Exactly solvable model with coupled channels

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Introduction

It is well known that exact analytical solutions can be found for a relatively small number of physically interesting systems in both the classic and quantum cases. But the exact solutions often serve as a starting point for approximate calculations. In addition, they can help to define the edge of applicability of different approximate methods.

The first solutions of Schrödinger equation were found for the simplest potentials which direct equation of motion to differential equations for the known elementary and special functions. For example, such potentials are

$$V(r) = ar^n,$$

where $n = 2$ correspond to oscillator, $n = 1$ – to linear, $n = 0$ – to constant, $n = -1$ – to Coulomb and $n = -2$ – to centrifugal potentials [1, 2].

The class of tasks, which assumes a solution in the exact analytical kind, for the systems of coupled equations is considerably poorer, than at the single channel [3, 4]. At the same time a necessity for simple models for research of complex physical systems (with coupled channels) is yet more vital. Multichannel approach shows by itself universal and power mean of microscopic description of the many-particle and multidimensional systems, of processes connected with excitation of different degrees in them, and also by the method of solution of tasks with mixing of the spin states [5]. In this article an exact analytical solution is presented which is found for the system of light-heavy quarks with mixed spin states [6].

Physical states of two fermions system

The state of two fermions system is characterized by energy of system E, total angular momentum J ($\vec{J} = \vec{L} + \vec{S}$), projection of momentum M_J and spatial parity $P = (-1)^{L+1}$. Orbital momentum L and total spin of the system S ($\vec{S} = \vec{s}_1 + \vec{s}_2$) aren't "good" quantum numbers in general case, but it is convenient to use them for denotation of the concrete states. If the system consists of particle and its anti-particle ($q\bar{q}$), the state of such system is characterized additionally by charge–conjugation parity $C = (-1)^{L+S}$ and has a definite spin S. The possible states of the two-fermions system are determined by the rules of addition of angular momentum and resulted in a table 1, where we used spectroscopy denotations $^{2S+1}L_J$.

Table 1. The states of two fermions with equal masses.

J ↓ \ P⟶	Singlet (S = 0)		Triplet (S = 1)	
	+	−	+	−
0	—	$^1S_0$	$^3P_0$	—
1	$^1P_1$	—	$^3P_1$	$^3S_1+^3D_1$
2	—	$^1D_2$	$^3P_2+^3F_2$	$^3D_2$

Such states are characteristic for quarkonium $q\bar{q}$, positronium e^+e^- and other similar systems. Evidently, that in this case the states $^1S_0$, $^3P_0$, $^1P_1$, $^3P_1$, are the pure states, but states $^3S_1+^3D_1$, $^3P_2+^3F_2$ and so on are mixed ones. We note that all states have definite spin.

In the case when the two fermions system consists of particles of different mass (for example, $q\overline{Q}$), a total spin of the system already is not simply certain and the physical states of such system can be got imposition of two parts a table 1, and that is resulted in a table 2.

Table 2. The states of two fermions with different masses.

J ↓ \ P⟶	+	−
0	${}^3P_0$	${}^1S_0$
1	${}^1P_1+{}^3P_1$	${}^3S_1+{}^3D_1$
2	${}^3P_2+{}^3F_2$	${}^1D_2+{}^3D_2$

In this situation only two states ${}^1S_0$ and ${}^3P_0$ are the pure states, but all others are mixed on an orbital moment (but definite on the spin), or on a spin (but definite on the orbital momentum).

Potential models with coupled channels

By analogy with the well studied structure of nucleon - nucleon potential [7], Hamiltonian of the quark-quark system with different masses ($m_1 \neq m_2$) can be written down in a kind [8, 9]

$$H = \frac{p^2}{2\mu} + V_0(r) + V_{LS}(r)(\vec{L}\vec{S}) + V_{LS-}(r)(\vec{L}\vec{S}_-) + V_{SS}(r)(\vec{s}_1\vec{s}_2) + V_T(r)S_{12}, \tag{1}$$

where μ is reduced mass of the system, $\vec{S}_- = \vec{s}_1 - \vec{s}_2$. In Hamiltonian (1) it is taken into account only members which are linear in relation to a spin and orbital moments, namely, symmetric ($\vec{L}\vec{S}$) and antisymmetric ($\vec{L}\vec{S}_-$) spin-orbital operators, spin-spin ($\vec{s}_1\vec{s}_2$) and spin-tensor $S_{12} = 12\left[(\vec{s}_1\vec{r})(\vec{s}_2\vec{r})/r^2 - (\vec{s}_1\vec{s}_2)/3\right]$ operators.

According to the table 2, for the system of different constituent flavors ($m_q \neq m_{\bar{Q}}$), the states with the complete moment $J = L$ are described by a double-base wave function

$$\Psi(\vec{r}) = \frac{u(r)}{r}\Im^M_{JJ0}(\theta, \varphi) + \frac{w(r)}{r}\Im^M_{JJ1}(\theta, \varphi), \tag{2}$$

where $u(r)$ and $w(r)$ are radial wave functions, which correspond to singlet and triplet spin configuration, and $\Im^M_{JLS}(\theta, \varphi)$ is spin-orbital part of the wave function.

In work [6] it was shown, that similarly to mixing of orbital momentums by tensor forces in a deuteron [10] and in a quarkonium [11], it could be got the system of coupled Schrödinger equations for mesons with quarks of different flavors and total momentum $J = L$

$$\begin{cases} u'' + \left(k^2 - \frac{J(J+1)}{r^2} - U_1(r)\right)u = U_3(r)w \\ w'' + \left(k^2 - \frac{J(J+1)}{r^2} - U_2(r)\right)w = U_3(r)u \end{cases}, \tag{3}$$

where $k^2 = 2\mu E$ and $U_i(r)$ are channels' potentials, which are expressed through the components of quark-quark potential

$$U_1(r) = 2\mu\left(V(r) - \frac{3}{4}V_{SS}(r)\right),$$

$$U_2(r) = 2\mu\left(V(r) - V_{LS}(r) + \frac{1}{4}V_{SS}(r) + 2V_T(r)\right), \tag{4}$$

$$U_3(r) = 2\mu\left(\sqrt{J(J+1)}V_{LS-}(r)\right).$$

The numerical solution of the system (3) was realized in work [6]. For the receipt of analytical solution we will choose channel potentials at a Coulomb kind, namely, $U_1(r) = \frac{A}{r}$, $U_2(r) = \frac{B}{r}$ and $U_3(r) = \frac{C}{r}$. If one substitutes them in (3), we obtain such coupled system of equations

$$\begin{cases} u'' + \left(k^2 - \frac{J(J+1)}{r^2} - \frac{A}{r}\right)u = \frac{C}{r}w; \\ w'' + \left(k^2 - \frac{J(J+1)}{r^2} - \frac{B}{r}\right)w = \frac{C}{r}u. \end{cases}, \tag{5}$$

If potential parameter C is set to zero then system (5) will contain two uncoupled equations. Because for Coulomb potential the bound energies are negative the value k^2 in (5) is connected with the energy by expression

$$k^2 = -2\mu E. \tag{6}$$

Analytical solution

Similarly to a task for the hydrogen atom, an asymptotic of wave functions looks like e^{-kr} on the infinity $(r \longrightarrow \infty)$. Therefore we will search the solution of the system (5) in a kind

$$u = f(r) \cdot e^{-kr}; \quad w = g(r) \cdot e^{-kr}. \tag{7}$$

After a substitution (7) in the system (5) and obvious simplifications, we will obtain:

$$\begin{cases} r^2 f''(r) - Lf(r) - L^2 f(r) - Arf(r) - 2kr^2 f(r) = Crg(r) \\ r^2 g''(r) - Lg(r) - L^2 g(r) - Brg(r) - 2kr^2 g(r) = Crf(r) \end{cases} \tag{8}$$

Farther it is convenient to convert the system of two differential equations of the second order to the single equation of the fourth order. For this purpose we will define the function $g(r)$ from the first equation of the system (8) and substitute it in the second equation. For the function $f(r)$ we will get differential equation

$$r^4 f^{(4)} + r^3 p_3(r) f^{(3)} + r^2 p_2(r) f'' + +rp_1(r)f' + p_0(r)f = 0, \tag{9}$$

where $p_i(r)$ are polynomial functions whose coefficients contain the potential parameters A, B, C, orbital momentum L and energy parameter k. We will search the solution of equation (9) in the kind of power series

$$f(r) = r^s(a_0 + a_1 r + a_2 r^2 + a_3 r^3 + a_4 r^4 + a_5 r^5 + ...), \tag{10}$$

where s and a_i are unknown coefficients $(a_0 \neq 0)$. We will substitute (10) in (9) and collect together terms with like powers of r and set equal to zero. A fourth power equation in s (indicial equation) arises when the coefficient of r^0 is set to zero and a_0 is left arbitrary

$$r^0 : -2a_0 L - a_0 L^2 + 2a_0 L^3 + a_0 L^4 - 2a_0 s + +4a_0 Ls + 5a_0 s^2 + 2a_0^L s^2 - 4a_0 s^3 + a_0 s^4 = 0.$$

This indicial equation gives four real roots

$$((s_1 = L + 2), (s_2 = L + 1), (s_3 = -L + 1), (s_4 = -L)). \tag{11}$$

The first two solutions $f(r) \sim r^{L+2}$ and $f(r) \sim r^{L+1}$ obey the boundary condition in the origin $u(0) = w(0) = 0$, but the other don't. It can be noticed that a point of $r = 0$ is the erroneous special point [12] of equation (9) and so the both noted solution are independent ones. Namely, function

$$f(r) = a_0 r^{L+1} + a_1 r^{L+2} + a_2 r^{L+3} + ... \tag{12}$$

with two free constants a_0 and a_1 will be a general solution which obeys the first boundary condition.

At the definite power parameter of $s = L + 1$, setting coefficients at the next degrees of r equal to zero pozvolyae consistently to define while unknown coefficients a_2, a_3, a_4... of function (12):

$$\begin{aligned} (AB - C^2 + 2Ak + 2Bk + 4k^2 + 2AkL + 2BkL + 8k^2 L + 4k^2 L^2)a_0 + \\ +(-2A - 2B - 12k - 2AL - 2BL - 20kL - 8kL^2)a_1 + (12 + 20L + 8L^2)a_2 = 0 \end{aligned} \tag{13}$$

$$\begin{aligned} (AB - C^2 + 4Ak + 4Bk + 16k^2 + 2AkL + 2BkL + 16k^2 L + 4k^2 L^2)a_1 + \\ (-6A - 6B - 60k - 4AL - 4BL - 64kL - 16kL^2)a_2 + (72 + +84L + 24L^2)a_3 = 0 \end{aligned} \tag{14}$$

At set a_0 and a_1 from (13) we will find a_2, then from (14) it is possible to define $a_3$et cetera. From the structure of equations (13) and (14) evidently, that the system of equations for determination of coefficients of power series (12) has a tridiagonal form of the following kind:

$$\begin{cases} b_{11} \cdot a_0 + b_{12} \cdot a_1 + b_{13} \cdot a_2 = 0 \\ \cdots\cdots\cdots b_{22} \cdot a_1 + b_{23} \cdot a_2 + b_{24} \cdot a_3 = 0 \\ \cdots\cdots\cdots\cdots\cdots b_{33} \cdot a_2 + b_{34} \cdot a_3 + b_{35} \cdot a_4 = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots b_{44} \cdot a_3 + b_{45} \cdot a_4 + b_{46} \cdot a_5 = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{cases} , \qquad (15)$$

where the first two coefficients b_{ij} in each the row depend on the parameters of potentials of A, B, C, orbital number L and eigenvalue parameter k of system (3), but the third – only from an orbital number.

As a partial decision (12) is determined by two satisfactory constants, we can dispose of them thus, to build polynomial functions. Namely, we get the first polynomial, taking

$$a_0 = 1, \quad a_1 = 0 \qquad (16)$$

but need set to zero a coefficient b_{11}

$$\left(4 + 8L + 4L^2\right) k^2 + (2A + B + 2AL + 2BL)\, k + AB - C^2 = 0. \qquad (17)$$

The condition (16) is an equation on an eigenvalue k

$$k_{1,2} = -\frac{A + B \pm \sqrt{(A - B)^2 + 4C^2}}{4(L + 1)}. \qquad (18)$$

Substitution of (16) and (18) in the system (15) gives the next values of coefficients of power series (12)

$$a_2 = a_3 = ... = 0.$$

So, according to (7), (8) and (12), we get the first two pairs of radial functions

$$\begin{aligned} u(r) &= r^{L+1} e^{-kr} \\ w(r) &= -\frac{A + 2k(L+1)}{C} r^{L+1} e^{-kr} \end{aligned} , \qquad (19)$$

where k takes on two values (18).

An algorithm (16-18) can be repeated, namely, to take satisfactory a_1 and a_2

$$a_1 = 1, \quad a_2 = 0$$

and to need set to zero a coefficient b_{22}

$$\left(16 + 16L + 4L^2\right) k^2 + (4A + 4B + 2AL + 2BL)\, k + AB - C^2 = 0,$$

that determines the next two eigenvalues of wave-number k

$$k_{3,4} = -\frac{A + B \pm \sqrt{(A - B)^2 + 4C^2}}{4(L + 2)}. \qquad (20)$$

Such initial choice gives us next two solutions

$$\begin{aligned} u(r) &= r^{L+1}(a_0 + r) e^{-kr} \\ w(r) &= r^{L+1} \left(\frac{2(L+1)(1 - a_0 k) - a_0 A}{C} - \frac{A + 2k(L+2)}{C} r \right) e^{-kr} \end{aligned} , \qquad (21)$$

where a_0 is determined from equation (13) and eigenvalues (20) and looks like

$$a_0 = \frac{4(L + 1)(L + 2)}{A + B \pm \sqrt{(A - B)^2 + 4C^2}}. \qquad (22)$$

Results and discussions

The main result we have obtained is that the model of coupled channels with Coulomb like potentials (5) has exact analytical solution which is very similar to the solution of single channel case (hydrogen atom). The radial wave functions are exponent function multiplied by polynomials. In the case of coupled channels the eigenvalues k are twofold (see (18) and (20)). The greater eigenvalue we indexed α and the second one as β

$$k_{n\alpha} = -\frac{A + B - \sqrt{(A-B)^2 + 4C^2}}{4(L+n)}, \quad k_{n\beta} = -\frac{A + B + \sqrt{(A-B)^2 + 4C^2}}{4(L+n)}, \tag{23}$$

where $n = 1, 2, \ldots$ And according to (6) the correspondent energies are

$$E_{n\alpha} = \frac{-(A^2 + B^2 + 2C^2) + (A + B)\sqrt{(A-B)^2 + 4C^2}}{16\mu(L+n)^2}, \tag{24}$$

$$E_{n\beta} = \frac{-(A^2 + B^2 + 2C^2) - (A + B)\sqrt{(A-B)^2 + 4C^2}}{16\mu(L+n)^2}. \tag{25}$$

The energy $E_{n\alpha}$ is the ground state of the system. According to (7) the physical values of value k must be positive and from (23) it follows that if a condition

$$C < \sqrt{A \cdot B} \tag{26}$$

is correct (the case of weak coupling) then we have two eigenvalues of energy (24) and (25) but if the condition (26) isn't correct (strong coupling) we have only one eigenvalue of energy (24).

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