GENERALIZED PLANE WAVES FOR SCHRÖDINGER AND DIRAC PARTICLES ON THE BACKGROUND OF BOLYAI–LOBACHEVSKY GEOMETRY: SIMULATING OF A SPECIAL MEDIUM

Bolyai–Lobachevsky geometry substantially affects quantum-mechanical particles, simulating a medium with special reflecting properties of an ideal mirror.

For Schrödinger particle the problem reduces to a second order differential equation which can be associated with one-dimensional Schrödinger problem for a particle in external potential field \( U(z) = U_0 e^{2z} \). In quantum mechanics, curved geometry acts as an effective potential barrier with reflection coefficient \( R = 1 \). Hyperbolic geometry simulates a medium that effectively acts as an ideal mirror. Penetration of the particle into the effective medium, depends on the parameters of quantum solutions \( \varepsilon, k^2 + k^2 \), and the curvature radius \( \rho \). Similar analysis is performed for the case of a Dirac spin 1/2 particle; additional to the quantum numbers \( \varepsilon, k^2 + k^2 \) for the spin 0 particle here is a quantum number related with an extended helicity operator.

Key words: geometry Bolyai–Lobachevsky, plane wave, Schrödinger equation, Dirac equation, hypergeometric function.

Introduction

It is known that in the field theory of elementary particles, the basis of plane waves is of the most use. However, in presence of a curvature, any common plane wave solutions do not exist. Therefore, of a special interest are examples non-Euclidean spaces in which some analogues of such solutions can be constructed. In the paper [1], it was shown that in the Lobachevsky space there are such solutions for particles with spin 0; also see the books by Gelfand–Graev–Vilenkin [2], [3]. An analog of plane waves in a space of constant positive curvature was studied by Volobuev [4]. The later treatment of this problem was given in [5]. Solutions of the plane wave type for Maxwell’s equations have been considered in [6]–[9]. In [10], the problem of constructing solutions of the Dirac equation in the Lobachevsky space was studied on the base of the method of squaring; in particular, it was pointed out the possibility of constructing solutions of the Dirac plane waves starting with Shapiro’s scalar waves. In this paper we will construct a complete basis of solutions of the plane wave type for Dirac and Weyl particles in the Lobachevsky space, applying the method of separation of the variables in a special system of quasi-cartesian coordinates closely related to horospherical coordinates.

To understand the physical meaning of the system under consideration, it should be mentioned that Lobachevsky geometry simulates a medium with special constitutive relations. The situation being specified in quasi-cartesian coordinates \((x, y, z)\) was treated in [9]. Exact solutions of the Maxwell equations in complex 3-vector form, extended to curved space models within the tetrad formalism, have been found in Lobachevsky space. The problem reduces to a second order differential equation which can be associated with an 1-dimensional Schrödinger problem for a particle in external potential field \( U(z) = U_0 e^{2z} \). In quantum mechanics, curved geometry acts as an effective potential barrier with reflection coefficient \( R = 1 \); in electro-
dynamics context results similar to quantum-mechanical ones arise: the Lobachevsky geometry simulates a medium that effectively acts as an ideal mirror. Penetration of the electromagnetic field into the effective medium, depends on the parameters of an electromagnetic wave, frequency $\omega$, $k_1^2 + k_2^2$, and the curvature radius $\rho$.

In the present paper, that analysis will be extended to the case of particles with spin 1/2, described by equations of Dirac and Weyl. The generalized spinor plane waves can find application in the analysis of the behavior of fermions particles on cosmological scales, or in simulating special media affecting the spinor particles.

1. On the solutions of the Schrödinger equation

In the Lobachevsky space–time parameterized by quasi-cartesian coordinates

$$dS^2 = dt^2 - e^{2z}(dx^2 + dy^2) - dz^2,$$

the element of volume given by $dV$ and the sign of the $z$ are substantial, in particular when referring to the probabilistic interpretation of the wave functions

$$dW = |\Psi|^2 \, dV = |\Psi|^2 \, e^{-2z} \, dx \, dy \, dz.$$  

Let us describe some details of the parametrization of the space by coordinates $(x, y, z)$. It is known that this model can be identified with a branch of hyperboloid in 4-dimensional flat space

$$u_0^2 - u_1^2 - u_2^2 - u_3^2 = \rho^2, u_0 = +\sqrt{\rho^2 + u^2}.$$ 

Coordinates $x, y, z$ are referred to $u_a$ by relations

$$u_i = xe^{-z}, u_2 = ye^{-z},$$

$$u_3 = \frac{1}{2}[(e^z - e^{-z}) + (x^2 + y^2)e^{-z}],$$

$$u_0 = \frac{1}{2}[(e^z + e^{-z}) + (x^2 + y^2)e^{-z}].$$  \hspace{1cm} (1.1a)

It is convenient to employ 3-dimensional Poincaré realization for Lobachevsky space as an inside part of 3-sphere:

$$q_i = \frac{u_i}{\sqrt{\rho^2 + u_1^2 + u_2^2 + u_3^2}}, q_i q_i < +1.$$ \hspace{1cm} (1.1b)

Quasi-cartesian coordinates $(x, y, z)$ are referred to $q_i$ as follows

$$x = \frac{q_1}{1-q_3}, y = \frac{q_2}{1-q_3}, e^z = \frac{\sqrt{1-q^2}}{1-q_3}. \hspace{1cm} (1.1c)$$

In particular, note that on the axis $q_i = 0, q = 0, q \in (-1,+1)$ relations (1.1c) assume the form

$$q_3 \rightarrow +1, e^z \rightarrow +\infty, z \rightarrow +\infty;$$

$$q_3 \rightarrow -1, e^z \rightarrow +0, z \rightarrow -\infty. \hspace{1cm} (1.2)$$

Schrödinger equation in Riemannian space [11], in quasi-cartesian coordinates (1.1a) takes the form

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2M} \left( e^{2z} \frac{\partial^2 \Psi}{\partial x^2} + e^{2z} \frac{\partial^2 \Psi}{\partial y^2} + e^{2z} \frac{\partial \Psi}{\partial z} \right) |e^{-2z} \frac{\partial \Psi}{\partial z} \right).$$

The variables are separated by the substitution $\Psi = e^{-ikx} e^{iz} f(z)$:

$$\left( \frac{d^2}{dz^2} - 2 \frac{d}{dz} + \varepsilon - e^{2z} (k_1^2 + k_2^2) \right) f(z) = 0,$$  \hspace{1cm} (1.3a)

where a dimensionless quantity used $\varepsilon = 2ME\rho^2/\hbar^2$, $\rho$ - curvature radius of the space. Elementary substitution $f = \phi(z)$ in equation (1.3a) gives a Schrödinger-like equation

$$\left( \frac{d^2}{dz^2} + \varepsilon - 1 - (k_1^2 + k_2^2)e^{2z} \right) \phi(z) = 0 \hspace{1cm} (1.3b)$$

with potential function

$$U(z) = 1 + (k_1^2 + k_2^2)e^{2z}. \hspace{1cm} (1.3c)$$

Note that the probabilistic interpretation of the wave function after the transformation to $\phi$ reads

$$dW = |\Psi|^2 \, dV = |\phi|^2 \, dx dy dz. \hspace{1cm} (1.4)$$

An easily interpretable physical solution for $\varepsilon > 1$ is the following: on the left we have the superposition of two waves, falling from the
left and reflected. On the right behind the barrier, the wave function must sharply
decrease to zero.

It should be noted that the case

\[ k_1 = 0, \quad k_2 = 0 \]

is special: the equation (1.3a) is very much changed because the potential function

\[ f = e^{(x_1^2 - 1)x}, \quad \phi = e^{(x_2^2 - 1)x}, \]

and the function \( \phi \) is a solution of the type of ordinary plane wave.

Let us turn to the general case and in eq. (1.3a) introduce the variable

\[ \sqrt{k_1^2 + k_2^2} e^{\varphi} = Z, Z \in (0, +\infty); \]

the equation takes the form

\[
\left( \frac{d^2}{dz^2} - \frac{1}{Z} \frac{d}{dz} + \frac{\varepsilon}{Z^2} - 1 \right) f(Z) = 0;
\]

with the help of a substitution \( f = \sqrt{Z} F \), one can remove the term with the first derivative

\[
\left( \frac{d^2}{dz^2} + \frac{\varepsilon - 3/4}{Z^2} - 1 \right) F(Z) = 0.
\]

This form makes it easy to find the asymptotical behavior of solutions

\[
(z \to -\infty) Z \to 0, F \propto Z^{1/2+i\sqrt{-\varepsilon}},
\]

\[
f \propto e^{1/2+i\sqrt{-\varepsilon}x}, \]

\[
(z \to +\infty) Z \to +\infty, F \propto e^{iZ},
\]

\[
f \propto e^{-iZ/2} \exp(\pm \sqrt{k_1^2 + k_2^2} e^{\varphi}). \]

We now turn to the construction of exact solutions of (1.6) in the entire range of coordinate \( z \). We seek solutions in the form of

\[ f(Z) = Z^A e^{BZ} F(Z); \] at \( A, B \) chosen according (for definiteness, we take the minus sign before the root in the expression for \( A \); assuming \( \varepsilon > 1 \))

\[ A = 1 - i\sqrt{\varepsilon - 1}, \quad B^2 = 1, \]

the equation (1.6) gives (let us make another change \( Z = y/2 \) and let \( B = -1 \)) an equation for the confluent hypergeometric function

\[
y \frac{d^2Y}{dY^2} + (c-y) \frac{dY}{dy} - aY = 0,
\]

\[ c = 2a, a = A - 1/2 = 1/2 - i\sqrt{\varepsilon - 1}, \]

\[ f(Z) = y^{a + 1/2} e^{-y/2} Y(y). \]

We use two pairs of linearly independent solutions [12]

\[ Y_1 = \Phi(a,2a,y), Y_2 = y^{1-2a} \Phi(1-a,2-2a,y); \]

\[ Y_5 = \Psi(a,2a,y), Y_7 = e^a \Psi(a,2a,-y). \]

These pairs of solutions are related by Kummer linear relations [12]

\[ Y_5 = \frac{\Gamma(1-2a)}{\Gamma(1-a)} Y_1 + \frac{\Gamma(2a-1)}{\Gamma(a)} Y_2, \]

\[ Y_7 = \frac{\Gamma(1-2a)}{\Gamma(1-a)} Y_1 - \frac{\Gamma(2a-1)}{\Gamma(a)} Y_2, \]

which after multiplication by \( y^{a + 1/2} e^{-y/2} \) take the form

\[ f_5 = \frac{\Gamma(1-2a)}{\Gamma(1-a)} f_1 + \frac{\Gamma(2a-1)}{\Gamma(a)} f_2, \]

\[ f_7 = \frac{\Gamma(1-2a)}{\Gamma(1-a)} f_1 - \frac{\Gamma(2a-1)}{\Gamma(a)} f_2. \]

Note that the solutions \( Y_1 \) and \( Y_2 \) describe the waves with asymptotic behavior

\[ z \to -\infty, (y \to 0) \]

\[ f_1 \propto \left( 2\sqrt{k_1^2 + k_2^2} \right)^{1-i\sqrt{-\varepsilon}} e^{z/2} e^{-i\sqrt{-\varepsilon}x}, \]

\[ f_2 \propto \left( 2\sqrt{k_1^2 + k_2^2} \right)^{1+i\sqrt{-\varepsilon}} e^{z/2} e^{i\sqrt{-\varepsilon}x}. \]

Thus, for example, the function \( Y_5 \) (and the related \( \phi_5 \)) at negative \( z \to -\infty \) behaves as a superposition of two plane waves according to

\[ \phi_5 \propto \frac{\Gamma(1-2a)}{\Gamma(1-a)} \frac{\Gamma(2a-1)}{\Gamma(a)} \left( 2\sqrt{k_1^2 + k_2^2} \right)^{1-i\sqrt{-\varepsilon}} e^{-\sqrt{-\varepsilon}x} + \frac{\Gamma(2a-1)}{\Gamma(a)} \left( 2\sqrt{k_1^2 + k_2^2} \right)^{1+i\sqrt{-\varepsilon}} e^{\sqrt{-\varepsilon}x}. \]

We define the reflection coefficient as the square modulus of the amplitude ratio in a superposition of plane waves.
\[ M e^{i\sqrt{\varepsilon}z} + M e^{i\sqrt{\varepsilon}z}, R = \left| \frac{M}{M_0} \right|^2, \]

\[ R = \left| \frac{\Gamma(1-2a) \Gamma(a)}{\Gamma(2a-1) \Gamma(1-a)} \right|^2. \]  

(1.15a)

We take into account

\[ 1-2a = +2i\sqrt{\varepsilon-1}, \]

\[ a = 1/2-i\sqrt{\varepsilon-1}, a = 1/2+i\sqrt{\varepsilon-1}, \]

then

\[ R = \left| \frac{\Gamma(1+2i\sqrt{\varepsilon-1})}{\Gamma(2i\sqrt{\varepsilon-1})} \right|^2 \times \]

\[ \times \left| \frac{\Gamma(1/2-i\sqrt{\varepsilon-1})}{\Gamma(1/2+i\sqrt{\varepsilon-1})} \right|^2 \equiv 1. \]  

(1.15b)

We find the behavior of \( Y_i \) at large \( y \). Using the known asymptotic relation [12]

\[ Y_5 = \Psi'(a,c,y) \square y^{-a}, \]

we get at \( z \rightarrow +\infty \):

\[ f_5 = y^{\alpha+1/2} e^{\gamma y/2} Y_5 \square y^{1/2} e^{-\gamma y/2} \]

\[ \square (2k^2 + \varepsilon^2 + 1/2 \exp(-\sqrt{k^2 + \varepsilon^2}) \]

\[ \exp \varepsilon \gamma \alpha = 0. \]  

(1.16)

Thus, the solution \( f_5 \) describes the expected situation: wave going from the left is reflected with probability 1 on the effective barrier; behind the barrier the solutions sharply decrease to zero. It is easy to find the critical point, after which wave function must sharply decrease

\[ \varepsilon - 1 = (k^2 + \varepsilon^2) \varepsilon^{i\gamma} \Rightarrow z_0 = \ln \frac{\varepsilon - 1}{k^2 + \varepsilon^2}, \]

in the usual units, this critical point is described by the relation

\[ z_0 = \rho \ln \frac{2mE\varepsilon^2/h^2 - 1}{(K^2 + \varepsilon^2)\rho^2}. \]  

(1.17)

Next we consider the analogue of this situation for particles with spin 1/2, described by the relativistic Dirac equation, when analysis is much more complicated.

2. The Dirac equation, separation of the variables

We start with the general covariant form of the Dirac equation [11]; in the coordinate system (1.1) we use the diagonal tetrad then the Dirac equation takes the form

\[ \begin{bmatrix} y^0 \frac{\partial}{\partial t} + y^1 \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y} + y^3 \frac{\partial}{\partial z} + \\
+ y^4 \left( \frac{\partial}{\partial z} - 1 \right) + im \end{bmatrix} \Psi = 0. \]  

(2.1)

Note that the addition of -1 about the operator \( \partial_z \) can be removed by substituting \( \Psi = e^{i\varphi} \phi \). The following three operators \( i\partial_1, i\partial_2, i\partial_3 \) commute with the wave operator: solutions can be searched in the form

\[ \Psi(x,\varphi) = \begin{pmatrix} f_1(z) \\ f_2(z) \\ f_3(z) \\ f_4(z) \end{pmatrix}. \]  

(2.2)

Using the Dirac matrices in spinor basis, from (2.1) we find equations for \( f_i(z) \)

\[ -i\varepsilon f_1 - ik_1 e_1 f_4 - k_2 e^1 f_4 - (\frac{d}{dz} - 1)f_1 + imf_1 = 0, \]

\[ -i\varepsilon f_2 - ik_1 e_1 f_3 + k_2 e^1 f_3 + (\frac{d}{dz} - 1)f_2 + imf_2 = 0, \]

\[ -i\varepsilon f_3 + ik_1 e_1 f_2 + k_2 e^1 f_2 + (\frac{d}{dz} - 1)f_3 + imf_3 = 0, \]

\[ -i\varepsilon f_4 + ik_1 e_1 f_1 - k_2 e^1 f_1 - (\frac{d}{dz} - 1)f_4 + imf_4 = 0. \]  

(2.3)

There is a generalized helicity operator which commutes with the operator of the wave equation:

\[ \Sigma = \frac{1}{2} \left( e^\gamma \gamma^3 \frac{\partial}{\partial x} + e^\gamma \gamma^1 \frac{\partial}{\partial y} + \\
+ y^4 \gamma^2 \left( \frac{\partial}{\partial z} - 1 \right) \right). \]  

(2.4)

Using the substitution (2.2) in the eigenvalues equation \( \Sigma \Psi = \rho \Psi \) we obtain

\[ k_1 e_1 f_2 - ik_2 e^1 f_4 - i(\frac{d}{dz} - 1)f_2 = pf_2, \]

\[ k_1 e_1 f_3 + ik_2 e^1 f_3 + i(\frac{d}{dz} - 1)f_3 = pf_3, \]

\[ k_1 e_1 f_4 - ik_2 e^1 f_4 - i(\frac{d}{dz} - 1)f_4 = pf_4. \]  

(2.5)
From equations (2.5) and (2.3), considered together, it follows a linear homogeneous system with respect to \( f_i \):
\[
-\imath \varepsilon f_3 - ip f_3 + im f_1 = 0,
-\imath \varepsilon f_4 - ip f_4 + im f_2 = 0,
-\imath \varepsilon f_3 + ip f_3 + im f_3 = 0,
-\imath \varepsilon f_2 + ip f_2 + im f_4 = 0.
\] (2.6)

We find two values for the \( p \) and the corresponding restrictions on the functions \( f_i \):
\[
p = \pm \sqrt{\varepsilon^2 - m^2}, f_3 = \frac{\varepsilon - p}{m} f_1, f_4 = \frac{\varepsilon - p}{m} f_2.
\] (2.7)

Thus, we have three continuous quantum number \( \varepsilon, k_1, k_2 \) and one discrete, which distinguishes the values \( p = \pm \sqrt{\varepsilon^2 - m^2} \). In view of (2.7), from four equations (2.3) we arrive at two equations for \( f_1, f_2 \)
\[
\frac{d}{dz} (1 - ip) f_1 + e^{i\varepsilon (ik_1 + k_2)} f_2 = 0,
\frac{d}{dz} (1 + ip) f_2 - e^{i\varepsilon (ik_1 - k_2)} f_1 = 0.
\] (2.8)

Note the symmetry of the equations with respect to change
\[
f_1 \Rightarrow f_2, \quad p \Rightarrow -p.
\] (2.9)

It is convenient to obtain solutions of similar equations in the flat space
\[
\frac{d}{dz} (1 - ip) f_1 + (ik_1 + k_2) f_2 = 0,
\frac{d}{dz} (1 + ip) f_2 - (ik_1 - k_2) f_1 = 0,
\] so that
\[
f_2 = \frac{1}{ik_1 + k_2} (\frac{d}{dz} - ip) f_1,
\frac{d^2}{dz^2} + \varepsilon^2 - m^2 - k_1^2 - k_2^2 \frac{d}{dz} f_1 = 0.
\] (2.10)

Here, there exist two independent solutions (let \( k_3 = \sqrt{\varepsilon^2 - m^2 - k_1^2 - k_2^2} \))
\[
f^{(1)}_1 = e^{i k_3 z}, f^{(1)}_2 = \frac{(-ik_3 + ip)}{ik_1 + k_2} e^{i k_3 z},
f^{(2)}_1 = e^{-i k_3 z}, f^{(2)}_2 = \frac{(ik_3 - ip)}{ik_1 + k_2} e^{-i k_3 z}.
\] (2.12)

The sign before \( k_3 \) determines the direction of the wave propagation, the sign of \( p \) defines the state of polarization. Generalized analogue of these solution are to be investigated in the hyperbolic space \( H_3 \).

### 3. A special case of the waves along the \( z \)-axis

There exists a special case when \( k_1, k_2 = 0 \):
\[
\Psi^{0,0}(t, z) = e^{-i\varepsilon t}.
\] (3.1)

The equations change substantially, and the problem reduces to
\[
\frac{\partial}{\partial z} (1 - ip) f_1 = 0, f_1 = C_1 e^{i e^{i\varepsilon t}},
\frac{\partial}{\partial z} (1 + ip) f_2 = 0, f_2 = C_2 e^{i e^{-i\varepsilon t}}.
\] (3.2)

Solutions more simple to interpret are
\[
\Psi^{0,0,p}(t, z) = \frac{1}{\varepsilon - p} e^{i e^{i\varepsilon t}}; \quad (3.3a)
\]
\[
\Psi^{0,0,p}(t, z) = \frac{0}{\varepsilon - p} e^{-i e^{-i\varepsilon t}}. \quad (3.3b)
\]

Obviously, the factor \( e^{i \varepsilon} \) in the solutions will be compensated when considering any bilinear structure of the wave functions (with their subsequent multiplication by \( \sqrt{g} \ dx \ dy \ dz \)).

### 4. Construction of solutions in the general case

Let us turn to (2.8) and introduce a new
variable \( \sqrt{k_1^2 + k_2^2} e^z = Z, Z \in (0, +\infty), \):
\[
\left( \frac{Z}{dZ} - 1 - ip \right) f_1 + Z \frac{k_2 + ik_1}{k_2 - ik_1} f_2 = 0, \quad (4.1a)
\]
\[
\left( \frac{Z}{dZ} + 1 + ip \right) f_2 + Z \frac{k_2 - ik_1}{k_2 + ik_1} f_1 = 0. \quad (4.1b)
\]
From (4.1) we get two second order differential equations for \( f_1 \) and \( f_2 \):
\[
Z \frac{d^2 f_1}{dZ^2} - 2 \frac{d f_1}{dZ} + \left( \frac{p^2 + ip + 2}{Z} - Z \right) f_1 = 0, \quad (4.2)
\]
\[
Z \frac{d^2 f_2}{dZ^2} - 2 \frac{d f_2}{dZ} + \left( \frac{p^2 - ip + 2}{Z} - Z \right) f_2 = 0. \quad (4.3)
\]
Considering eq. (4.2), let us use a substitution \( f_1(Z) = Z^A e^{BZ} F_1(Z) \); at \( A \) and \( B \) chosen according
\[
A = +ip + 1, -ip + 2, B = \pm 1, \quad (4.4)
\]
equation for \( F_1 \) becomes simpler (we use else one change \( Z = y/2 \) and let \( B = -1 \)), then we arrive at equation for the confluent hypergeometric function (for definiteness let it be \( A = +ip + 1 \))
\[
y \frac{d^2 \Phi}{dy^2} + (c - y) \frac{d \Phi}{dy} - a \Phi = 0, \quad a = +ip, c = 2a = +2ip. \quad (4.5)
\]
Two linearly independent solutions are [12]
\[
F_1^{(1)}(y) = \Phi(a, c, y), \quad F_1^{(2)}(y) = y^{1-c} \Phi(a - c + 1, 2 - c, y). \quad (4.6)
\]
Consider the equation (4.3). Using the above-noted symmetry, we obtain
\[
f_2 = y^{-a + 1} e^{-y/2} F_2(y), \quad a' = -ip, c' = -2ip,
F_2^{(1)}(y) = \Phi(a', c', y), \quad F_2^{(2)}(y) = y^{1-c'} \Phi(a' - c' + 1, 2 - c', y). \quad (4.7)
\]
It is convenient to employ one independent parameter \( a \):
\[
f_1 = y^{a+1} e^{-y/2} F_1(y), \quad F_1^{(1)}(y) = \Phi(a, 2a, y), \quad F_1^{(2)}(y) = y^{1-2a} \Phi(1-a, 2-2a, y); \quad (4.8a)
\]
\[
f_2 = y^{-a+1} e^{-y/2} F_2(y), \quad F_2^{(1)}(y) = \Phi(-a, -2a, y), \quad F_2^{(2)}(y) = y^{1+2a} \Phi(1+a, 2+2a, y). \quad (4.8b)
\]
The functions \( f_1, f_2 \) (note that before now we did not find possible numerical factors at them) must be related by the first-order operators (4.1). These equations relate functions in the following pairs
\[
F_1^{(1)}(y) - F_2^{(2)}(y), \quad F_1^{(2)}(y) - F_2^{(1)}(y).
\]
Corresponding relative factors are calculated – the result is
\[
I \quad f_1 = M_e^{-y/2} y^{1-a} \Phi(a, 2a, y), \quad f_2 = e^{-y/2} y^{2-a} \Phi(a+1, 2+2a, y), \quad M_+ = 2 e^{-y/2} (1+2a); \quad (4.9)
\]
\[
II \quad f_1 = M_e^{-y/2} y^{2-a} \Phi(1-a, 2-2a, y), \quad f_2 = e^{-y/2} y^{-a} \Phi(-a, -2a, y), \quad M_- = 2 e^{-y/2} (-1-2a), \quad (4.10)
\]
where
\[
e^{-y/2} = \sqrt{\frac{k_2 + ik_1}{k_2 - ik_1}}.
\]
Remind that \( a = ip = \pm \sqrt{\kappa^2 - m^2} \); the sign of \( p \) is associated with the polarization state of the spinor waves; types \( I \) and \( II \) are supposed to be associated with the directions of wave propagation: to the left or to the right.

To conclude this section we consider the limiting process in the constructed solutions to the case of the flat space. This will allow a better understanding of the obtained results in the Lobachevsky space. To this end, we first need to go to the usual dimensional quantities:
\[
z = \frac{z}{\bar{z}}, \quad m = \frac{M_c R}{\hbar}, \quad \varepsilon = \frac{E}{c \hbar},
\]
\[
p = +R \sqrt{E^2/c^2 \hbar^2 - M^2 c^2/\hbar^2} = R_p, \quad (4.11)
\]
\[
k_1 = \frac{P_R}{c \hbar}, k_2 = \frac{P_R}{c \hbar}, k_1^2 + k_2^2 = R \sqrt{P_1^2 + P_2^2} = R_k, \quad (4.12)
\]
\[
a = ip = \pm R_p, \quad c = 2a = \pm 2R_p, \quad (4.13)
\]
\[
y = 2 \sqrt{k_1^2 + k_2^2} e^z = 2 R_k (1 + \frac{\chi}{R} + ...) = 2 R_k \quad (4.14)
\]
Let us consider the solutions (4.9)
\[
I \quad f_1 = M_e^{-y/2} y^{1-a} \Phi(a, 2a, y),
\]
\[
II \quad f_1 = M_e^{-y/2} y^{2-a} \Phi(1-a, 2-2a, y).
\]

138
$f_2 = e^{-y/2}y^{2+ia}\Phi(a+1,2+2a,y)$,
$M_+ = \left[2e^{ia}(1+2a)\right]$;

taking into account
\[
\frac{a}{c} \gamma \Rightarrow RK_+,
\]
\[
\frac{1}{2!} \frac{a(a+1)}{c(c+1)} y^2 \Rightarrow \frac{1}{2!} (RK_+)^2,
\]
\[
\frac{1}{3!} \frac{a(a+1)(a+2)}{c(c+1)(c+2)} y^2 \Rightarrow \frac{1}{3!} (RK_+)^3 \ldots,
\]
we get
\[
e^{-y/2} \Rightarrow e^{-RK_+}, \Phi(a,2a,y) \Rightarrow e^{RK_+},
\]
\[
e^{-y/2} \Rightarrow e^{-RK_+}, \Phi(a+1,2a+2,y) \Rightarrow e^{RK_+},
\]
and further
\[
I \quad f_1 = M_+(2RK_+e^{ia+ip_0}) \Phi^{ii_p},
\]
\[
I \quad f_2 = (2RK_+e^{ia+ip_0}) \Phi^{ii_p}.
\]
\[
(4.11)
\]
Similarly, we find
\[
II \quad f_2 = e^{-y/2}y^{1-ia}\Phi(-a,-2a,y) \Phi^{ii_p},
\]
\[
f_1 = M_+e^{-y/2}y^{2-ia}\Phi(1-a,2-2a,y) \Phi^{ii_p}.
\]
\[
(4.12)
\]
We may conclude that solutions of the type $I$ (in curved model $H_3$) provide us with extension for the flat waves in Minkowski space of the type $e^{ikz}$; whereas solutions of the type $II$ represent extension for the flat waves in Minkowski space of the type $e^{-ikz}$.

**Conclusions**

In the paper complete systems of exact solutions for Schrödinger and Dirac equations in the hyperbolic space $H_3$ are constructed on the base of the method of separation of the variables in quasi-cartezian coordinates. An extended helicity operator is introduced. It is shown that solutions constructed when translating to the limit of vanishing curvature coincide with common plane wave solutions on Minkowski space going in opposite $z$-directions. Results are much the same for 2-component Weyl equation.

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УЗАГАЛЬНЕНІ ПЛОСКІ ХВИЛІ ДЛЯ ЧАСТИНКОК ШРЕДІНГЕРА І ДИРАКА НА ФОНА ГЕОМЕТРІЇ БОЯЙ–ЛОБАЧЕВСЬКОГО

Показано, що геометрія Бояй–Лобачевського робить істотний вплив на квантово-механічну частинку, моделюючи середовище зі спеціальними відбиваючими властивостями ідеального дзеркала. Для скалярної частинки Шредінгера задача зведена до диференціального рівняння другого порядку, яке може бути зіставлено з однорідним рівнянням Шредінгера для частинки в зовнішньому потенціальному полі \( U(z) = U_0 e^{2\xi} \). У квантовій механіці викривлена геометрія виступає як ефективний потенціальний бар'єр з коефіцієнтом відбиття \( R = 1 \). Геометрія моделює середовище, ефективно діюче як ідеальне дзеркало. Проникнення частинок в ефективне середовище залежить від параметрів хвильових рішень \( \xi, k_x^2 + k_y^2 \), а також від радіусу кривизни \( \rho \). Аналогічний аналіз проведено для діраківської частинки зі спином \( \frac{1}{2} \). В цьому випадку виникає додаткове квантове число, пов’язане з узагальненим оператором спіральності.

Ключові слова: геометрія Бояй–Лобачевського, плоска хвиля, рівняння Шредінгера, рівняння Дірака, гіпергеометричні функції.

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ОБОБЩЕННЫЕ ПЛОСКИЕ ВОЛНЫ ДЛЯ ЧАСТИЦ ШРЕДИНГЕРА И ДИРАКА НА ФОНЕ ГЕОМЕТРИИ БОЯЙ–ЛОБАЧЕВСКОГО

Показано, что геометрия Бояи–Лобачевского оказывает существенное влияние на квантово-механическую частицу, моделируя среду со специальным отражающим свойствами идеального зеркала. Для скалярной частицы Шредингера задача сведена к дифференциальному уравнению второго порядка, которое может быть сопоставлено с однородным уравнением Шредингера для частицы во внешнем потенциальном поле \( U(z) = U_0 e^{2\xi} \). В квантовой механике Искривленная геометрия выступает как эффективный потенциальный барьер с коэффициентом отражения \( R = 1 \). Геометрия моделирует среду, эффективно действующую как идеальное зеркало. Проникновение частиц в эффективную среду зависит от параметров волновых решений \( \xi, k_x^2 + k_y^2 \), а также радиуса кривизны \( \rho \). Аналогичный анализ проведен для дирацковской частицы со спином \( \frac{1}{2} \). В этом случае возникает дополнительное квантовое число, связанное с обобщенным оператором спиральности.

Ключевые слова: геометрия Бояи–Лобачевского, плоская волна, уравнение Шредингера, уравнение Дирака, гипергеометрические функции.