Adv. Appl. Clifford Algebras

DOI 10.1007/s00006-016-0717-3

Advances in Applied Clifford Algebras

Extension of the Standard CD Algebra in the Axiomatic Approach for Spinor Field and Fermi–Bose Duality

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Abstract. The exclusive representations of the extended (29-dimensional) real-number Clifford–Dirac algebra are constructed for the spinor field. In the canonical Foldy–Wouthuysen representation for a $e^{-}e^{+}$-dublet these representations contain physically justified on equal footing and conserved in time fermion and boson spins, and the canonical equation of motion for a dublet coincides with the quantum mechanical equation in the Hilbert space $L_2(\mathcal{R}^3) \times \mathcal{C}^4 \equiv H^{3,4}$ with definite metrics. In $H^{3,4}$ the experimentally observed dublet energy is always positive. The Fermi and Bose spins define sets of both equal status Fermi and Bose states, which univocally elucidate the physical content of the Fermi–Bose (FB)-dualism of the $e^{-}e^{+}$ microobject. We briefly review the ad hoc boson object in the same space as partner of the $e^{-}e^{+}$-dublet and treat issues, related to its BF-dualism. The mathematical correctness of the technique is acquired by the application of the simplified variant of the axiomatic approach (A-approach) to the spinor field.

Keywords. Fermi–Bose duality, CD algebra representations, Real number algebra, Dirac equation, Dirac matrices, Foldy–Wouthuysen representation, Positive energy solutions, Spin, Particle-antiparticle dublet, Positronium, SUSY.

1. Introduction

This work aims at (i) construction of a physically motivated extension of the standard 16-dimensional Clifford–Dirac (CD) algebra, generated by 4 primary 4×4 Dirac matrices to an extended 29-dimensional CD algebra over real numbers (ERCD) in the same complex space $\mathcal{C}^4$ of 4-component complex vectors and (ii) application of this extension to the clarification and realization of the Fermi–Bose (FB) and Bose–Fermi (BF) dualism for physical objects: ad hoc spinor field ψ of the dublet of particles with SU(2)-spin $s = \frac{1}{2}$ and ad hoc boson field with SU(2)-spin $s = (1, 0)$ (tensor-scalar multiplet).

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Published online: 19 August 2016

The subject of FB-dualism of spinor and other fields (including SUSY partners) has been for a long time treated by many authors. The possibility of both Fermi- and Bose-quantization of particle fields of different spins was considered in [1]. In [2,3] (see also references therein) the applicability of both anticommutative and commutative quantization using the same Fock space was demonstrated for amplitudes of various fields. Recently various aspects of FB dualism have been treated in [4,5]. A nonsingular relation between massless Dirac equation and Maxwell equations for field strengths with gradient currents was considered in many works (see, for example, [6,7] and references therein). The generalization of this treatment for arbitrary masses has been started in [8]. However up to date no unambiguous understanding of concepts, related to various aspects of FB dualism, is demonstrated (cf. terms "FB-equivalence, FB-duality, FB-transmutations" etc) [5].

The present contribution treats the question of *univocacy* of involved concepts within the frame of FB and BF-dualism of SUSY partners in first line in terms of the most commonly used in elementary particles theory object—an ad hoc F-object, namely, the electron-positron (e^-e^+)-dublet. This univocacy is achieved in all three known forms, namely: local model of spinor field ψ (Loc-ψ-model), canonical Foldy–Wouthuysen form (FW-ϕ-model) and canonical quantum mechanics (CQM) model for e^-e^+-dublet. It requires the account for experimental fact of mirror symmetry of particle and antiparticle in the e^-e^+-dublet, which prescribes the unique form of the operator $\vec{s}$ of the dublet spin [see below (51)], which is different from commonly used form $2\vec{s} = \vec{\Sigma}$ (28). The proof of this univocacy uses the extension of the standard 16-dimensional CD-algebra as a *real-number algebra in the complex space* $\mathcal{C}^4$-codomain of the spinor field ψ—up to 29-dimensional algebra (ERCD $\supset$ CD) in the same $\mathcal{C}^4$ space.

The mathematical correctness of the technique is acquired by the application of the simplified variant of the axiomatic approach (A-approach) to the field models (free fields) instead of the customary Wightman A-approach (considered, for example, in [9]). The simplified A-approach considers as the state space for a field of a physical object not the space of generalized Schwartz functions S^*, but the space of *test Schwartz functions* $\mathcal{S}$. This choice can be motivated from mathematical and physical point of view as follows:

(i) S is dense in S^*, therefore for practical purposes any generalized function $f^G \in S^*$ can be approximated (with prescribed accuracy) with elements of some Cauchy sequence in S, convergent to $f^G \in S^*$. This limitation is sufficient for ensuring the experimental verifiability of the obtained consequences of the A-approach by means of physical devices with arbitrarily high (but not absolute) precision.

(ii) Functions $f \in S$ are infinitely differentiable and rapidly decreasing at infinity together with their derivatives of any order, and the set of these functions is invariant with respect to the Fourier transform. Therefore all calculations are performed within classical calculus without involving the technique of generalized calculus of functionals in S^*.

The field models in the simplified A-approach share the principal features of axiomatic models in field theories: the definiteness of the state space where solutions to the equations of motions of physical models are constructed, and definiteness of domains and codomains of all involved operators. It should be stressed, that this approach treats the Schwartz space S as *common domain and codomain* of all used operators in the state space $f \in S$.

We shall use custom relativistic notations, in particular:

$$M(1,3) \doteq \left\{x \equiv (x^{\mu}) \equiv (x^0 = t, \vec{x}); \vec{x} \equiv (x^j) \in R^3 \; \mu = \overline{0,3},\; j = \overline{1,3}\right\}. \quad (1)$$

The set $(x^{\mu}) \doteq x$ contains contravariant (Carthesian) coordinates x^{μ} (setting $h = c = 1$) of points of the physical space-time space in arbitrarily fixed ($\forall$) inertial frame of reference (IFR). The metric tensor in the Minkowski space M(1,3) is

$$g^{\mu\nu} = g_{\mu\nu} = g^{\mu}_{\nu} \; : \quad (g^{\mu}_{\nu}) = \mathrm{diag}(1,-1,-1,-1);\; g_{\mu\nu}x^{\nu} \doteq x_{\mu}\,, \quad (2)$$

where (x_{μ}) is a covariant vector in M(1,3).

The relativistic invariance of any field model within SRT assumes involvement of minimal relativistic groups—eigen orthochronous Lorentz $L^{\uparrow}_{+} \doteq \mathrm{SO}(1,3)$ and Poincaré groups $P^{\uparrow}_{+} \supset L^{\uparrow}_{+}$ with the following mathematical refinement: these groups being a "representation" of more fundamental groups—universal covering groups $\mathcal{P} \supset \mathcal{L} \doteq \mathrm{SL}(2,\mathrm{C})$ as real-number topological Lie groups. As *real parameters* of these Lie groups we use the translation parameters $(a^{\mu}) \equiv a \in \mathrm{M}(1,3)$ and rotation angles $\omega^{\mu\nu} = -\omega^{\nu\mu}$ in planes $\mu\nu$ of the pseudo-Euclidian space M(1,3). Therefore the $\mathcal{P}$-generators $(p_{\mu}, j_{\rho\sigma})$ (derivatives by real parameters $(a^{\mu}, \omega^{\rho\sigma})$ of the elements of the topological group $\mathcal{P}$) are called *primary* $\mathcal{P}$-generators (see, e.g. [10,11] on the use of such generators of the symmetry groups). They satisfy the following explicitly covariant commutative relations:

$$[p_{\mu}, p_{\nu}] = 0, \quad [p_{\mu}, j_{\rho\sigma}] = g_{\mu\rho}p_{\sigma} - g_{\mu\sigma}p_{\rho}, \quad (3a)$$

$$[j_{\mu\nu}, j_{\rho\sigma}] = -g_{(\mu\rho}j_{\nu\sigma)} \doteq -g_{\mu\rho}j_{\nu\sigma} - g_{\rho\nu}j_{\sigma\mu} - g_{\nu\sigma}j_{\mu\rho} - g_{\sigma\mu}j_{\rho\nu}. \quad (3b)$$

The relations (3b) hold also for the Lorentz spin—the primary generators $s_{\mu\nu}$ of the purely matrix representations of the group $\mathcal{L}$, containing two independent 3-component sets

$$\left\{(s_{\mu\nu}) : (s_{jk}) \equiv (s^{l}) \doteq \vec{s};\; (s_{0j}) \doteq \vec{\eta} \equiv (\eta^{j});\; j,k,l = \overline{1,3}\right\}. \quad (4)$$

The matrices s_{jk} of the set $\vec{s}$ in (4) satisfy the SU(2) algebraic relations

$$[s_{jk}, s_{ln}] = \delta_{(jl}s_{kn)} \Longleftrightarrow [s^{j}, s^{k}] = s^{l}\,;\; j,k,l \in 123!, \quad (5)$$

where (123!) denotes cyclic permutation of indices j,k,l).

The set $(s_{\mu\nu})$ is called evidently *the Lorentz spin operator*, and its subset $\vec{s} \equiv (s_{ln})$ is the SU(2)-spin (let us recall, that SU(2) group is a universal covering group for the SO(3) group of rotations in the Euclidean space $R^3 \subset \mathrm{M}(1,3)$).

In what follows we will use various matrix representations in $\mathcal{C}^4$ of algebras of real-number parametric groups $\mathcal{L} = \mathrm{SL(2,C)} \rightsquigarrow \mathrm{SO(1,3)}$ and SU(2)

$\rightsquigarrow$ SO(3) $\subset$ SO(1,3). Therefore it is appropriate for our purposes to interpret the algebras of the said groups as real-number algebras in the complex space $\mathcal{C}^4$ (identical symbols are employed for groups and their algebras). Besides that, we will use various matrix representations in $\mathcal{C}^4$ of algebras of SO(1,N) $\supset$ SO(N) groups with $N > 3$ as real-number parametric Lie groups.

The choice of uniform with $\omega^{\mu\nu} = -\omega^{\nu\mu}$ and $\omega^{jl} = -\omega^{lj}$ real-number parameters of the groups SO(1,N) and SO(N), on one hand, ensures the sorting in their embeddings SO(1,3) $\subset$ SO(1,4) $\subset \cdots$ and SO(3) $\subset$ SO(4) $\subset \cdots$. On the other hand, this choice leads to uniformity of commutational relations for the primary generators $s_{\mu_1\mu_2}$ and $s_{j_1j_2}$ of the groups SO(1,N) $\supset$ SO(N), that is generators of the real-number algebras in $\mathcal{C}^4$ of the said groups, namely:

$$\begin{aligned}[s_{\mu_1\mu_3}, s_{\mu_2\mu_4}] = -g_{(\mu_1\mu_2}s_{\mu_3\mu_4)} &\doteq -g_{\mu_1\mu_2}s_{\mu_3\mu_4} - g_{\mu_2\mu_3}s_{\mu_4\mu_1} \\ &\quad -g_{\mu_3\mu_4}s_{\mu_1\mu_2} - g_{\mu_4\mu_1}s_{\mu_2\mu_3}\end{aligned} \tag{6}$$

$$[s_{j_1j_3}, s_{j_2j_4}] = \delta_{(j_1j_2}s_{j_3j_4)} \doteq \delta_{j_1j_2}s_{j_3j_4} + \delta_{j_2j_3}s_{j_4j_1} + \delta_{j_3j_4}s_{j_1j_2} + \delta_{j_4j_1}s_{j_2j_3} \tag{7}$$

Here parentheses for indices μ_n and j_n denote their cyclic permutations 1234! by numbers $n = \overline{1,4}$. Therefore we will call such representations of the SO(1,N) $\supset$ SO(N) algebras and their primary generators (generators of real number algebras) *cyclic* (ζ-generators, for brevity). These properties of embedding and ordering are consequences of generic choice of the structure coefficients of the corresponding Lie algebras. The explicit forms of ζ-relations in (6), (7) reflect the special choice of coefficients of the Lie groups SO(1,N) and SO(N), as a consequence of this choice of real number parameters $\omega^{\mu_1\mu_2}$ and $\omega^{j_1j_2}$ of these groups. This choice is convenient, for example, since if there are no coincident pairs among the sets $j_{1,2}$ and $j_{3,4}$, then $[s_{j_1j_2}, s_{j_3j_4}] = 0$.

The relevance of the suggested detailed description of different aspects of the FB and BF-dualism for *free* (isolated) objects like e^-e^+-dublets of spin $s = \frac{1}{2}$ and B-compound field with $s = (1, 0)$ consists in the following. The detailed study of the model of free (isolated) microobjects (fields or particles) is important, since the experimental verification of physical quantities for a microobject including their principal patterns, necessary for the formulation of the axioms, a part of which *ad hoc* would yield the definition of the object itself (axioms of the object definition), can be realized with *experimental measurements of characteristics of the state for such object only in the finite domains of asymptotically large space-time distances.* In that regions both in- and out- states of a microobject can be fairly treated as free states. Besides that, all interaction models for microobject fields are built of the models of their free fields.

Within the frame of the SRT requirements we formulate in Sect. 2 the basics of the simplified A-approach for the spinor field ψ starting exclusively from three axioms and valid regardless to the representation of matrices γ_μ. It is shown, that such A-approach yields the principal properties of the spinor field: relativistic invariance (in the form of $P_+^\uparrow$-invariance) of Dirac equation and 10 principal conservation laws for the field ψ as direct consequence of these axioms. We recall the results of [12], where for the first time the impossibility of interpreting the operator $\vec{x} \in R^3 \subset \mathrm{M}(1,3)$ and the set $\vec{s} = (s_{ln})$ of the Lorentz SL(2,C)-spin $(s_{\mu\nu})$ as operators of coordinate and spin of a e^-e^+-dublet was pointed out. We show, that the singled-out character of the time variable $t = x^0$ as opposed to 3-coordinate $(x^j) \equiv \vec{x}$ does not contradict the relativistic invariance in any representations of the spinor field including nonlocal ones.

In Sect. 3 we implement the real-number extension of the 16-dimensional Clifford–Dirac algebra (CD) towards the extended real-number ERCD $\rightsquigarrow$ SO(8) algebra in the Pauli–Dirac (PD) representation, as well as in other used representations of the Dirac matrices γ_μ. In Sect. 3.1 we point at the possibility (taking the time variable on a singled-out basis) of choosing the generators of the standard CD algebra as generators of the representation of the SO(6) algebra in $\mathcal{C}^4$ (similarly to the standard CD algebra $\rightsquigarrow$ SO(3,3), used in [14,15]). Starting from the PD-representation of matrices γ_μ, we will perform explicitely in Sect. 3.2 the extension of the standard 16-dimensional CD $\rightsquigarrow$ SO(6)-algebra to a 29-dimensional extended (ERCD $\rightsquigarrow$ SO(8)) algebra.

Using nonsingular operators T of the similarity transformations for algebra $\mathrm{SO}(8)^{\mathrm{PD}}$, we derive in Sect. 3.3 two matrix representations of the SO(8) algebra as a carrier of the internal degrees of freedom for the $s = \frac{1}{2}$-dublet, where the Foldy–Wouthuysen equation of motion for a e^-e^+-dublet will acquire a canonical quantum mechanical (CQM) form. This will construct a base of the CQM model with definite, $L_2(R^3) \times \mathcal{C}^4 \doteq H^{3,4}$-metric. It is the $H^{3,4}$ model, which is able to unequivocally elucidate the FB dualism of an *ad hoc* $s = \frac{1}{2}$-dublet in terms of equal status of quantum mechanical sets $\{f^F\}$, $\{f^B\} \subset H^{3,4}$ of Fermi f^F and Bose f^B-states for the same dublet (as an elementary microobject), defined by Fermi Q^F and Bose Q^B stationary complete sets of observables in $H^{3,4}$. The exclusive FW representations of the CQM-model (in particular, the equations of motion and states f^F and f^B) can be transformed into corresponding Loc-ψ-model. However the physical clearness of equal status of F- and B-states for a dublet in this picture would remain somehow shaded. We will discuss briefly the way of constructing the quantum model of a FB-dual spinor field in the CQM model, analogously to the secondary quantization technique in the nonrelativistic quantum theory. Finally, in Sect. 3.4 we sketch a complementary scheme of the BF-dualism of an ad hoc Bose multiplet as a SUSY partner of a e^-e^+-dublet in the same space $S^{3,4}$, based on an additional boson representation of the SO(8) algebra. The states of this object are related by a nonunitary transformation with the states of an ad hoc fermion object e^-e^+.

2. Axiomatic Approach to the Spinor Field in the Schwartz Test Functions Space

The Axiom *A1* for the spinor field $\psi : M(1,3) \to C^4$ (C^4, topological space of 4 complex numbers) is cast as the attribution of the field function ψ to the *Schwartz test functions space* $S^{4,4}$, as opposed to common attribution to the Schwartz generalized functions space $S^{4,4*}$. As will be shown below in the Loc-ψ-model, the space $S^{4,4} \supset C^4$ is a common domain and codomain space of all involved operators, which ensures the consistency of the simplified A-approach for the field ψ in this model.

The Axiom *A2* postulates the Dirac equation for a free field in $\forall$ inertial frame of reference (IFR):

$$(i\gamma^\mu \partial_\mu - m)\psi(x) = 0\,; \quad \mu = \overline{0,3}, x \in M(1,3), \psi \in S^{4,4}, \tag{8}$$

where nonsingular 4×4 matrices γ^μ (operators in C^4 in $\forall$ representation) satisfy the generic relations

$$\begin{aligned} \gamma^\mu \ : \quad & \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}, \quad g_{\mu\nu}\gamma^\nu \doteq \gamma_\mu \Rightarrow \gamma_0 = \gamma^0 \\ & = \gamma_0^{-1}, \gamma_j = -\gamma^j = -\gamma_j^{-1}, \end{aligned} \tag{9}$$

Besides (8), we will use the *Schrödinger form* of the Dirac equation (SD-equation):

$$(\partial_0 - H \equiv \partial_0 - \gamma_{0j}\partial_j + i\gamma_0 m)\psi(x) = 0; \quad \psi \in S^{4,4}, \gamma_{\mu\nu\ldots} \equiv \gamma_\mu\gamma_\nu\ldots, \tag{10}$$

where H is the Hamiltonian, composed of primary operators; Eq. (10) can be derived from (8) via multiplication by the nonsingular operator $-i\gamma_0$. The SD-form of the Dirac equation unambiguously determines the Hamiltonian of the spinor field in $\forall$ representations of γ_μ matrices.

Axiom *A3* (about $\mathcal{P}$-covariance of the field ψ) is stated as follows: at $P_+^\uparrow \supset L_+^\uparrow$-transformations in M(1,3),

$$\begin{aligned} x \to x_{a\omega} = \Lambda(\omega)x + a \Rrightarrow x^\mu \to x^\mu_{a\omega} \doteq \Lambda^\mu_\nu x^\nu + a^\mu\,, (a^\mu) \\ \equiv a, x, x'_{a\omega} \in M(1,3), (\Lambda^\mu_\nu) \equiv \Lambda(\omega) \in \mathrm{L}_+^\uparrow\,, \end{aligned} \tag{11}$$

the field ψ in $S^{4,4}$ will transform as follows

$$\begin{aligned} \psi(x) \to \psi_{a\omega}(x) &\doteq F^\gamma(\omega)\psi(\Lambda^{-1}(x-a)) \\ &\overset{inf}{=} \left(1 + a^\mu p_\mu + \frac{1}{2}\omega^{\mu\nu} j_{\mu\nu}\right)\psi(x); \psi, \psi_{a\omega} \in S^{4,4}, \end{aligned} \tag{12}$$

$$F^\gamma(\omega) \overset{inf}{=} 1 + \frac{1}{2}\omega^{\mu\nu}s_{\mu\nu}; \quad s_{\mu\nu} \doteq \frac{1}{4}[\gamma_\mu, \gamma_\nu]. \tag{13}$$

The notation $\overset{inf}{=}$ means infinitesimal neighborhood around the unity element in a correspondent group.

State now simple consequences of the axioms.

The axioms *A1*, *A3* define explicit form of operators for principal physical quantities for the field ψ in $S^{4,4}$,

$$p_\mu = \partial_\mu, \quad j_{\rho\sigma} = m_{\rho\sigma} + s_{\rho\sigma}; \quad m_{\rho\sigma} \equiv x_\rho\partial_\sigma - x_\sigma\partial_\rho, \quad s_{\rho\sigma} \doteq \frac{1}{4}[\gamma_\rho, \gamma_\sigma], \tag{14}$$

(here the Lorentz spin $s_{\mu\nu}$ (4) defines a purely matrix, spinor $\left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right)$ representation $F^\gamma(\omega)$ of the group $\mathcal{L} \subset \mathcal{P}$), that is energy-momentum operators p_μ and 4-angular momentum $j_{\rho\sigma}$ as derivatives of $\psi_{a\omega}$ by real parameters a^μ, $\omega^{\rho\sigma}$, therefore termed as primary operators (see [10,11] about the application of primary generators of real Lie groups).).

The operators (14) are functions in $S^{4,4}$ of 14 independent operators (including the Lorentz spin, defined by (4)):

$$x_\mu, p_\nu \doteq \partial_\nu, s_{\rho\sigma} = \frac{1}{4}[\gamma_\rho, \gamma_\sigma] \, : \, [x_\mu, p_\nu] = -g_{\mu\nu} \, , [(x_\mu, p_\nu), s_{\rho\sigma}] = 0 \tag{15}$$

Any other observables for the field ψ in this model are merely functions in $S^{4,4}$ of these operators (15). Therefore, the axioms *A1*, *A3* define all *real algebra* A_S^L of "observables" for the field ψ (as operator functions in $S^{4,4}$, as common domain and codomain of the algebra A_S^L in the Loc-ψ-model).

Next, $\mathcal{P}$-generators (14) (Lie operators because of relations $[(\partial_\mu, m_{\rho\sigma}), s_{\alpha\beta}] = 0$), satisfy $\mathcal{P}$-relations (3), and commute with the operator of the Dirac equation (8) [and with Hamiltonian H in (10)]. So, the convergent in $S^{4,4}$ operator exp-series

$$\begin{aligned}(a,\omega) \to g^{\gamma L}(a,\omega) \doteq \exp(a^\mu\partial_\mu + \tfrac{1}{2}\omega^{\rho\sigma} j_{\rho\sigma}) \equiv F^\gamma(\omega)\exp(a^\mu\partial_\mu + \tfrac{1}{2}\omega^{\rho\sigma} m_{\rho\sigma})\\ \Rightarrow g^{\gamma L}(a,\omega)\psi(x) = F^\gamma(\omega)\psi(\Lambda^{-1}(x-a))\end{aligned} \tag{16}$$

defines a *local representation* $\mathcal{P}^{\gamma L}$ of the group $\mathcal{P}$ [which, according to (14), is uniquely defined by the Lorentz spin $s_{\mu\nu}$ (4)]; this representation is the invariance group for the Dirac equation (8)=(10), and determines the minimal relativistic symmetry of the free field ψ within STR requirements.

The axiom *A2*, that is the equation (8)=(10) identically leads to the Klein–Gordon (KG) equation in any representation

$$\begin{aligned}(\partial_0 - H)^2\psi \equiv \left(\partial_0^2 - H^2\right)\psi \equiv \left(\partial_0^2 - \Delta + m^2 \equiv \partial_\mu\partial^\mu + m^2\right)\psi(x) = 0;\\ \psi = (\psi^\alpha) \in S^{4,4}\end{aligned} \tag{17}$$

This means, that (i) any component ψ^α of the field ψ identically satisfies the Eq. (17), (ii) the Eq. (17) does not impose any additional restrictions on the spinor field ψ, beyond that, stated in axioms *A1*–*A3*, and (iii) the common statement about the Dirac equation (8)=(10) as a linearization of the KG operator from (17), is inappropriate. However, from (17) it is evident, that if $m^2 > 0$ (for arbitrary $m \in (-\infty, +\infty)$) the field ψ states are non-tachyonic.

Next, Eq. (17) is the 2nd order parabolic (and not elliptic) equation. Therefore it confirms a *mathematical nonequivalence* of the time $x_0 = t$ and space $(x^j) \equiv \vec{x} \in R^3 \subset \mathrm{M}(1,3)$ coordinates. This observation allows to cast the Axiom *A1* as follows: the field ψ by the variable $\vec{x}$ belongs to the space $S^{3,4} \doteq S(R^3) \times \mathcal{C}^4 \subset S^{3,4*}$, and the dependence of ψ on time $x^0 = t$ is given parametrically either through a unitary integral operator $U(t,t_0) \doteq \exp[i(t-t_0)H] : \psi(t-t_0,\vec{x}) = U(t,t_0)\psi(t,\vec{x})$ in $S^{3,4}$ (we set $t_0 = 0$), or via Eq. (10). It is evident from Eq. (17), that any general solution to the Dirac equation (8)=(10), related to an arbitrarily fixed stationary complete set $Q = (\hat{q}_1, \hat{q}_2, \ldots)$ (i.e. a set of all independent mutually commuting operators, which commute with the Hamiltonian H in (10)), will have the form

$$\begin{aligned}\psi_Q(t,\vec{x}) = \int \frac{d^3k}{(2\pi)^{3/2}}\left(e^{-ikx}A^-(\vec{k}) + e^{ikx}A^+(\vec{k})\right) \in S^{4,4}; kx \equiv \underset{\sim}{\omega} t - \vec{k}\vec{x}, \underset{\sim}{\omega}\\ \equiv \sqrt{\vec{k}^2 + m^2},\end{aligned} \tag{18}$$

where the decomposition of complex 4-component amplitudes $A^{\mp}(\vec{k})$ by the common eigenvectors of the $\forall$ stationary complete set Q of operators in $\vec{k}$-representation is unique. Let us note by passing, that the integration in the solutions like (18) is performed over the 3-dimensional Lebesgue measure d^3k (as it is done in the classical continuum theory, and in the nonrelativistic quantum mechanics), and the commonly encountering factor $1/\underset{\sim}{\omega}$ is assumed to be absorbed by mere definition of amplitudes $A^{\pm}(\vec{k})$. This convention differs from the custom notation, which is the consequence of integration over the Lebesgue–Stieltjes measure $d^4k\delta(k^2-m^2)$. The representation (18) of the general solution of Dirac equation (8)=(10) stresses the nonequivalence of the time $x^0=t$ and space 3-coordinate $\vec{x}\in R^3\subset M(1,3)$, seen from mere form of the KG equation (17). This does not contradict to the relativistic invariance of the field model ψ, which is interpreted as the minimal STR requirement of the invariance of equation (8)=(10) for the field ψ (as the invariance of the solution set (18) of this equation) with respect to the representation (16) in $S^{3,4}$ of the universal covering group $\mathcal{P}\supset\mathcal{L}=SL(2,C)$ of the proper orthochronous Poincaré group $P_+^{\uparrow}\supset L_+^{\uparrow}=SO(1,3)$. These statements can be justified, if one notices, that the contraction $(p_\mu,j_{\rho\sigma})^{\mathrm{Ind}}$ of Lie operators (14) to the set of solutions (18) of equation (10)=(8) has the following form

$$(p_\mu,j_{\rho\sigma})^{\mathrm{Loc}}_{|\{\psi_Q\}}\doteq(p_\mu,j_{\rho\sigma})^{\mathrm{Ind}}:\quad p_0^{\mathrm{I}}=H\equiv\gamma_{0j}\partial_j+i\gamma_0 m\quad p_j=\partial_j,$$
$$j_{ln}=m_{ln}+s_{ln}, \tag{19a}$$
$$j_{0l}^{\mathrm{I}}=-j_{l0}^{\mathrm{I}}\doteq t\partial_l-\frac{1}{2}\{x_l,H\};\quad\{A,B\}\equiv AB+BA,$$
$$(s_{ln}\equiv\frac{1}{4}[\gamma_l,\gamma_n])\equiv\vec{s}. \tag{19b}$$

A part of these operators (p_0^I,j_{0l}^I), as operators in $S^{3,4}\doteq S(\mathrm{R}^3)\times\mathcal{C}^4\subset S^{3,4*}$ are no more Lie operators (they belong to the class of Lie–Backlund operators) and do not have explicitely covariant form. Besides that, they are operator functions of 10 primary independent elementary time-independent operators

$$x_j,p_l=\partial_l,s_{\rho\sigma}\,:\,[x_j,p_l]=-\delta_{jl}\,,[(x_j,p_l),s_{\rho\sigma}]=0, \tag{20}$$

that is functions of the operators $\vec{x}$, $\vec{p}$ (satisfying Heisenberg commutation rules) and operators $s_{\rho\sigma}$ of the Lorentz spin (4), commuting with them. The operators (20) with common domain and codomain $S^{3,4}\subset S^{4,4}$ define the contraction A_S^I of the algebra A_S^L to $S^{3,4}$, and the operators (19a, 19b) belong to this algebra as well. The operators $(p_\mu,j_{\rho\sigma})^{\mathrm{Ind}}$ are determined (in terms of matrix operators) by the same Lorentz spin (4) and satisfy the commutation relations for $\mathcal{P}$-generators in the explicitely covariant form (3), and commute with the Dirac equation (10). Therefore, in terms of the convergent in $S^{3,4}$ exp-series

$$g^{\gamma L}(a,\omega)_{|\{\psi_Q\}}\doteq g^{\gamma\mathrm{I}}(a,\omega)=\exp(a^\mu p_\mu^I+\frac{1}{2}\omega^{\rho\sigma}j_{\rho\sigma}^I)\overset{inf}{=}\left(1+a^\mu p_\mu^I+\frac{1}{2}\omega^{\rho\sigma}j_{\rho\sigma}^I\right) \tag{21}$$

$\mathcal{P}$-generators (19a, 19b) define the so-called *induced representation* $\mathcal{P}^{\gamma I}$ of the group $\mathcal{P}$ in $S^{3,4}$, which is the invariance group of the Dirac equation (10)=(8). The general consequence of $\mathcal{P}$-invariance of the equations (8)=(10) is the existence of 10 common conservation laws for the spinor field ψ (see Eq.(26) below).

The common physical limitations of the Loc-ψ-model of the spinor field were discussed for the first time in [1,12,20] (see [13] for more details). We note here, that the Ind-ψ-form in fact shares the same limitations, as the commonly used explicitely covariant Loc-ψ-model, treating only Lie operators in the $\mathcal{P}^L$ representation. In particular, no one of the set of elementary operators (15) or (20), besides the 3-momentum operator $\vec{p} \equiv (p^j) = -\nabla$, would represent experimentally observable elementary physical properties for the spinor field ψ. Even the spatial part $(s_{ln}) \subset (s_{\mu\nu})$ of the Lorentz spin $s_{\mu\nu}$, that is SU(2)-spin $\vec{s}$ from (4), since it does not commute with the Hamiltonian H, does not conserve in the Loc-ψ-model of a spinor field (although no spin-flip processes in free states of spinor fields have been observed so far).

The most important Foldy–Wouthuysen analysis [12] of shortcomings of the explicitely covariant model of the spinor field (Loc-ψ-model) is the conclusion about non-reliability of interpretation of operators $\vec{x} \in R^3 \subset$ M(1,3) and $\vec{s} = (s_{jn}) \sim$ SU(2) from (4) as operators of the 3-coordinate and SU(2)-spin of an isolated (free) dublet of particles.

In [1,12,20] a nonlocal canonical model of the free spinor field was proposed (in what follows - the FW-ϕ-model). The simplified A-approach to the said model can be deduced from the statement, that for the field $\phi \in S^{3,4} \subset S^{3,4*}$ the space $S^{3,4}$ is a common domain and codomain for all operators, introduced in [1,12,20]. The objects in FW-ϕ- and Loc-ψ-models of the spinor field are mutually related by a pair of nonsingular in $S^{3,4}$ operators:

$$\begin{gathered} NV^{\pm} \doteq \pm i\gamma_l\partial_l + \hat{\omega} + m; \quad \hat{\omega} \doteq \sqrt{-\Delta + m^2}, \quad N^{-1} \equiv \sqrt{2\hat{\omega}(\hat{\omega}+m)}, \\ V^-V^+ = I_4 \in S^{3,4}. \end{gathered} \tag{22}$$

In particular,

$$V^+\left(\partial_0 - \gamma_{0l}\partial_l + i\gamma_0 m\right)\psi(x) \equiv \left(\partial_0 + i\gamma_0\hat{\omega}\right)\phi(x) = 0; \quad \psi, \phi \doteq V^+\psi \in S^{4,4}; \tag{23}$$

$$V^+(p_\mu, j_{\rho\sigma})^{\mathrm{Ind}}V^- \doteq (p_\mu, j_{\rho\sigma})^{\mathrm{FW}} \equiv (\hat{p}_\mu, \hat{j}_{\rho\sigma}), \tag{24}$$

where $\mathcal{P}^{\mathrm{FW}}$-generators (as operators in $S^{3,4}$):

$$\hat{p}_0 \doteq -i\gamma_0\hat{\omega}, \hat{p}_l = \partial_l, \hat{j}_{ln} = m_{ln} + s_{ln}; m_{ln} \doteq x_l\partial_n - x_n\partial_l; \tag{25a}$$

$$\hat{j}_{0l} = -\hat{j}_{l0} \doteq t\partial_l + i\frac{\gamma_0}{2}\{x_l, \hat{\omega}\} + i\gamma_0\frac{s_{ln}\partial_n}{\hat{\omega}+m}, \tag{25b}$$

that is the operators of 3-momentum $\vec{p}$ and total angular 3-momentum $\vec{j}$ coincide with those in (19a), and $\hat{p}_0$ with $\hat{j}_{0l}$ are expressed through γ_0 and SU(2)-part s_{ln} of the Lorentz spin $s_{\mu\nu}$ (4) (the operators (25a, 25b) do not depend on the boost components s_{0j} of the Lorentz spin operator $s_{\mu\nu}$ (4)).

The nonsingularity of operators[1] $V^{\pm}$ (22) results in $\mathcal{P}$-generators (25a, 25b) satisfying the same relations (3) in an explicitely covariant form and commuting with Hamiltonian $H^{FW} = -i\gamma^0\hat{\omega}$ of (23).

Since the three $\mathcal{P}$-representations, that is local $\mathcal{P}^{\gamma L}$ (16), induced $\mathcal{P}^{\gamma I}$ (21) and $\mathcal{P}^{\mathrm{FW}}$-representation of $\mathcal{P}$ are invariance groups of the Dirac equation in their respective forms, general statements follow: 10 principal integral physical quantities P_μ, $J_{\rho\sigma}$:

$$\begin{aligned}(P_\mu, J_{\rho\sigma}) &\doteq \int d^3x \psi^+(t,\vec{x}) i\gamma^0 (p_\mu, j_{\rho\sigma})\psi(t,\vec{x}) \\ &= \int d^3x \phi^+(t,\vec{x}) i(\hat{p}_\mu, \hat{j}_{\rho\sigma})\phi(t,\vec{x}) = \mathrm{const}(t)\,. \end{aligned} \tag{26}$$

conserve in time in any inertial reference frame and for all states ψ as solutions to (8)=(10) or $\phi = V^+\psi$ as solutions to the FW equation (23).

Following advantage of the FW-ϕ-model is to be noted: not only the generators $\mathcal{P}^{\mathrm{FW}}$ (25a, 25b), but the whole algebra A_S^{FW} is constructed from Heisenberg operators $\vec{x}$, $\vec{p} = -\nabla$, and conserved in time SU(2)-spin $\vec{s}$ in (19b). This construction is consistent with the principle of inheritability with nonrelativistic quantum mechanics and ensures, that all experimentally observable quantities for this field are functions of experimentally observable Heisenberg operators and SU(2)-spin:

$$\vec{x}, \vec{p}, \vec{s}: \quad [x^j, p^l] = \delta^{jl}, \quad [(\vec{x}, \vec{p}), \vec{s}] = 0, \tag{27}$$

where in PD-representation of matrices γ_μ

$$\vec{s} \equiv (s^j) \doteq \frac{1}{2}(\gamma_{23}, \gamma_{31}, \gamma_{12}) = -\frac{i}{2}\vec{\Sigma} \overset{PD}{\equiv} \frac{1}{2}\begin{vmatrix} -i\vec{\sigma} & 0 \\ 0 & -i\vec{\sigma}\end{vmatrix}. \tag{28}$$

These simplest 9 operators (27) generate the whole real-number algebra A_S^{FW} of observables in FW-ϕ-model of the spinor field. An analogy with the nonrelativistic quantum mechanics here should be mentioned, where the algebra of observables is defined by the Heisenberg operators $\vec{x}$, $\vec{p} = -\nabla$ in Hilbert space $L_2(R^3)$ of states $f \in L_2(R^3)$ of a "point mass").

For the problem of determining the (experimentally observed) average values of operators of physical quantities or probability distribution amplitudes over the stationary complete sets $\hat{Q}$ of experimentally observed physical characteristics of a dublet there is no need to transform the operators from algebra A_S^{FW} in FW-ϕ-model into algebra A_S^L of the Loc-ψ-model of a spinor field. The monograph [16] describes a technique for finding different "nongeometric" and "geometric" hidden symmetries of the Dirac equation (complementary to the $\mathcal{P}$-symmetry) in the FW-representation, which are generated by various operators, commuting with the Hamiltonian $H^{\mathrm{FW}} = -i\gamma^0\hat{\omega}$ of Eq. (23) for $s = \frac{1}{2}$ dublet.

[1] FW-transformation (22) is the similarity transformation for the standard CD algebra in $\mathrm{SO}(6)^{\mathrm{PD}}$-form (see below Sect. 3), that is $V^{\pm}A_0V^{\mp} \equiv A_0^{\pm}$ generates $\mathrm{SO}(6)^{\pm} = V^{\pm}\mathrm{SO}(6)^{\mathrm{PD}}V^{\mp}$. Matrices $\gamma_{\underline{j}}^{\pm}$ as operators in $S^{4,4}$ are functions of not only $\gamma_j = \gamma_j^{\mathrm{PD}}$, but also differential ∂_j and pseudodifferential $\hat{\omega}$ operators in $S^{4,4}$. Here these realizations of CD algebra are not used.

From the explicit form (25a, 25b) of $\mathcal{P}^{\rm FW}$-generators it is seen, that their principal Casimir operators are $p^\mu p_\mu = -m^2 < 0$ and $\vec{s}^2 = -\frac{3}{4}\cdot 1_4 \Rightarrow s = \frac{1}{2}$, which means, that the symmetry, generated by operators (25a, 25b), is a Fermi-type $\mathcal{P}^{\rm F}$-symmetry). Therefore the revealing of the FB-dualism of a spinor field in the present work will be performed in the FW-ϕ-model of the spinor field, starting from the real number algebra $A_S^{\rm FW}$ and extending it to cover the boson SU(2)-spin. This extension of $A_S^{\rm FW}$ is performed in the following section through the extension of its matrix subalgebra, that is of the standard CD-algebra, treated as a real-number algebra in the complex space $\mathcal{C}^4 \subset S^{3,4}$.

In conclusion we point out at two essential physical limitations of the FW-ϕ-models, namely: the average of the spinor field energy for the FW-ϕ-model (similarly to Loc-ψ-model):

$$P_0[\phi] \doteq \int d^3x \phi^+(x)\gamma_0\hat{\omega}\phi(x) \equiv \int d^3x \bar{\psi}(x) iH\psi(x) \gtrless 0 \tag{29}$$

in $\forall$-representation of matrices γ_μ, does not have generally a definite sign. On the other hand, in the PD representation the SU(2)-spin (28) of the e^-e^+-dublet does not reflect the experimentally stated fact, that the particle and the antiparticle in a dublet are mutually mirror reflected not only by the charge sign $g = i\gamma_0$, but also by chirality. It will be shown below, that in certain exclusive representations of matrices γ_μ the FW-ϕ-model will acquire the features of a quantum mechanical model without these two shortcomings.

3. Real-Number Extension of the Clifford–Dirac Algebra and its Application

3.1. Choice of Appropriate Generators of a Standard CD-Algebra

The constitutive relations (9) for matrices γ^μ ($\mu = \overline{0,3}$) (as nonsingular operators in $\mathcal{C}^4 \subset S^{3,4}$, complemented with the matrix $\gamma^4 \doteq \gamma^{0123}$ in their $\forall$ representation will identically yield the equations ($\underline{\mu} = \overline{0,4}$):

$$\prod_0^4 \gamma_{\underline{\mu}} = -1_4 \Rightarrow \prod_1^4 \gamma_{\underline{j}} = -\gamma_0;\ \gamma_{234} = -\gamma_{01}, \gamma_{123} = \gamma_{04},$$
$$\gamma_{034} = -\gamma_{12}, \gamma_{012} = \gamma_{34}, \gamma_{0234} = \gamma_1, \gamma_{0123} = \gamma_4, (123!), \tag{30}$$

with arbitrary cyclic permutation 123! of spatial indices 1,2,3 (the equalities (30) hold for both contra- and covariant $\gamma^{\underline{\mu}}$, $\gamma_{\underline{\mu}}$ Dirac matrices). It is seen from these equalities, that any product $\gamma_{\underline{\mu}\underline{\nu}\ldots} \doteq \gamma_{\underline{\mu}} \cdot \gamma_{\underline{\nu}} \ldots$ can be expressed through the matrices $\gamma_{\underline{\mu}}$ and their binary products $\gamma_{\underline{\mu}\underline{\nu}}$.

Introduce the following notations for two different sets of generators of SO(1,5) and SO(6):

$$\mathrm{SO}(1,5)$$
$$\rightsquigarrow \left\{ s_{\mu_1\mu_2}(\mu_{1,2}{=}\overline{0,5})\colon 2s_{\underline{\mu}5} = -2s_{5\underline{\mu}} \doteq \gamma_{\underline{\mu}}, 2s_{\underline{\mu}\underline{\nu}} \doteq \frac{1}{2}[\gamma_{\underline{\mu}}, \gamma_{\underline{\nu}}]; \underline{\mu}{=}\overline{0,4}) \right\}, \tag{31}$$

TABLE 1. Matrices $2s_{j_1 j_2} \equiv -2s_{j_2 j_1}$ $(j_{1,2} = \overline{1,6})$ of CD-algebra SO(6)

γ_{12}	γ_{13}	γ_{14}	$15 \doteq i\gamma_{01}$	$16 \doteq \gamma_1$
	γ_{23}	γ_{24}	$25 \doteq i\gamma_{02}$	$26 \doteq \gamma_2$
		γ_{34}	$35 \doteq i\gamma_{03}$	$36 \doteq \gamma_3$
$\leftarrow$ SO(4) $\rightarrow$			$45 \doteq i\gamma_{04}$	$46 \doteq \gamma_4$
				$56 \doteq -i\gamma_0$

$$\mathrm{SO}(6) \rightsquigarrow \Big\{ s_{j_1 j_2}(j_{1,2} = \overline{1,6}) : 2s_{\underline{j}6} \doteq -2s_{6\underline{j}} \doteq \gamma_{\underline{j}}, \quad 2s_{56} \doteq -2s_{65} \doteq -i\gamma_0;$$
$$2s_{\underline{j}\underline{l}} \doteq \frac{1}{2}[\gamma_{\underline{j}}, \gamma_{\underline{l}}]; \underline{j}, \underline{l} = \overline{1,4} \Big\}. \tag{32}$$

Using the relations $(9)_{\mu \to \underline{\mu} = \overline{0,4}}$, it is straightforward to show, that non-singular sets $s_{\mu_1 \mu_2}$ (31) and $s_{j_1 j_2}$ (32) of matrices in $\mathcal{C}^4$ satisfy the relations (6) and respectively (7), that is the said sets are ζ-generators of representations of these algebras in $\mathcal{C}^4$. This means, that these representations are isomorphous to the standard 16-dimensional CD-algebra in $\mathcal{C}^4$ (cf. [15], where the algebra SO(3,3) was used as isomorphous to the CD-algebra).

For our purposes, taking into account the concept of single-out t_0 variable it is convenient to use the SO(6) form of CD algebra.

As it is seen from Table 1, the standard CD-algebra in SO(6)-form contains all matrices, used for construction of both Dirac equations (10=8), and operators $V^{\pm}$ (22), relating Loc-ψ- and FW-ϕ-forms of a spinor field. The real-number algebra SO(6) contains matrices for constructing the 4-component Dirac equation in both forms in a 5-dimensional space, that is for the Minkowski space M(1,4), however, this goes beyond the scope of the present paper. It is convenient to associate the CD $\rightsquigarrow$ SO(6)-algebra with FW-ϕ-form of the spinor field: indeed, this SO(6)-algebra contains the SU(2)-spin $\vec{s}$ (28), and the whole subalgebra $\rightsquigarrow \{\gamma_{12}, \gamma_{13}, \gamma_{23}, \gamma_{14}, \gamma_{24}, \gamma_{34}\} \subset \mathrm{SO}(6)$, and its element $2s_{56} \doteq -i\gamma_0$ is the Casimir operator of the SO(4)-algebra, thus determining the Hamiltonian $H^{\mathrm{FW}} = -i\gamma_0\hat{\omega}$.

The standard CD $\rightsquigarrow$ SO(6)-algebra as a carrier of the internal degrees of freedom for the particle-antiparticle dublet (including purely matrix SO(4) $\supset$ SU(2)-symmetries of the FW-ϕ-field), is the subalgebra of the real algebra A_S^{FW}, therefore the SO(6)-algebra should be interpreted as well as a real purely matrix algebra in the complex space $\mathcal{C}^4 \subset S^{3,4}$.

Note A Likewise the FW-ϕ-model, the CD $\rightsquigarrow$ SO(6) algebra (as opposed to the CD $\rightsquigarrow$ SO(1,5)-algebra) explicitely stresses the concept of single-out time variable. Thats why its primary generators are not 4 matrices γ_μ $(\mu = \overline{0,3})$, but 4 matrices γ_j, which in the $\forall$ representation satisfy the generic relations

$$\gamma_j(j = \overline{1,4}) : \ \gamma_j\gamma_l + \gamma_l\gamma_j = -2\delta_{jl} \Rightarrow \gamma_j^{-1} = -\gamma_j = \gamma^j \, . \tag{33}$$

Taken as a set together with matrix $i\prod_1^4 \gamma_j \equiv -i\gamma_0 \doteq 2s_{56}$, matrices $\gamma_{\underline{j}} = 2s_{\underline{j}6}$ satisfy the "extended" relations $(33)_j \to \underline{j} = \overline{1,5}$. As it can be seen from Table 1, the matrices $\gamma_{\underline{j}} \doteq 2s_{\underline{j}6}$ ($\underline{j} = \overline{1,5}$) of the last column of the table determine all remaining unit vectors—paired products $\gamma_{j\underline{l}} = 2s_{j\underline{l}}$ of CD $\rightsquigarrow$ SO(6) algebra.

3.2. Extension of standard CD $\rightsquigarrow$ SO(6)-algebra to ERCD $\rightsquigarrow$ SO(8)

For the construction of representations, overcoming the mentioned problems of the FW-ϕ model, we first emphasize several concepts to be used. In particular, we single out the time variable [in SD (10) and FW-ϕ (46) equations], therefore initial matrices for the standard CD $\rightsquigarrow$ SO(6) algebra are chosen as γ^j (33). Then, we take into account the concept of physical meaning of *real number* algebras of observables in field models, and the indispensability of the *extension* of the standard CD algebra as carrier of the internal degrees of freedom. Such an algebra, included into the algebra of observables of the FW-ϕ model will contain as initial independent operators over the set $\{\phi\} \in S^{3,4}$ not only 4 initial independent matrices $\gamma^j (j = \overline{1,4})$, but also the operator of the complex conjugation $\hat{C}$ and the "imaginary unit" $i = \sqrt{-1}$. They, being operators in $\mathcal{C}^4$, have a trivial form $\hat{C} \doteq c1_4$, $\hat{i} \doteq i1_4$ only in the standard Pauli–Dirac representation. These additional operators are known since long ago in the theory of the spinor field; thus $\hat{C}$ was first used in [18, 19], and $\hat{i}$ is known as the Heaviside–Larmor–Rainic operator [26]. The use of $\hat{C} = c1_4$, $\hat{i} = \sqrt{-1} = i1_4$ (in PD representation of γ^j matrices) gives a possibility of performing a simple and unambiguous technique of extension of the 16-dimensional CD$\rightsquigarrow$ SO(6) algebra to the 29-dimensional "extended" algebra ERCD $\rightsquigarrow$ SO(8) in the PD representation, and then—in an arbitrarily fixed representation of the generating matrices (cf. first versions of this extension technique for ERCD construction in [23, 27], presented there in an unreasonably complicated manner).

Let us amend the initial independent Dirac matrices γ^j with two additional matrices (which are independent in the real number algebra), defined as follows:

$$\gamma^5 \doteq \gamma^{13}\hat{C}, \quad \gamma^6 \doteq i\gamma^5 = -\gamma^5 i. \tag{34}$$

It is straightforward to check, that 7 independent matrices γ^j, $j = \overline{1,6}$ and $\gamma^7 \doteq \prod_{j=1}^6 \equiv -i\gamma^0$ anticommute, and their squares equal -1. (Let us note, that the matrices γ^4 and γ^5 in the notations of the present paper should not be confused with definitions of γ^4, γ^5, commonly used in the field theory but having a different meaning). Likewise 5 matrices γ^j, $j = \overline{1,4}$ and $-i\gamma^0$ generate the unit vectors of the 16-dimensional CD$\rightsquigarrow$ SO(6) algebra (cf. Table 2), now 7 matrices $\gamma^{\underset{\sim}{j}}$ ($\underset{\sim}{j} = \overline{1,7}$) generate the unit vectors of 29-dimensional ERCD $\rightsquigarrow$ SO(8) algebra (see below the Table 2) in the PD representation of γ^j matrices and operators $\hat{C} = c1_4$, $\hat{i} = i1_4$ (in other representations the operators $\hat{C}$ and $\hat{i}$ generally will not have the same form as in the PD representation), by the following formulas

TABLE 2. Indices of matrices $2s_{j_1 j_2} = -2s_{j_2 j_1}$ $(j_{1,2} = \overline{1,8})$ as generators of 29-dimensional ERCD $\rightsquigarrow$ SO(8) algebra

12	13	14	15	16	17	$18 \doteq \gamma_1$
	23	24	25	26	27	$28 \doteq \gamma_2$
		34	35	36	37	$38 \doteq \gamma_3$
$\leftarrow$ SO(4) $\rightarrow$			45	46	47	$48 \doteq \gamma_4 \equiv \gamma_{0123}$
				56	57	$58 \doteq \gamma_5 \equiv \gamma_{31}\hat{C}$
$\longleftarrow$ SO(6) $\longrightarrow$					67	$68 \doteq \gamma_6 \equiv i\gamma_{31}\hat{C}$
						$78 \doteq \gamma_7 \equiv -i\gamma_0$

$$\mathrm{SO}(8) \rightsquigarrow \left\{ 2s_{j_1 j_2} : 2s_{\underline{j}8} \doteq -2s_{8\underline{j}} \doteq \gamma_{\underline{j}},\, 2s_{\underline{j}\underline{l}} \doteq \frac{1}{2}\left[\gamma_{\underline{j}}, \gamma_{\underline{l}}\right]; \underline{j} = \overline{1,7} \right\} \tag{35}$$

This technique can be performed in the explicitely covariant form in the spaces M(1, N≤6) with the use of 7 independent matrices

$$\gamma^{\mu}(\mu = \overline{0,6}) : \gamma^{\mu} = \gamma^{\overline{0,3}}, \gamma^4 \doteq \gamma^{0123}, \gamma^5 \doteq \gamma^{13}\hat{C}, \gamma^6 \doteq i\gamma^5 \,. \tag{36}$$

Initial matrices here are 6 matrices $\gamma^{\overline{0,3}}$ and $\gamma^{5,6}$. They satisfy the "extended" anticommutation relations and define the generators $s_{\mu_1\mu_2}$ $(\mu_{1,2} = \overline{0,7})$ of the ERCD $\rightsquigarrow$ SO(1,7)$^{\mathrm{PD}}$ algebra as follows:

$$SO(1,7) \stackrel{PLC}{=} \left\{ s_{\mu_1\mu_2} : 2s_{\mu 7} \doteq -2s_{7\mu} \doteq \gamma_{\mu}, s_{\mu\nu} = \frac{1}{2}\left[\gamma_{\mu}, \gamma_{\nu}\right] (\mu,\nu = \overline{0,6}) \right\}. \tag{37}$$

Evidently, the 29-dimensional algebra SO(8) in $\mathcal{C}^4$ has more wide implications, compared to the standard algebra CD $\rightsquigarrow$ SO(6). In particular, seven 4×4-matrices—operators $\gamma_{\underline{j}}$ in $\mathcal{C}^4$ anticommute, defining all remaining generators of SO(8), i.e. paired products $\gamma_{\underline{j}\underline{l}}$. They generate 4-component Dirac equations in Minkowski spaces M(1, N≤ 6), belonging to $S^{N,4} \doteq$ S(M(1,N))×$\mathcal{C}^4$ in both Loc-ψ and FW-ϕ forms. In the last case the equation in M(1,6) acquires the following form

$$(\partial_0 - \gamma_7\underline{\hat{\omega}})\,\phi(\underline{x}) \equiv (\partial_0 + i\gamma_0\underline{\hat{\omega}})\,\phi(\underline{x}) = 0\,,\ \underline{x} \in \mathrm{M}(1,6)\,, \phi(\underline{x}) \in S^{6,4}\,. \tag{38}$$

The FW operator

$$\underline{NV}^{+} \doteq i\left(\gamma_a\partial_a + \gamma_4\partial_4 + \gamma_5\partial_5 + \gamma_6\partial_6\right) + \underline{\hat{\omega}} + m;\ \underline{\hat{\omega}} \equiv \sqrt{-\Delta + \partial_4^2 + \partial_5^2 + \partial_6^2 + m^2} \tag{39}$$

transforms the Eq. (38) into Loc-ψ-form in $S^{6,4}$:

$$\left(\partial_0 + \gamma_{0\underline{j}}\partial_{\underline{j}} + i\gamma_0 m\right)\psi(\underline{x}) = 0 \stackrel{\gamma_7}{\rightarrow} \left(i\gamma^{\underline{\mu}}\partial_{\underline{\mu}} - m\right)\psi(\underline{x}) = 0\,;\ \psi(\underline{x}) \in S^{6,4}\,,$$
$$\underline{j} = \overline{1,6}, \underline{\mu} = \overline{0,6}. \tag{40}$$

It is worth noting that for each $N = \overline{3,6}$ the Dirac equation in $S^{N,4}$ is invariant with respect to $\mathcal{P}(1, N \leq 6)$ representation of the Poincaré group in $S^{N,4}$ of respective dimension and form. But the most descriptive is the *extended purely matrix symmetry*—the invariance of each of $N = \overline{3,6}$ FW-ϕ-forms of Eq. (38) with respect to the same 16-dimensional subalgebra $\mathrm{SO}(6)^{\mathrm{I}}$ of the SO(8) algebra, which is defined by the unit vectors from the columns of Table 2 up to the column $2s_{j6}$ ($j = \overline{1,5}$). It is evident from the fact, that the matrix $\gamma_7 = -i\gamma_0$ (that is the matrix part of each of the Hamiltonians $H^{\mathrm{FW}}_{N\leq 6} = -i\underline{\hat{\omega}}\gamma_0$) is the Casimir operator for the whole subalgebra $\mathrm{SO}(6)^{\mathrm{I}} \subset \mathrm{SO}(8)$.

We note by passing, that nontrivial reductions of Eqs. (38), (40) (defined with equalities x_6, x_5, $x_4 = \mathrm{const}$ respectively) would lead to corresponding equations in flat spaces M(1,N=5,4,3). Meanwhile the nontrivial reductions (setting in the general solution of the Eq. (40) k_6, k_5, $k_4 = \mathrm{const}$ for the canonically conjugated momenta) would lead to corresponding Loc-ψ Dirac equations in M(1,N=5,4,3). In particular, the nontrivial reduction (projection) of the Eq. (40) onto the real number Minkowski space M(1,3) will ensure the experimental verifiability of the consequences of the spinor field model, which satisfies the 4-parametric Dirac equation

$$(\partial_0 + \gamma_{0a}\partial_a + \gamma_{04}im_2 + \gamma_{24}Cm_3 + i\gamma_{24}Cm_4 + i\gamma_0 m_1)\,\psi(x) = 0\,,\ \ \psi \in S^{3,4}\,. \tag{41}$$

This equation for $m_1^2+m_2^2-m_3^2-m_4^2 > 0$ is a 4-parametric Dirac equation in M(1,3) for a non-tachyonic field $\psi \in S^{3,4}$. The detailed treatment of Dirac equations in M(1,N$\leq$ 6) however is not covered by this work.

In the PD representation the initial independent matrices γ^j, satisfying (33), have the following 2×2-block form:

$$\gamma^a \doteq \begin{vmatrix} 0 & \sigma^a \\ -\sigma^a & 0 \end{vmatrix}\ (a = 1,2,3), \gamma^4 \doteq \begin{vmatrix} 0 & -i \\ -i & 0 \end{vmatrix} \Rightarrow \prod_1^4 \gamma^j \equiv -\gamma^0 \doteq \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix}, \tag{42}$$

where 2×2 Pauli matrices:

$$\sigma^1 = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}, \quad \sigma^2 = \begin{vmatrix} 0 & -i \\ i & 0 \end{vmatrix}, \quad \sigma^3 = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} \Rightarrow \sigma^a\sigma^b$$
$$= \delta^{ab} + i\varepsilon_{abc}\sigma^c\,(\varepsilon_{123} = +1)\,. \tag{43}$$

The extended ERCD algebra contains a subalgebra $\mathrm{SO}(6)^{\mathrm{PD}} \subset \mathrm{SO}(8)^{\mathrm{PD}}$, which is different from the SO(6) algebra in Table 1. It is a more wide invariance algebra of the FW equation (23), compared to the formerly known SO(4) algebra as a subalgebra of the "unextended" SO(6) (see Table 1), which is an analog of the invariance algebra for the Hydrogen atom, introduced in the Fock's paper. The carriers of the internal degrees of freedom of a e^-e^+-dublet in the algebra $\overline{A}_S^{\mathrm{FW}}$ are now represented by the subalgebra $\mathrm{SO}(6) \subset \mathrm{SO}(8)$ as invariance algebra of the FW equation (23). This subalgebra contains three samples of the conserved in time SU(2)-spins, which in the PD representation have the form $\vec{s}$ (28),

$$\vec{s}' \doteq \frac{1}{2}\,(\gamma_{54}, \gamma_{46}, \gamma_{65}) \overset{PD}{\equiv} \frac{1}{2}\left(-\gamma_{02}\hat{C}, i\gamma_{02}\hat{C} - i\right) \Rightarrow s = \frac{1}{2}\,;\ \vec{s} + \vec{s}' \doteq \underline{\vec{s}}. \tag{44}$$

It is straightforward to verify, that the components of each pair of the SU(2)-spin set $\vec{s}$ (28), $\vec{s}'$, $\underline{\vec{s}}$ (44) mutually commute (it is evident from the

ζ-relations (7): if indices do not coincide, then $[s_{j_1j_2}, s_{j_3j_4}] = 0$). However, contrary to $\vec{s}'^2 = \vec{s}^2 \equiv -\frac{1}{2}(\frac{1}{2}+1)1_4 \Rightarrow s = \frac{1}{2}$, the operator $\underline{\vec{s}}^2$ is nondiagonal in the PD representation.

Besides that it is necessary to stress, that the explicit form of the main spin $\vec{s}$ (28) in the PD representation does not reflect the experimental comprehension of the e^-e^+-dublet as a pair of mirror reflected particles by both the charge and spirality. In this sense such a form of the spin $\vec{s}$ (28) is nonphysical for the description of the e^-e^+-dublet. Therefore the problem is to find such representations of the ERCD $\rightsquigarrow$ SO(8) algebra, which overcome the remaining two shortcomings of the FW-ϕ model in the PD representation, namely, indefinite sign of the energy $P_0[\phi]$ and lack of mirror symmetry of the spin (28).

3.3. Constructing Exclusive Representations of Matrix Algebra in $\overline{A}_S^{\mathbf{FW}} \supset$ SO(8)

For the search of the ERCD representations without mentioned shortcomings of the FW-ϕ model we construct firstly the ERCD $\rightsquigarrow$ SO(8) algebra in $\forall$ representation, starting from the introduced technique of constructing the ERCD $\rightsquigarrow$ SO(8)$^{\mathrm{PD}}$ algebra in the PD representation. This construction requires introducing the following

Definition. A nonsingular in $\mathcal{C}^4$ operator T of the transformation of the set $A_0 \doteq \{\hat{C}, \hat{i} = \sqrt{-1}, \gamma_j, j = \overline{1,7}\}$ in the PD representation (defining the whole algebra SO(8)$^{\mathrm{PD}}$) is called the *similarity transformation* of the algebra SO(8)$^{\mathrm{PD}}$, if T preserves following commutation and anticommutation relations:

$$[\hat{C}, \gamma_{1,3,5}] = [i, \gamma_j] = \{\hat{C}, (\hat{i}, \gamma_2, \gamma_4, \gamma_6)\} = 0\,;\ \{\gamma_j, \gamma_l\} = -\delta_{jl}\,(j, l = \overline{1,7}) \tag{45}$$

for corresponding operators from the set $TA_0T^{-1} = A'_0$.

This definition ensures that the extension CD$'$ $\rightsquigarrow$ ERCD$'$ in $\forall$ representation of the set A'_0 is performed by the same technique, as introduced above for the extension of CD^{PD} $\Rightarrow$ ERCD^{PD} $\rightsquigarrow$ SO(8)^{PD}. It can be verified, that not every similarity transformation for the subalgebra SO($N<8$)^{PD} will be the similarity transformation for the whole algebra SO(8)^{PD}, which justifies the viability of introducing this definition.

3.3.1. Quantum Mechanical PD Representation of SO(8) Algebra.

Let us construct the simplest exclusive representation of the SO(8) algebra, different from the PD representation. It is evident, that in the FW-ϕ^{PD} equation (23) the field $\phi(x)$ is the direct sum of the 2-component quantum mechanical wave function of the electron $f_{e^-}(x)$ and the function $f^*_{e^+}$, complex conjugated to the wave function f_{e^+} of the positron. This means, that the nonsingular operator

$$\begin{vmatrix} 1_2 & 0 \\ 0 & c1_2 \end{vmatrix} \equiv \begin{vmatrix} 1 & 0 \\ 0 & c_2 \end{vmatrix} \doteq v_0 = v_0^{-1} \equiv C_+ - \gamma^0 C_-; C_\pm = \hat{C} \pm 1_4 \tag{46}$$

transforms the FW-ϕ-equation (23) for a e^-e^+-dublet into a 4-component quantum mechanical equation

$$v_0\left(\partial_0 + i\gamma^0\hat{\omega}\right)\phi^{\mathrm{PD}}(x) \equiv \left(\partial_0 + i\hat{\omega}\right) f(x) = 0; v_0\phi^{\mathrm{PD}}(x) \equiv \phi^{\mathrm{ex}}(x) \doteq f(x)$$
$$\equiv \begin{vmatrix} f_{e^-} \\ f_{e^+} \end{vmatrix} \in S^{3,4}. \tag{47}$$

This simplest nonsingular operator v_0 (46) is the similarity transformation operator for the whole algebra $\mathrm{SO}(8)^{\mathrm{PD}}$, which can be checked directly. In particular, it transforms the set $A^{\mathrm{PD}} \doteq \left\{\hat{C}, \hat{i}, \gamma^{\underline{j}}(\underline{j} = \overline{1,7})\right\}$ into the set $A^{\mathrm{QPD}} \doteq v_0 A^{\mathrm{PD}} v_0 \equiv \{\check{O}\}$, whose elements $\check{O}$ are expressed through PD operators as follows:

$$\check{C} \equiv \hat{C}, \check{i} \equiv i\gamma^0, \check{\gamma}^{1,3} \equiv \gamma^1\hat{C}, \gamma^3\hat{C}; \check{\gamma}^{2,4} \equiv \gamma^{02}\hat{C}, \gamma^{04}\hat{C};$$
$$\check{\gamma}^5 \doteq \check{\gamma}^{13}\check{C} \equiv \gamma^{13}\hat{C}, \check{\gamma}^6 \doteq \check{i}\check{\gamma}^5 \equiv -\gamma^{24}\hat{C}, \check{\gamma}^7 \doteq -\check{i}\check{\gamma}^0 \tag{48}$$

(we recall that the unlabeled operators $\hat{C}$, $\hat{i}$, γ^μ refer to the PD representation). Seven matrices $\check{\gamma}^j$ in (48) can be used to write down all generators of the algebra $\mathrm{SO}(8) \doteq v_0\mathrm{SO}(8)^{\mathrm{PD}}v_0$ by formulas in (35). The representation $\mathrm{SO}(8)^{\mathrm{QPD}}$ of the algebra is called *fermionic quantum mechanical Pauli-Dirac representation* (or QPD) for evident reasons.

In the $\mathrm{SO}(8)^{\mathrm{QPD}}$ representation of the ERCD algebra the Hermitian Hamiltonian of the equation for $f(x)$ (47) is expressed through $\check{\gamma}^7$: $iH^{\mathrm{FW}}_{ex} \doteq i\check{\gamma}^7\hat{\omega} \equiv \hat{\omega}$. Thats why the Eq. (47) is the quantum mechanical equation with only positive energy solutions. The spinor field of the FW-ϕ model $\phi^{ex} = f$ in the space $S^{3,4} \subset S^{3,4*}$ is extended to the four component wave function field f in the quantum mechanical Hilbert space $H^{3,4} = L_2(R^3) \times \mathcal{C}^4$, defined via the same Lebesgue measures d^3x in $\vec{x}$- or d^3p in $\vec{p}$ representations, which are used in the nonrelativistic quantum mechanics. The metrics in the set $\{f(x)\} \subset S^{3,4}$ of the solutions to (47) is positively defined and hence the experimentally measured energy of the dublet in $\forall f(x)$ will be positive as well:

$$||f||^2 = \int d^3x f^\dagger(x) f(x) \geq 0\,,\ P_0[f] \doteq \int d^3x f^\dagger(x)\hat{\omega} f(x) \geq 0\,. \tag{49}$$

More, the object of the Eq. (47) is defined in the extended Hilbert space

$$f \in S^{3,4} \subset H^{3,4} \doteq L_2(\mathcal{R}^3) \times C^4 \subset S^{3,4*}\,. \tag{50}$$

This set of three spaces in (50), named *rigged* quantum mechanical Hilbert space, revokes interest by the fact, that the space $S^{3,4}$ in the set (50) is a *kernel* space. The latter means, that $S^{3,4}$ is dense in both quantum mechanical $H^{3,4}$, and generalized Schwartz function spaces $S^{3,4*}$ (by corresponding topologies in these spaces). Such refinement of the A1 axiom means, that in the *canonical quantum mechanics* (CQM) for the e^-e^+-dublet any (generalized) state from the standard quantum mechanical $H^{3,4}$ or $S^{3,4*}$ space can be approximated (to any pre-specified degree of precision) with a Cauchy sequence, completely pertaining to the test Schwartz function space $S^{3,4}$.

Next, in the ERCD^{ex}=SO(8)$^{\mathrm{QPD}}$ algebra a purely matrix invariance algebra of the Eq. (47) for e^-e^+ is a subalgebra SO(6)$^{\mathrm{QPD}}$ and the main SU(2)-spin for the e^-e^+-dublet has the form

$$\check{\vec{s}} \doteq \frac{1}{2}\left(\check{\gamma}_{23}, \check{\gamma}_{31}, \check{\gamma}_{12}\right) \equiv \frac{1}{2}\left(\gamma_{023}, \gamma_{31}, \gamma_{012}\right) \equiv \frac{-i}{2}\begin{vmatrix} \vec{\sigma} & 0 \\ 0 & -c_2\vec{\sigma}c_2 \end{vmatrix} \equiv v_0\vec{s}v_0\,. \qquad (51)$$

This (*quantum mechanical*) form of the spin for a e^-e^+-dublet has a *physical meaning*, since it describes the mutual mirror reflection of the dublet particles by the charge sign and spirality. It is interesting to note that not only the main spin (51), but also two complimentary independent SU(2)-spins

$$\check{\vec{s}}' \doteq \frac{1}{2}\left(\check{\gamma}_{54}, \check{\gamma}_{46}, \check{\gamma}_{65}\right) \equiv \frac{1}{2}\left(\gamma_{2}, -i\gamma_{02}, -i\gamma_{0}\right), \check{\underline{\vec{s}}} \doteq \check{\vec{s}} + \check{\vec{s}}' \qquad (52)$$

for the e^-e^+-dublet in the QPD representation do not contain the $\hat{C}$ operator.

The componentwise commutativity of the SU(2) spins $\vec{s}$ (51) and $\vec{s}'$ (52) implies, that the components of the sum $\check{\underline{\vec{s}}} = \check{\vec{s}} + \check{\vec{s}}'$ in (52) and of the main spin $\check{\vec{s}}$ also mutually commute, so $[\check{\vec{s}}^2, \check{\underline{\vec{s}}}^2] = 0$. However, in contrast to $\check{\vec{s}}^2$, the square $\check{\underline{\vec{s}}}^2$ of the independent spin is non-diagonal. By direct solving the Sturm–Liouville problem for $\check{\underline{\vec{s}}}^2$ it can be shown, that the spin number $\underline{s} = (1, 0)$ (it will be evident from Sect. 3.3.2 below in a novel exclusive representation of the SO(8) algebra). So, the time-conserved spin $\check{\underline{\vec{s}}} = \check{\vec{s}} + \check{\vec{s}}'$, as independent from the main spin $\check{\vec{s}}$ (51) carrier of the internal degrees of freedom, describes the bosonic compound states with $\underline{s} = (1, 0)$ (that is tensor-scalar states) of an object f of the Eq. (47) (that is of e^-e^+-dublet) on equal footing with the spin $\check{\vec{s}}$ (51) describing fermionic states with $s = \frac{1}{2}D$ dublet e^-e^+.

Further, two time-conserving spins, the F-spin $\check{\vec{s}}$ in (51) and B-spin $\check{\underline{\vec{s}}}$ in (52) define two sets of independent quantum mechanical $\mathcal{P}$-generators $\left(\check{p}_\mu, \check{j}_{\rho\sigma}\right)^{\mathrm{F,B}}$ in $S^{3,4} \subset H^{3,4}$, having the following form:

$$\check{p}_0 = -i\hat{\omega},\ \check{p}_l = \partial_l,\ \check{j}^{F,B}_{ln} = m_{ln} + s^{F,B}_{ln};\ m_{ln} \equiv x_l\partial_n - x_n\partial_l, \qquad (53a)$$

$$\check{j}^{F,B}_{0l} = -\check{j}^{F,B}_{l0} = t\partial_l + \frac{i}{2}\{x_l, \hat{\omega}\} + \frac{s^{F,B}_{ln}\partial_n}{\hat{\omega} + m} \qquad (53b)$$

(we omit the labels F, B at the generators where they are identical for Fermi and Bose cases). These $\check{\mathcal{P}}^{\mathrm{F,B}}$-generators satisfy $\mathcal{P}$-relations (3a, 3b) and commute with operators of the quantum mechanical equation (47). Therefore they define (by an exp-series, convergent in $S^{3,4}$) two independent quantum mechanical $\mathcal{P}^{\mathrm{F,B}}$-representations of the group $\mathcal{P}$ in the state space $S^{3,4} \in H^{3,4}$ for the dublet.

Let Q^F and Q^B are two stationary complete sets in the algebra of observables $\overline{A}^{ex}_S \supset \mathrm{SO}(8)^{\mathrm{QPD}}$, such that the internal degrees of freedom of the dublet are represented by the F-spin $\check{\vec{s}}^F \doteq \check{\vec{s}}$ (51) and, respectively, B-spin $\check{\vec{s}}^B \doteq \check{\underline{\vec{s}}}$ in (52). These $Q^{F,B}$ sets define unambiguously (due to absence of degeneration for eigenvectors and common spectra of any full set, see, e.g., [9]) general solutions of the Eq. (47) in $H^{3,4}$ as

$$f^F(x) = \int \frac{d^3k}{(2\pi)^{3/2}} e^{-i\omega t + i\vec{k}\vec{x}} a_{F\alpha}(\vec{k}) e^F_\alpha(\vec{k}),\ f^B(x) = \int \frac{d^3k}{(2\pi)^{3/2}} e^{-i\omega t + i\vec{k}\vec{x}} b_{B\beta}(\vec{k}) e^B_\beta(\vec{k});\ \omega \equiv \sqrt{\vec{k}^2 + m^2}, \tag{54}$$

where $e^F_\alpha(\vec{k})$, $\check{e}^B_\beta(\vec{k})$ are 4-component unit vectors of the sets $Q^{F,B}$ in the $\vec{k}$ representation, and $a_{F\alpha}(\vec{k})$, $b_{B\beta}(\vec{k})$ are probability distribution amplitudes by the momentum $\vec{k}$ and quantum numbers α and β of the internal F- and B degrees of freedom. These two types of quantum mechanical states sets $\{f^F\}$ and $\{f^B\}$ (54) are of equal importance for the understanding of the e^-e^+ dublet on the base of experimental measurements of two types of amplitude probability distributions (boson and fermion amplitudes) of full sets of eigenvalues for the dublet.

The interpretation of these statements is given at the end of Sect. 3.3.2, which positively answers a question about the possibility of other exclusive representations of the SO(8) algebra, where the FW-ϕ equation (23) has quantum mechanical form (47). Following fact is evident from this point: the interpretation of $s = \frac{1}{2}D$ "particle-antiparticle" dublets (based on the Bargmnan–Wigner classification [22]) solely as elementary relativistic fermions, is incomplete.

3.3.2. Quantum mechanical FTS representation of SO(8) algebra. Now we use the unitary operator in $H^{3,4}$

$$U = \frac{1}{\sqrt{2}} \begin{vmatrix} 1 & 0 & -1 & 0 \\ i & 0 & i & 0 \\ 0 & -1 & 0 & 1 \\ 0 & -1 & 0 & -1 \end{vmatrix} \Rightarrow U^\dagger = \frac{1}{\sqrt{2}} \begin{vmatrix} 1 & -i & 0 & 0 \\ 0 & 0 & -1 & -1 \\ -1 & -i & 0 & 0 \\ 0 & 0 & 1 & -1 \end{vmatrix}. \tag{55}$$

This operator does not affect the quantum mechanical equation in (47)

$$(\partial_0 + i\omega) f(x) \xrightarrow{U} (\partial_0 + i\omega) \underset{\sim}{f}(x) = 0,\ \underset{\sim}{f} = Uf \in S^{3,4}. \tag{56}$$

As the similarity transformation operator for the whole algebra SO $(8)^{\mathrm{QPD}}$, the operator (56) transforms the quantum mechanical set $A^{\mathrm{QPD}} \doteq \{\check{C}, \check{i}, \check{\gamma}^{\underline{j}}, (\underline{j} = \overline{1,7})\}$ into the set $A^{\mathrm{FTS}} \doteq U A^{\mathrm{QPD}} U^\dagger \equiv \{\underset{\sim}{Q}\}$, whose elements $\underset{\sim}{Q}$ have the form:

$$\underset{\sim}{C} = \begin{vmatrix} c & 0 & 0 & 0 \\ 0 & -c & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & c \end{vmatrix},\quad \underset{\sim}{i} = \begin{vmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{vmatrix},\quad \underset{\sim}{\gamma}_1 = \begin{vmatrix} 0 & 0 & 0 & c \\ 0 & 0 & -ic & 0 \\ 0 & ic & 0 & 0 \\ -c & 0 & 0 & 0 \end{vmatrix}, \tag{57}$$

$$\underset{\sim}{\gamma}_2 = \begin{vmatrix} 0 & 0 & ic & 0 \\ 0 & 0 & 0 & c \\ -ic & 0 & 0 & 0 \\ 0 & -c & 0 & 0 \end{vmatrix},\quad \underset{\sim}{\gamma}_3 = \begin{vmatrix} 0 & -ic & 0 & 0 \\ ic & 0 & 0 & 0 \\ 0 & 0 & 0 & c \\ 0 & 0 & -c & 0 \end{vmatrix},\quad \underset{\sim}{\gamma}_4 = \begin{vmatrix} 0 & -c & 0 & 0 \\ c & 0 & 0 & 0 \\ 0 & 0 & 0 & ic \\ 0 & 0 & -ic & 0 \end{vmatrix} \tag{58}$$

$$\underset{\sim}{\gamma}_5 \equiv \underset{\sim}{\gamma}_{31}\underset{\sim}{C} = \begin{vmatrix} 0 & 0 & c & 0 \\ 0 & 0 & 0 & ic \\ -c & 0 & 0 & 0 \\ 0 & -ic & 0 & 0 \end{vmatrix}, \quad \underset{\sim}{\gamma}_6 \equiv i\underset{\sim}{\gamma}_5 = \begin{vmatrix} 0 & 0 & 0 & ic \\ 0 & 0 & -c & 0 \\ 0 & c & 0 & 0 \\ -ic & 0 & 0 & 0 \end{vmatrix},$$
$$\underset{\sim}{\gamma}_7 \doteq -i\underset{\sim}{\gamma}_0 \equiv -i \cdot 1_4 \tag{59}$$

Starting from the column $2\underset{\sim}{s}_{\underline{j}8} \doteq \gamma_{\underline{j}}$ of the Table 2, it is straightforward to derive explicit forms of all remaining generators $2\underset{\sim}{s}_{\underline{j}\underline{l}} \doteq \underset{\sim}{\gamma}_{\underline{j}\underline{l}}$ and their subsets in this representation. The expression $2\underset{\sim}{s}_{78}\underset{\sim}{\gamma}_7 \doteq -i1_4 \Rightarrow H^{\mathrm{FW}}_{\mathrm{FTS}} = -i\hat{\omega}$ reflects the fact, that the equation of motion for a e^-e^+dublet in the FTS representation has the form (56), i.e. that of the quantum mechanical Schroedinger–Foldy equation (47).

Further, the quantum mechanical spins of the QPD representation have in the FTS representation the following form:

$$\vec{s} \equiv \left(s_{ln}\right) \doteq \frac{1}{2}\left(\underset{\sim}{\gamma}_{23}, \underset{\sim}{\gamma}_{31}, \underset{\sim}{\gamma}_{12}\right):$$
$$\underset{\sim}{\gamma}_{23} = \begin{vmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ i & 0 & 0 & 0 \end{vmatrix}, \underset{\sim}{\gamma}_{31} = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \\ -1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{vmatrix}, \underset{\sim}{\gamma}_{12} = \begin{vmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{vmatrix}$$
$$\Rightarrow \vec{s}^{\,2} \equiv -\frac{1}{2}\left(\frac{1}{2}+1\right) \cdot 1_4 \Rightarrow s = \frac{1}{2}. \tag{60}$$

$$\underset{\sim}{\vec{s}}' \doteq \frac{1}{2}\left(\underset{\approx}{\gamma_{54}}, \underset{\approx}{\gamma_{46}}, \underset{\approx}{\gamma_{65}}\right) \equiv C\underset{\sim}{\vec{s}}C, \underset{\sim}{\vec{s}}'^{\,2} = \underset{\sim}{\vec{s}}^{\,2} \Rightarrow s = \frac{1}{2}; \tag{61}$$

$$\underset{\sim}{\vec{s}} + \underset{\sim}{\vec{s}}' \doteq \vec{s}^{\,\mathrm{TS}} \equiv \begin{vmatrix} \vec{s}_\zeta & 0 \\ 0 & 0 \end{vmatrix} \Rightarrow \left(\vec{s}^{\,\mathrm{TS}}\right)^2 = -\begin{vmatrix} 1(1+1)\cdot 1_3 & 0 \\ 0 & 0 \end{vmatrix} \Rightarrow \underline{s} = (1,0). \tag{62}$$

The explicit form of the Casimir operator in (62) is seen from the observation, that both $\underset{\sim}{\vec{s}}$ (60) and $\underset{\sim}{\vec{s}}'$ (61) contain a 3 × 3 block—SU(2)-spin

$$\vec{s}_\zeta \equiv \left(s^j_\zeta\right) : s^1_\zeta = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{vmatrix}, s^2_\zeta = \begin{vmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{vmatrix}, s^3_\zeta = \begin{vmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} \Rightarrow \vec{s}^{\,2}_\zeta$$
$$= -1(1+1)\cdot 1_3 \Rightarrow s = 1. \tag{63}$$

This SU(2)-spin is the spin operator of a complex antisymmetric self-dual tensor field in an *exclusive* cyclic representation: it satisfies the SU(2)-relations (5), but in addition all its components are cyclic, namely: $\left(s^j_\zeta\right)^k_l = -\varepsilon^{jkl}$ (where ε^{jkl} is the Levi–Civita tensor, $\varepsilon^{123} = +1$). This clarifies the exclusive nature of the operator $\underset{\sim}{T}$ in (64) and the term “tensor-scalar spin” $\underset{\sim}{\vec{s}}^{\,\mathrm{TS}}$ (62).

Note B The similarity transformation for the $\mathrm{SO}(8)^{FTS}$-algebra at the transition to the commonly used PD representation is given by the operator

$$\underset{\sim}{T}^{-1} \doteq v_0 U^\dagger = \frac{1}{\sqrt{2}} \begin{vmatrix} 1 & -i & 0 & 0 \\ 0 & 0 & -1 & -1 \\ -c & ic & 0 & 0 \\ 0 & 0 & c & -c \end{vmatrix} \Rightarrow \underset{\sim}{T} = \frac{1}{\sqrt{2}} \begin{vmatrix} 1 & 0 & -c & 0 \\ i & 0 & ic & 0 \\ 0 & -1 & 0 & c \\ 0 & -1 & 0 & -c \end{vmatrix} . \tag{64}$$

It is straightforward to verify:

$$\underset{\sim}{T}^{-1} \left(A^{\mathrm{FTS}} \doteq \left\{ \underset{\sim}{C}, i, \underset{\sim}{\gamma}^{\underline{j}}, (\underline{j} = \overline{1,7}) \right\} \right) \underset{\sim}{T} \equiv A^{\mathrm{PD}} \doteq \left\{ \hat{C}, \hat{i}, \gamma^{\underline{j}}, (\underline{j} = \overline{1,7}) \right\} , \tag{65}$$

where, in particular, $\gamma^5 = \gamma^{13}\hat{C}$ and $\gamma^6 = i\gamma^5$, hat is they coincide with definitions (34) in the PD representation. This proves the uniqueness of the extension procedure CD $\rightsquigarrow$ $\mathrm{ERCD}^{\mathrm{PD}}$ from Sect. 3.2 in the PD representation. Indeed, any other choice of the explicit form of the 3×3-spin with $\vec{s}_\zeta$ with $s = 1$ in $\vec{s}^{\mathrm{TS}}$ (62), different from the *exclusively cyclic form* (63) under the transformation $\underset{\sim}{T}^{-1}\underset{\sim}{\gamma}^{5,6}\underset{\sim}{T}$ would give for γ^5 and γ^6 the results, contradicting the definition (34).

To the end of this chapter let us emphasize, that the $\mathcal{P}$-generators $(p_\mu, j_{\rho\sigma})^{\mathrm{F,B}}$ and the sets $\{f^F, f^B\}$ of F- and B-solutions to the quantum mechanical equation (47) in QPD- and FTS representations are related by a unitary operator U (55) (which does not alter the equation (47)=(56) for the e^-e^+ object). The FTS images of QPD expressions (53) and (54) alter only explicit forms of spin matrices $\underset{\sim}{s}^{\mathrm{F,B}} = U\check{\vec{s}}^{\mathrm{F,B}}U^\dagger$ and 4-component unit vectors

$$\left(e^{\mathrm{F}}_{\alpha}(\vec{k}), \check{e}^{\mathrm{B}}_{\beta}(\vec{k}) \right)^{\mathrm{QPD}} \to \left(\underset{\sim}{e}^{\mathrm{F}}_{\alpha}(\vec{k}), e^{\mathrm{B}}_{\beta}(\vec{k}) \right)^{\mathrm{FTS}} \doteq U \left(e^{\mathrm{F}}_{\alpha}(\vec{k}), \check{e}^{\mathrm{B}}_{\beta}(\vec{k}) \right)^{\mathrm{QPD}} , \tag{66}$$

but the experimentally measured quantum mechanical amplitudes $a_{F\alpha}(\vec{k})$ and $b_{B\beta}(\vec{k})$ remain the same. This means, that the physical interpretation (end of Sect. 3.3.1) of the results (53) and (54) about the status of $\mathcal{P}^{\mathrm{F,B}}$ representations and sets of $\{f^F, f^B\}$-solutions to the SF equation (47) for the e^-e^+-dublet in QPD and FTS representations (as in any other representation, derived by means of a unitary similarity transformation operator) remains the same. Hence, the Bargmann and Wigner's classification [22] stating the "fermion nature of the e^-e^+-dublet" can be replaced by the statement, that the said dublet as an elementary relativistic object in any exclusive FW-$\phi^{ex} = f$-models is a FB-dual object, described by corresponding F- and B-spins, which define the equally important 4-component sets $\{f^F\}$ and $\{f^B\}$ of the solutions to the SF equation (47).

Note C All statements about FB-dual properties of the e^-e^+-dublet and their physical interpretation from two FW-$\phi^{ex} = \phi^{\mathrm{QPD,FTS}}$-models of the spinor field can be cast via an appropriate nonsingular FW-operator (23) in these two representations as Loc-ψ^{ex}-models of the spinor field. However these statements, formulated in the language of Loc-ψ^{ex}-models, have a rather complicated form, and, even presented in such a form, would not yield any additional physical and mathematical knowledge about the FB-dual ψ^{ex} field. Therefore we do not write such transformations here.

The same refers to the theory of the FB-dual quantum (quantized) spinor field ψ^{ex}, which is constructed in the Loc-ψ-model version on the base of the anticommutative quantization of the field $\psi(x)$. We sketch briefly the construction algorithm of the theory of the quantum FB-dual spinor field, using the quantum mechanical model (FW-ϕ^{ex}-model) for the e^-e^+-dublet.

There is no need to apply the Grassmanian coordinates for constructing the quantum FB and BF-dual fields in the forms Loc-ψ and FW-ϕ: the "second quantization", either by Fermi, or by Bose, is performed by constructing the Fock space $\mathcal{H}^{\mathcal{F}}$ over the "one-particle" $\mathcal{H}^{3,4}$-space and transition in the space $\mathcal{H}^{\mathcal{F}}$ from the configuration representation for a system of n identical Fermi or Bose states towards the representation of occupation numbers for these states, in the comprehensive analogy with the custom "second quantization" technique of the nonrelativistic quantum mechanics for a system of identical elementary objects. The operator amplitudes of the quantum mechanical F-solutions to equation (47)=(56) will satisfy the systems of anticommuting relations (ACR), whilst the operator B-amplitudes of solutions to SF equation (47)=(56)—the CCR-relations. N-particle F-states will fill the antisymmetric part of $\mathcal{H}^{\mathcal{F}}$, and symmetric states will fill the symmetric part of $\mathcal{H}^{\mathcal{F}}$.

In brief, this "quantization" is performed as follows. The sets $\{a_{F\alpha}, a^*_{F\alpha'}\}$ and $\{b_{B\beta}, b^*_{B\beta'}\}$ of stationary quantum mechanical probability amplitudes (as functions of $\vec{k}$) for the sets $\{f_{Q^F} \equiv f^F\}$ and $\{f_{Q^B} \equiv f^B\}$ of solutions to SF equation (47) play part of the generalized field coordinates in the Hamilton approach for a quantum spinor field. Passing to the occupation number representation $N(\vec{k}_1, \vec{k}_2 \ldots)$ these sets of amplitudes will transform to corresponding operator functions $\hat{a}, \hat{a}^\dagger$ and $\hat{b}, \hat{b}^\dagger$, which in the space $\mathcal{H}^{\mathcal{F}}$ satisfy ACR and CCR-relations respectively:

$$\{\hat{a}_{F\alpha}(\vec{k}), \hat{a}^*_{F\alpha'}(\vec{k})'\} = \delta_{\alpha\alpha'}\delta(\vec{k}-\vec{k}'); \ \left[\hat{b}_{B\beta}(\vec{k}), \hat{b}^*_{B\beta'}(\vec{k})'\right] = \delta_{\beta\beta'}\delta(\vec{k}-\vec{k}') \tag{67}$$

(all remaining $\hat{a}$ or $\hat{a}^\dagger$-pairs anticommute, mixed pairs commute). There is a single vacuum in $\mathcal{H}^{\mathcal{F}}$ (as a state with $N \equiv 0$); $0 \neq N$-"particle" F-states fill the antisymmetric sectors in $\mathcal{H}^{\mathcal{F}}$, and $0 \neq N$-"particle" B-states fill the symmetric sectors in $\mathcal{H}^{\mathcal{F}}$.

The possibility for both ACR-, and CCR-quantization of spinor and other fields has been for the first time set forth in Garbaczewski's articles (see, for example, [2,3] and references therein). We have presented above the outline of the "secondary" F- and B-quantization to stress the following: only applying the group analysis (at least for $\mathcal{P}^\uparrow_+$-group) of the spinor field theory, involving the extended 29-dimensional algebra ERCD $\rightsquigarrow$ SO(8) and its exclusive quantum mechanical representations, and taking into account the role of the stationary complete sets of operators in $H^{3,4}$, it is possible to elucidate unambiguously the sense of FB-dualism for particle-antiparticle dublets and to perform the construction of the quantum spinor field model for such objects, in complete analogy to the "secondary quantization" technique of the nonrelativistic quantum mechanics.

3.4. Brief Analysis of SUSY Partner for e^-e^+-Dublet

Now let us introduce and analyze briefly an another, namely, B-representation of the ERCD $\rightsquigarrow$ SO(8)-algebra, denoted in what follows as BTS and related with an ad hoc Bose-multiplet as partner of a e^-e^+-dublet in the same 4-component state space $S^{3,4}$. The principal physical quantities in the $\mathrm{SO}(8)^{\mathrm{BTS}}$-representation are three sets of SU(2)-spins, and the canonical equation of motion for a B-object. They can be obtained from the corresponding elements of the algebra of the $\mathrm{SO}(8)^{\mathrm{FTS}}$-representation and from the quantum mechanical equation (56) with use of the simplest nonunitary operator:

$$v_1 \doteq \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & c \end{vmatrix} \equiv \mathrm{diag}(1,1,1,c) \Rightarrow v_1^{-1} = v_1. \tag{68}$$

From the explicit form of SO(8) SU(2)-spins $\underset{\sim}{\vec{s}}$ (60), $\underset{\sim}{\vec{s}'}$ (61) and $\underset{\sim}{\vec{s}}+\underset{\sim}{\vec{s}'} = \vec{s}^{\mathrm{TS}}$ (62) by direct calculations we obtain their v_1-transformed images:

$$\tilde{\vec{s}} \doteq \frac{1}{2}\left(\tilde\gamma_{23},\tilde\gamma_{31},\tilde\gamma_{12}\right):\ \tilde\gamma_{23} = \begin{vmatrix} 0 & 0 & 0 & ic \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -ic & 0 & 0 & 0 \end{vmatrix},\ \tilde\gamma_{31} = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & ic \\ -1 & 0 & 0 & 0 \\ 0 & -ic & 0 & 0 \end{vmatrix},$$

$$\tilde\gamma_{12} = \begin{vmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & ic \\ 0 & 0 & -ic & 0 \end{vmatrix}; \tag{69}$$

$$\tilde{\vec{s}}' \doteq \frac{1}{2}\left(\tilde\gamma_{54},\tilde\gamma_{46},\tilde\gamma_{65}\right) \equiv -i\tilde{\vec{s}} \Rightarrow \tilde{\vec{s}}^2 = \tilde{\vec{s}}'^2 \equiv -\frac{1}{2}\left(\frac{1}{2}+1\right)\cdot 1_4 \Rightarrow s = \frac{1}{2}; \tag{70}$$

$$\tilde{\vec{s}} + \tilde{\vec{s}}' \doteq \vec{s}^{\mathrm{TS}} \equiv \begin{vmatrix} \vec{s}_\zeta & 0 \\ 0 & 0 \end{vmatrix} \Rightarrow \underline{s} = (1,0). \tag{71}$$

It is evident, that the operator v_1 (68) changes 2 duplicate sets of FTS-SU(2)-spins with $s=\frac{1}{2}$, however not affecting their sum: $\underset{\sim}{\vec{s}}+\underset{\sim}{\vec{s}'} = \tilde{\vec{s}}+\tilde{\vec{s}}' \equiv \vec{s}^{\mathrm{TS}}$; but the nonunitary operator v_1 changes the Hamiltonian of the quantum mechanical SF equation (56) in the FTS representation:

$$v_1(\underset{\sim}{\gamma}^7\hat\omega \equiv -i\hat\omega)v_1 \equiv (-\tilde{i}\tilde\gamma_0 \equiv -i\Gamma)\hat\omega \doteq H^{FW}_{BTS};\ \Gamma \equiv \mathrm{diag}(1,1,1,-1), \tag{72}$$

for the states of $\mathcal{E}(x) \doteq v_1\underset{\sim}{f}(x)$. This means, that the operator v_1 changes the object, described by the quantum mechanical equation (47) = (65) for FTS states of the e^-e^+-dublet: the equation of motion for BTS states of the $\mathcal{E}(x)$-object has the form

$$(\partial_0 + i\Gamma\hat\omega)\mathcal{E}(x) = 0,\ \mathcal{E}(x) \doteq v_1\underset{\sim}{f}(x) \in S^{3,4}. \tag{73}$$

This also means, that if the function $\underset{\sim}{f}(x) \in S^{3,4}$ describes B-states (73) of the e^-e^+ object, then the function $\mathcal{E}(x)$ describes B-states of an ad hoc boson object $\mathcal{E}(x)$ with $\underline{s} = (1, 0)$, which is a SUSY partner of the e^-e^+-dublet. Naturally, the existence of the time-conserved F-SU(2)-spin $\vec{\tilde{s}}$ with $s = \frac{1}{2}$ in the $\mathrm{SO}(8)^{\mathrm{BTS}}$-algebra leads to the appearance for the B-SUSY-partner of the e^-e^+-dublet of not only BTS solutions $\mathcal{E}^B(x) = v_1 \underset{\sim}{f}^B$, but also $\mathcal{E}^F(x)$–solutions, defined by the F-spin $\vec{\tilde{s}}$ (60). The sets of $\{\mathcal{E}^B\}$- and $\{\mathcal{E}^F\}$-solutions of the Eq. (73) are invariant with respect to corresponding $\tilde{\mathcal{P}}^B$ and $\tilde{\mathcal{P}}^F$-representations of the group $\mathcal{P}$. Therefore, in the complete analogy to the FB-dual e^-e^+-dublet, an ad hoc B-object with the states, described by the FW-ϕ^{BTS} equation (73), is a BF-dual object and SUSY partner of the FB-dual e^-e^+. This concludes the elucidation of the FB-dualism for the spinor field $\phi^{ex} = f$ (for a e^-e^+-dublet) and BF-dualism of its SUSY partner $\tilde{\phi}^{ex} = \mathcal{E}$ (tensor-scalar field) in quantum mechanical FW-ϕ^{ex}-models, which by use of FW-ϕ-operators V^{+ex} can be easily translated into Loc-ψ-models of these SUSY partners.

The nonunitary relation v_1 (68) between corresponding characteristics of FB- and BF-dual SUSY partners implies that the initial rest masses of these partners can arbitrarily differ. A comprehensive treating of an ad hoc boson field $\mathcal{E}(x)$ with $\underline{s} = (1, 0)$ requires a separate publication. The model of the complex 4-component tensor-scalar field $\mathcal{E}(x)$ with an infinitely small mass, which in the FW-ϕ^{BTS} model satisfies the Eq. (73) with $m^2 \to 0$, is an appropriate model for an (asymptotically) free electromagnetic field in terms of field strengths $\vec{E}, \vec{H}$ (rather than potentials). The in- and out states of such a field $\mathcal{E}(x)$ will give all experimentally observed physical values coinciding with correspondent values of the transversal free electromagnetic field. A detailed treatment of the SUSY partner for the e^+e^- object goes beyond the scope of the present contribution.

4. Summary of Principal Results

In the Sect. 2 we formulate the basics of the simplified Wightman's A-approach for the free spinor field as a model for experimentally observable asymptotic (in- / out-) states of relativistic microobjects—$s = \frac{1}{2}$ particle-antiparticle dublets and their possible SUSY partners. This A-approach is based solely on three physically justified axioms, ensuring experimental verification of its consequences for both Loc-ψ and canonical FW-ϕ models for an arbitrary representation of the algebra of observables. The cruciality of adequate description of experimentally observable free field states is that all field interaction models are actually formulated in terms of corresponding constructions of free fields.

We use the fact, that the *singled-out character of the time variable* $t = x^0$ as opposed to 3-coordinate $\vec{x} \in R^3 \subset \mathrm{M}(1, 3)$ does not contradict the relativistic (Poincaré) invariance in any representation, required by the SRT principles. Therefore as a state space of the spinor field (and its SUSY partners) we consider the *Schwartz test functions space* $S^{3,4} = S(R^3) \times \mathcal{C}^4$, as opposed to Schwartz generalized functions space $S^{3,4*}$. As the consequence of this axiom, the space $S^{3,4}$ appears to be a *common domain and codomain space* of all involved operators (including operators of the equation of motion in its various forms). In particular, mathematically correct is the use of the pseudodifferential operator $\sqrt{-\Delta + m^2} \doteq \hat{\omega}$ in its combinations, including the operators $V^{\mp}$ (23) of the FW transformations of physical quantities in Loc-ψ and FW-ϕ models in arbitrary representations of Dirac matrices γ^{μ}, which ensures the mathematically correct detalization of the A-approach in these models. Taking into account widespread variants of the FW approach, applied to many particular problems of physics and chemistry (see, e.g., [28–30] and references therein), the presented simplified A-approach can be useful for ensuring the mathematical accuracy of solving these problems in FW settings.

The analysis [12,21] of limitations of both Loc-ϕ-, and FW-ϕ models for a free spinor field is supplemented by additional issues. In particular, we point at the fact, that the operator of the SU(2) spin $\vec{s} = \frac{1}{2}(\vec{\Sigma} \doteq i(\gamma_{23}, \gamma_{31}, \gamma_{12})$ (28) for a e^-e^+-dublet in the PD representation does not reflect the mirror reflectivity of the dublet particles by the charge sign and chirality. Moreover, the detailed analysis of different representations of the algebras of internal variables of a dublet shows, that the principal drawback of spinor fields in both Loc-ψ- and FW-ϕ models, the undefiniteness of the sign of the field energy $p_0[\psi] = P_0^{\mathrm{FW}}[\psi] \lesseqgtr 0$ (29), is related to the nonphysicality of commonly used representations of the algebra of observables.

In Sect. 3.2, starting from the commonly used Pauli–Dirac (PD) representation for the standard CD-algebra, we derive the transparent technique of constructing the 29-dimensional ERCD-algebra in the form of representation of the SO(8) algebra in $\mathcal{C}^4$ (numbers of 28 nontrivial ζ-generators of the SO(8) algebra in $\mathcal{C}^4$ are listed in Table 2).

From ERCD $\rightsquigarrow$ $\mathrm{SO}(8)^{\mathrm{PD}}$ in the PD representation using nonsingular operators of the similarity transformation of $\mathrm{SO}(8)^{\mathrm{PD}}$ we obtain the quantum mechanical (QPD) representation of this algebra—in a definite sense the simplest representation, different from the PD form, which nevertheless ensures the positive sign of the energy P_0. Besides that it also eliminates the mentioned shortcoming of the representation of the spin for a e^-e^+-dublet (see (51)). The extension of the algebra makes evident the appearance of two additional SU(2)-spins $\vec{s}'$ (52) and $\vec{s} + \vec{s'}$ with mutually commuting components. Introducing a novel representation (FTS, Sect. 3.3.2), diagonalizing the square of the independent spin $(\vec{s} + \vec{s'})^2$, shows, that this spin describes the bosonic compound states of a e^-e^+-dublet, thus pointing at intrinsic FB-duality of this object. So, the $\mathcal{P}^{\mathrm{F,B}}$-invariant sets of fermionic $\{f^F\}$ and bosonic $\{f^B\}$ solutions of the equation of motion (65) = (47) for the dublet have equal status of F- and B-states of a e^-e^+-dublet in the relativistic quantum mechanical space $H^{3,4} = L_2(R^3) \times \mathcal{C}^4$. This fact is a transparent manifestation of the FB-dualism for the spinor field illustrated by the e^-e^+-dublet. It is seen apparently: if we choose a full set in $H^{3,4}$ as momentum, B-chirality and charge—

$$Q^B \doteq \left(\vec{p} = -\nabla, h^B = \vec{s}^{\mathrm{TS}}\vec{p}/|\vec{p}|, g = i\gamma_0\right),$$

where B is the spin $\vec{s}^{\mathrm{TS}}$ (62), then B-states, defined by 4-component state vectors f^B, are free (asymptotic) states of the positronium, which are analogous to the states of the $\pi_0^{\pm}$-triplet, complemented with a scalar ρ-meson. At the end of Sect. 3.3 we revise briefly the technique of construction of the A-approach for a quantum spinor field, identical to the "second quantization" technique in the nonrelativistic quantum mechanics of identical states. Finally, in Sect. 3.4 we briefly analyze the BTS representation, obtained via application of a nonunitary nonsingular operator v_1 (68) from the FTS representation and describing B-states of an ad hoc 4-component tensor-scalar field as a B-partner of the e^-e^+-dublet. This object, of course, possesses the fermionic states of equal status, which reflects its BF-dualism.

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Received: April 16, 2016.
Accepted: August 6, 2016.