$D^1 = \left\{\left(\frac{dx_1}{dt},\frac{dx_2}{dt}\right) : -0.1 \le \frac{dx_1}{dt} \le 0.3,\ -0.1 \le \frac{dx_2}{dt} \le 0.3\right\}.$$

In this case $D_{a,b} = D_a = D_b$. For $\rho = col(\rho_1,\rho_2)$ involved in (1.12), we choose the vector $\rho = col(0.4;0.4)$. Then, in view of (2.13) the sets (1.8) and D_2 takes the form:

$$D = D_\rho = \{(x_1,x_2) : -0.5 \le x_1 \le 0.6,\ -0.6 \le x_2 \le 0.7\}$$

and

$$D_2 = \{(x_1,x_2) : 0.25 \le x_1 \le 0.36,\ -0.6 \le x_2 \le 0.7\}.$$

A direct computation shows that the conditions (1.3)-(1.5) hold with

$$K_1 = \begin{bmatrix} 0.3 & 0.7 \\ 0.15 & 1 \end{bmatrix}, K_2 = \begin{bmatrix} 0 & 0.6 \\ 0.3 & 0 \end{bmatrix}, K_3 = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.2 \end{bmatrix},$$

$$K_4 = \begin{bmatrix} 0 & 0.2 \\ 0.3 & 0 \end{bmatrix}, K_5 = \begin{bmatrix} 1.2 & 0 \\ 0 & 1 \end{bmatrix}, K_6 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

TABLE 1.

m	z_1	z_2	η_1	η_2
0	-0.089643967	-0.0002812586	0.03176891	0.25026338
1	-0.0994489263	0.00051937347	0.0255001973	0.2504687527
4	-0.0999998827	$7.744981 \cdot 10^{-8}$	0.02500007591	0.25000011
6	-0.1000000004	$-2.263731 \cdot 10^{-10}$	0.02499999973	0.2499999996
Exact	-0.1	0	0.025	0.25

and therefore $r(K_2) = \sqrt{0.18} < 1$, and in (1.10) the matrix

$$K = \begin{bmatrix} 0.4756097561 & 1.585365854 \\ 0..2926829268 & 1.475609756 \end{bmatrix},$$

$$Q = \begin{bmatrix} 0.07134146342 & 0.2378048781 \\ 0.04390243902 & 0.2213414634 \end{bmatrix}, \; r(Q) = 0.273090089272152 < 1.$$

Furthermore, in view of (1.11)

$$\delta_{[a,b],D,D^1}(f) := \frac{1}{2}\left[\max_{(t,x,y)\in[a,b]\times D\times D^1} f(t,x,y) - \min_{(t,x,y)\in[a,b]\times D\times D^1} f(t,x,y)\right] =$$

$$= \begin{bmatrix} 0.31 \\ 0.7325 \end{bmatrix},$$

$$\rho = \begin{bmatrix} 0.4 \\ 0.4 \end{bmatrix} \geq \frac{b-a}{2}\delta_{[a,b],D,D^1}(f) = \begin{bmatrix} 0.155 \\ 0.36625 \end{bmatrix}.$$

We thus see that all conditions of Theorem 1 are fulfilled, and the sequence of functions (2.2) for this example is uniformly convergent.

Applying Maple 14, we can carried out the calculations.

It is easy to check that

$$x_1^*(t) = \frac{t^2}{8} - \frac{1}{10}, \;\; x_2^*(t) = \frac{t}{4}$$

is an exact continuously differentiable solution of the problem (1.1), (1.2). For a different number of approximations m we obtain from (2.10) the following numerical values for the introduced parameters, which are presented in Table 3.

On the Figure 1 one can see the graphs of the exact solution (solid line) and its zero ($\diamond$) and sixth approximation ($\times$) for the first and second coordinates.

The error of the sixth approximation ($m = 6$) for the first and second components:

$$\max_{t\in[0,1]} \left|x_1^*(t) - x_{61}(t)\right| \leq 1 \cdot 10^{-9}, \;\; \max_{t\in[0,1]} \left|x_2^*(t) - x_{62}(t)\right| \leq 5 \cdot 10^{-9}.$$

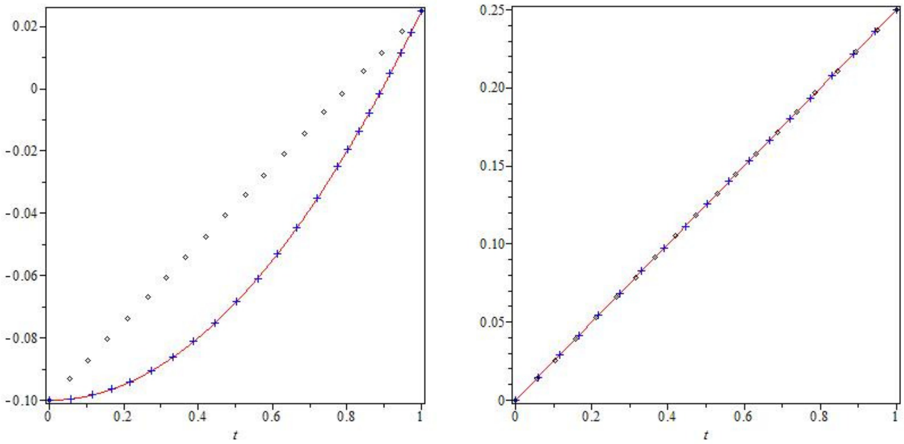

FIGURE 1.

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