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CONSISTENCY OF THE LEAST SQUARES ESTIMATES OF TRIGONOMETRIC REGRESSION MODEL PARAMETERS IN THE PRESENCE OF LINEAR RANDOM NOISE

This contribution is dedicated to the 80th anniversary of Professor Yuriy Vasyliovych Kozachenko.

Regression analysis is a huge part of mathematical and applied statistics. Nonlinear regression analysis is a significant extension and complication of classical linear regression analysis, due to the use of nonlinear or partially nonlinear in parameters models that describe more adequately than linear model phenomena requiring statistical analysis. A large number of applied problems in the numerical scientific, technical, and humanitarian fields of knowledge give impetus to the development of nonlinear regression analysis.

The task of estimation the vector signal parameter in the «signal + noise» observation models is a well-known problem of statistics of stochastic processes, and in the case of a nonlinear signal parameter is the problem of nonlinear regression analysis.

Among the variety of nonlinear regression analysis problems the problem of estimating amplitudes and angular frequencies of the sum of harmonic oscillations that are observed against the background of a random noise, takes significant place due to its numerous applications. Statistical model of such a type is said to be trigonometric regression, and the problem of statistical estimation is called the problem of detecting hidden periodicities.

The paper is devoted to the study of time continuous trigonometric regression model where the random noise is a linear Lévy driven stationary of the fourth order stochastic process with zero mean, integrable and square integrable impulse transmission function. This assumption leads to the integrability of the noise covariance function and cumulant function of the fourth order.

To estimate unknown amplitudes and angular frequencies of such a trigonometric model we use the least squares estimators in the Walker sense, that is special parametric set are considered to distinguish properly different angular frequencies in the sum of harmonic oscillations.

Theorem on strong consistency of the least squares estimators is proved in the paper under the assumption on the random noise described above.

To obtain such a result a very important lemma was proved on the uniform tending to zero almost surely of the average value of Lévy-driven linear stochastic process Fourier transform.

This Lemma is the main tool of the strong consistency Theorem proof. To prove the Theorem we, firstly, find some expressions for the least squares estimates of amplitudes via corresponding estimates of angular frequencies. Secondly, we substitute these formulas into the functional of the least squares method. The last step of the proof consists of the L_2-norm transformation of the difference between empirical trigonometric regression function and true regression function such that this norm tends to zero almost surely if and on if the estimators are strongly consistent.

Keywords: The detection of hidden periodicities, least squares estimator, consistency, Lévy-driven linear stochastic process.

1. Introduction. Let a stochastic process

$$X(t) = g(t, \theta^0) + \varepsilon(t), \qquad t \in [0, T], \tag{1}$$

be observed, where

$$g(t, \theta^0) = \sum_{k=1}^{N} (A_k^0 cos\varphi_k^0 t + B_k^0 sin\varphi_k^0 t), \tag{2}$$

$$\theta^0 = (\theta_1^0, \theta_2^0, \theta^0, \ldots, \theta_{3N-2}^0, \theta_{3N-1}^0, \theta_{3N}^0) = (A_1^0, B_1^0, \varphi_1^0, \ldots, A_N^0, B_N^0, \varphi_N^0), \tag{3}$$

$(A_K^0)^2 + (B_K^0)^2 > 0, k = \overline{1, N}$; here $\varepsilon(t), t \in \mathbb{R}^1$, is a stochastic process defined on a complete probability space $(\Omega, \mathcal{F}, P)$ and satisfying the condition introduced below.

The statistical estimation of unknown amplitudes and angular frequencies (3) of a sum of harmonic oscillations (2) observed in a random noise $\varepsilon(t)$ is a probabilistic setting of the hidden periodicities detection problem. Investigations of this problem as well as of its deterministic counterpart $\varepsilon(t) \equiv 0$ are initiated by Lagrange. Many applications of this problem in numerous scientific fields up to the middle of the 20-ies century are described in [1]. More recent applied aspects of the problem of detecting hidden periodicities are considered, for example, in the review [2] and monograph [3].

The literature on this problem is quite extensive. We mention only a few publications [4–9], where the consistency and asymptotic normality are studied for various statistical estimators of unknown amplitudes and angular frequencies under different assumptions concerning the stationary random noise $\varepsilon(t)$ in the model of observation (1), (2) with $N \geq 1$. Both cases of discrete and continuous time are studied in those papers.

To introduce the conditions on the stochastic process ε in (1) we need in some preliminary remarks [10].

A Lévy process $L(t), t \geq 0$, is a stochastic process, with independent and stationary increments, continuous in probability, with sample-paths which are right-continuous with left limits (*cádlág*) and $L(t) = 0$.

Let (a, b, Π) denote a characteristic triple of the Lévy process $L(t), t \geq 0$, that is for all $t \geq 0$

$$\log E \exp\{izL(t)\} = tk(z)$$

for all $z \in \mathbb{R}$, where

$$k(z) = iaz - \frac{1}{2}bz^2 + \int_{\mathbb{R}} \left(e^{izu} - 1 - iz\tau(u)\right) \Pi(du), z \in \mathbb{R}, \tag{4}$$

where $a \in \mathbb{R}, b \geq 0$, and

$$\tau(u) = \begin{cases} u, & |u| \leq 1; \\ \frac{u}{|u|}, & |u| > 1. \end{cases}$$

The Lévy measure Π in (4) is a Radon measure on $\mathbb{R}\backslash\{0\}$, such that $\Pi(\{0\}) = 0$, and

$$\int_{\mathbb{R}} \min(1, u^2)\,\Pi(du) < \infty.$$

It is known that $L(t)$ has finite p-th moment for $p > 0$ ($E|L(t)|^p < \infty$), if and only if

$$\int_{|u|\geq 1} |u|^p\,\Pi(du) < \infty, \tag{5}$$

see, i.e., Sato [11], Theorem 25.3.

If $L(t), t \geq 0$ is a Lévy process with characteristics (a, b, Π), then process $-L(t)$, $t \geq 0$ is also a Lévy process with characteristics $(-a, b, \breve{\Pi})$, where $\breve{\Pi}(A) = \Pi(-A)$ for each Borel set A, modifying it to be *cáglád* [12].

We introduce a two-sided Lévy process $L(t), t \in \mathbb{R}$, defined for $t < 0$ to be equal to independent copy of $-L(-t)$.

Let $\hat{a} : \mathbb{R} \to \mathbb{R}_+$ be a measurable function. We consider the Lévy-driven continuous time linear (or moving average) stochastic process

$$\varepsilon(t) = \int_{\mathbb{R}} \hat{a}(t - s)\,dL(s), t \in \mathbb{R}. \tag{6}$$

For causal process (6) $\hat{a} = 0, t < 0$.

In the sequel we assume that

$$\hat{a} \in L_1(\mathbb{R}) \cap L_2(\mathbb{R}),\ EL(1) = 0. \tag{7}$$

Under the condition (7) and

$$\int_{\mathbb{R}} u^2\,\Pi(du) < \infty,$$

the stochastic integral in (6) is well-defined in $L_2(\Omega)$ in the sense of stochastic integration introduced in Rajput and Rosinski [13].

The popular choices for the kernel in (6) are Gamma type kernels:

1. $\hat{a}(t) = t^{\alpha} e^{-\lambda t} \mathbb{I}_{[0,\infty)}(t), \lambda > 0, \alpha > -\frac{1}{2}$;
2. $\hat{a} = e^{-\lambda t} \mathbb{I}_{[0,\infty)}(t), \lambda > 0$ (Ornstein-Uhlenbeck process);
3. $\hat{a} = e^{-\lambda t}, \lambda > 0$ (well-balanced Ornstein-Uhlenbeck process).

A. The process ε in (1) is a measurable causal linear process of the form (6), where a two-sides Lévy process L and $\hat{a}$ satisfy (7). Moreover the Łévy measure Π of $L(1)$ satisfies (5) for p=4.

From the condition ***A*** it follows [12] for $r = 1, 2, 3, 4$

$$\log E \exp\{i \sum_{j=1}^{r} z_j \varepsilon(t_j)\} = \int_{\mathbb{R}} k\left(\sum_{j=1}^{r} z_j \hat{a}(t_j - s)\right) ds. \tag{8}$$

In turn from (8) it can be seen that the stochastic process ε is stationary of the 4th order.

Denote by

$$m_r(t_1,\dots,t_r)=E\varepsilon(t_1)\dots\varepsilon(t_r),$$
$$c_r(t_1,\dots,t_r)=i^{-r}\frac{\partial^r}{\partial z_1\dots\partial z_r}\log E\exp\{i\sum_{j=1}^{r}z_j\varepsilon(t_j)\}|_{z_1=\dots=z_r=0}$$

the moment and cumulant functions correspondingly of order r of the process ε. Thus $m_2(t_1,t_2)=B(t_1-t_2)$, where

$$B(t)=d_2\int_{\mathbb{R}}\hat{a}(t-s)\hat{a}(s)\,ds, t\in\mathbb{R},$$

is a covariance function of ε, and the 4th moment function

$$m_4(t_1,t_2,t_3,t_4)=c_4(t_1,t_2,t_3,t_4)+m_2(t_1,t_2)m_2(t_3,t_4)+ \\ +m_2(t_1,t_3)m_2(t_2,t_4)+m_2(t_1,t_4)m_2(t_2,t_4). \quad (9)$$

The explicit expression for cumulants of the stochastic process ε can be obtained from (8) by direct calculations:

$$c_r(t_1,\dots,t_r)=d_r\int_{\mathbb{R}}\prod_{j=1}^{r}\hat{a}(t_j-s)\ ds, \quad (10)$$

where d_r is the r-th comulant of the random variable $L(1)$. In particular,

$$d_2=EL^2(1)=-k^{(0)}, \qquad d_4=EL^4(1)-3(EL^2(1))^2.$$

2. Setting of the problem. The consistency of the least squares estimator (LSE) of the parameter θ^0 in the model (1) –(3) is studied in the paper under condition $\boldsymbol{A}$.

We arrange the frequencies $\varphi^0=(\varphi_1^0,\dots,\varphi_N^0)$ in ascending order. In other words, we assume that the parametric set where we search an estimator of unknown angular frequencies is of the following form:

$$\Phi(\underline{\varphi},\overline{\varphi})=\{\varphi=(\varphi_1,\dots,\varphi_N)\in\mathbb{R}^N:0\le\underline{\varphi}<\varphi_1<\dots<\varphi_N<\overline{\varphi}<+\infty\}.$$

Let

$$Q_T(\theta)=T^{-1}\int_0^T[X(t)-g(t,\theta)]^2\ dt. \quad (11)$$

According to the standard definition, the LSE of the parameter θ^0 constructed from observation $\{X(t),t\in[0,T]\}$ is any random vector

$$\theta_T=(A_{1T},B_{1T},\varphi_{1T},\dots,A_{NT},B_{NT},\varphi_{NT}) \quad (12)$$

that minimizes the functional $Q_T(\theta)$ in the set of parameters such that $\Theta\subset\mathbb{R}^{3N}$ the amplitudes $A_k,B_k,k=\overline{1,N}$, can take arbitrary values in Θ and $\varphi\in\Phi^c$, where Φ^c is the closure of the set $\Phi(\underline{\varphi},\overline{\varphi})$.

When proving the consistency of the estimator θ_T (see theorem below) we face the problem of studying the behavior, as $T \to \infty$, of the ratios

$$\frac{\sin T(\varphi_{kT} - \varphi_{jT})}{T(\varphi_{kT} - \varphi_{jT})}, \quad \frac{\sin T(\varphi_{kT} - \varphi_j^0)}{T(\varphi_{kT} - \varphi_j^0)}, \quad k \neq j, \quad \frac{\sin T\varphi_{kT}}{T\varphi_{kT}}, \quad k = \overline{1, N}. \tag{13}$$

However the above definition of the estimator $\varphi_T = (\varphi_{1T}, \ldots, \varphi_{NT})$ makes it impossible to determine the behavior of the differences $\varphi_{kT} - \varphi_{jT}$ and $\varphi_{kT} - \varphi_j^0$, $j \neq k$, as $T \to \infty$. Therefore the question of the behavior of ratios (13) remains open.

Walker [5] proposed a modification of the definition of the estimator φ^0 which guarantees the convergence to zero of the ratios (13). In turn, this implies the consistency of the LSE.

The Walker idea is to define the estimator (12) as a point of minimum of the functional (11) in a set where one can well distinguish the parameters φ_k.

Consider a nondecreasing family of open sets

$$\Phi_T \subset \Phi(\underline{\varphi}, \overline{\varphi}), \qquad T \geq T_0 > 0.$$

We assume that these sets contain the true value of the parameter φ^0 and satisfy the following conditions:

$$\lim_{T\to\infty} \inf_{1\leq j<k\leq N} \inf_{\varphi\in\Phi_T} T(\varphi_k - \varphi_j) = +\infty, \tag{14}$$

$$\lim_{T\to\infty} \inf_{\varphi\in\Phi_T} T\varphi_1 = +\infty. \tag{15}$$

In view of the above remark, we say that a vector θ_T is the LSE, if θ_T is a point of minimum of the functional $Q_T(\theta)$ in the set Θ_T for which (in contrast with Θ) $\varphi \in \Phi_T^c$.

Condition (15) obviously holds if $\underline{\varphi} > 0$. If $\Phi_T \subset \Phi(0, \overline{\varphi})$, then one can consider, for example, parametric sets such that

$$\inf_{1\leq j<k\leq N} \inf_{\varphi\in\Phi_T} T(\varphi_k - \varphi_j) = T^{-1/2}, \qquad \inf_{\varphi\in\Phi_T} T\varphi_1 = T^{-1/2}$$

in order to satisfy (14), (15).

Theorem 1. *Let assumption **A** holds. Then LSE θ_T is a strongly consistent estimator of the parameter θ^0, namely:*

$$A_{kT} \to A_k^0, B_{kT} \to B_k^0, T(\varphi_{kT} - \varphi_k^0) \to 0 \qquad a.s., \tag{16}$$

as $T \to \infty, k = \overline{1, N}$.

3. Lemma. The next lemma is the main part of the convergence (16) proof.

Lemma 1. *Under condition **A***

$$\xi(T) = \sup_{\lambda\in\mathbb{R}} \left| T^{-1} \int_0^T e^{-i\lambda t} \varepsilon(t)\, dt \right| \to 0 \quad a.s., \quad as \quad T \to \infty. \tag{17}$$

Proof. Since

$$\left|\int_0^T e^{-i\lambda t}\varepsilon(t)\ dt\right|^2 = \int_{-T}^{T} e^{-i\lambda t} \int_0^{T-|u|} \varepsilon(t+|u|)\varepsilon(t)\ dt\ du =$$

$$= 2\int_0^T \cos\lambda u \int_0^{T-u} \varepsilon(t+u)\varepsilon(t)\ dt\ du,$$

then

$$E\xi^2(t) \leq 2T^{-2}\int_0^T E\left|\int_0^{T-u} \varepsilon(t+u)\varepsilon(t)dt\right| du \leq 2T^{-2}\int_0^T K^{1frm-e}(u)du.$$

By formula (9)

$$K(u) = \int_0^{T-u}\int_0^{T-u} E\varepsilon(t+u)\varepsilon(s+u)\varepsilon(t)\varepsilon(s)dtds =$$

$$= \int_0^{T-u}\int_0^{T-u} c_4(t+u, s+u, t, s)dtds + (T-u)^2B^2(u)+$$

$$+\int_0^{T-u}\int_0^{T-u} B^2(t-s)dtds + \int_0^{T-u}\int_0^{T-u} B(t-s+u)B(t-s-u)dtds \leq$$

$$\leq K_1(u) + K_2(u) + K_3(u) + |K_4(u)|,$$

and

$$E\xi^2(T) \leq 2T^{-2}\int_0^T (K_1^{\frac{1}{2}}(u) + K_2^{\frac{1}{2}}(u) + K_3^{\frac{1}{2}}(u) + |K_4(u)|^{\frac{1}{2}})du. \qquad (18)$$

Accordind to (10)

$$K_1(u) = d_4\int_{\mathbb{R}}\left(\int_0^{T-u} \hat{a}(t+u-r)\hat{a}(t-r)dt\right)^2 dr \leq$$

$$\leq d_4\int_{\mathbb{R}}\left(\int_0^{T-u} \hat{a}^2(t+u-r)\int_0^{T-u} \hat{a}^2(t-r)dt\right) dr \leq$$

$$\leq d_4\|\hat{a}\|_2^2\int_0^{T-u} dt\int_{\mathbb{R}} \hat{a}^2(t+u-r)dr \leq d_4\|\hat{a}\|_4^2(T-u),$$

that is

$$T^{-2}\int_0^T K_1^{\frac{1}{2}}(u)du \leq d_4^{\frac{1}{2}}\|\hat{a}\|_2^2T^{-2}\int_0^T \sqrt{T-u}du = \frac{2}{3}d_4^{\frac{1}{2}}\|\hat{a}\|_2^2T^{-\frac{1}{2}}. \qquad (19)$$

From condition ***A*** it follows $\|B\|_1 = \int_{\mathbb{R}}|B(t)|dt < \infty$. Then

$$T^{-2}\int_0^T K_2^{\frac{1}{2}}(u)du = T^{-2}\int_0^T (T-u)|B(u)|du \leq \|B\|_1T^{-1}. \qquad (20)$$

Moreover,

$$K_3(u) \leq B(0) \int_0^{T-u} \int_0^{T-u} |B(t-s)| dt ds \leq B(0) \|B\|_1 (T-u),$$

$$T^{-2} \int_0^T K_3^{\frac{1}{2}}(u) du \leq \frac{2}{3} B^{\frac{1}{2}}(0) \|B\|_1^{\frac{1}{2}} T^{-\frac{1}{2}}. \tag{21}$$

Similarly,

$$T^{-2} \int_0^T K_4^{\frac{1}{2}}(u) du \leq \frac{2}{3} B^{\frac{1}{2}}(0) \|B\|_1^{\frac{1}{2}} T^{-\frac{1}{2}}. \tag{22}$$

From the inequalities (18) – (22) we get $E\xi^2(T) = O(T^{-\frac{1}{2}})$, as $T \to \infty$.
Let $T_n = n^{2+\delta}$ for some $\delta > 0$. Then

$$\sum_{n=1}^{\infty} E\xi^2(T_n) < \infty,$$

that is

$$\xi(T_n) \to 0 \ a.s., as \ n \to \infty.$$

Consider a sequence of random variables

$$\zeta_n = \sup_{T_n \leq T < T_{n+1}} |\xi(T) - \xi(T_n)| =$$

$$= \sup_{T_n \leq T < T_{n+1}} \left| \sup_{\lambda \in \mathbb{R}^1} \left| \frac{1}{T} \int_0^T e^{-i\lambda t} \varepsilon(t) dt \right| - \sup_{\lambda \in \mathbb{R}^1} \left| \frac{1}{T_n} \int_0^{T_n} e^{-i\lambda t} \varepsilon(t) dt \right| \right| \leq$$

$$\leq \sup_{T_n \leq T < T_{n+1}} \sup_{\lambda \in \mathbb{R}^1} \left| \frac{1}{T} \int_0^T e^{-i\lambda t} \varepsilon(t) dt - \frac{1}{T_n} \int_0^{T_n} e^{-i\lambda t} \varepsilon(t) dt \right| \leq$$

$$\leq \sup_{T_n \leq T < T_{n+1}} \left[\sup_{\lambda \in \mathbb{R}^1} \left| \left(\frac{1}{T} - \frac{1}{T_n} \right) \int_0^{T_n} e^{-i\lambda t} \varepsilon(t) dt \right| + \sup_{\lambda \in \mathbb{R}^1} \left| \frac{1}{T} \int_{T_n}^T e^{-i\lambda t} \varepsilon(t) dt \right| \right] \leq$$

$$\leq \frac{T_{n-+1} - T_n}{T_n} \xi(T_n) + \frac{1}{T_n} \int_{T_n}^{T_{n+1}} |\xi(t)| dt = \zeta_n^{(1)} + \zeta_n^{(2)}.$$

It is clear that $\zeta_n^{(1)} \to 0$ a.s., as $n \to \infty$.
Consider

$$E(\zeta_n^{(2)})^2 = \frac{1}{T_n^2} \int_{T_n}^{T_{n+1}} \int_{T_n}^{T_{n+1}} E|\xi(t_1)\xi(t_2)| dt_1 dt_2 \leq$$

$$\leq B(0) \left(\frac{T_{n-+1} - T_n}{T_n} \right)^2 = O(n^{-2}), \ as \ n \to \infty.$$

Thus, $\zeta_n^{(2)} \to 0$ a.s., as $n \to \infty$, also.

4. The proof of the Theorem. The proof of the theorem uses the ideas of the paper [8].

Let

$$x_{kT} = \frac{\sin T(\varphi_{kT} - \varphi_k^0)}{T(\varphi_{kT} - \varphi_k^0))}, \qquad y_{kT} = \frac{1 - \cos T(\varphi_{kT} - \varphi_k^0))}{T(\varphi_{kT} - \varphi_k^0)}.$$

We show that

$$A_{kT} = A_k^0 x_{kT} - B_k^0 y_{kT} + o(1), \qquad B_{kT} = A_k^0 y_{kT} - B_k^0 x_{kT} + o(1), \tag{23}$$

for $k = \overline{1, N}$, where $o(1)$ denotes, generally speaking, different stochastic processes approaching zero a.s., as $T \to \infty$.

Differentiating the functional $Q_T(\theta)$ with respect to the variables $A_1, \ldots, A_N$ and $B_1, \ldots, B_N$ we obtain the following system of linear equations for the LSE A_{kT} and B_{kT}, $k = \overline{1, N}$:

$$\begin{cases} \sum_{k=1}^N a_{kj}^{(1)}(T) A_{kT} + \sum_{k=1}^N b_{kj}^{(1)}(T) B_{kT} = c_j^{(1)}(T), & j = \overline{1, N}, \\ \sum_{k=1}^N a_{kj}^{(2)}(T) A_{kT} + \sum_{k=1}^N b_{kj}^{(2)}(T) B_{kT} = c_j^{(2)}(T), & j = \overline{1, N}, \end{cases} \tag{24}$$

where we used the notation

$$\langle u(t), v(t) \rangle = T^{-1} \int_0^T u(t) v(t) dt,$$

$$a_{kj}^{(1)}(T) = \langle \cos \varphi_{kT} t, \cos \varphi_{jT} t \rangle, \qquad a_{kj}^{(2)}(T) = \langle \cos \varphi_{kT} t, \sin \varphi_{jT} t \rangle,$$

$$b_{kj}^{(1)}(T) = \langle \sin \varphi_{kT} t, \cos \varphi_{jT} t \rangle, \qquad b_{kj}^{(2)}(T) = \langle \sin \varphi_{kT} t, \sin \varphi_{jT} t \rangle,$$

$$c_j^{(1)}(T) = \langle X(t), \cos \varphi_{jT} t \rangle, \qquad c_j^{(2)}(T) = \langle X(t), \sin \varphi_{jT} t \rangle,$$

$$k, j = \overline{1, N}.$$

Considering the properties (14) and (15) of the parametric set Φ_T (whose closure contains the value of the LSE $\varphi_T = (\varphi_{1T}, \ldots, \varphi_{NT})$) we derive the following relations:

$$a_{kj}^{(1)}(T) = o(1), \qquad k \neq j,$$

$$a_{kk}^{(1)}(T) = \frac{1}{2} + o(1), \qquad a_{kj}^{(2)}(T) = o(1), \qquad k, j = \overline{1, N}, \tag{25}$$

and

$$b_{kj}^{(1)}(T) = a_{kj}^{(2)}(T) = o(1),$$

$$b_{kj}^{(2)}(T) = o(1), \qquad k \neq j, \qquad b_{kk}^{(2)}(T) = \frac{1}{2} + o(1), \qquad k, j = \overline{1, N}. \tag{26}$$

Further,

$$c_j^{(1)}(T) = \langle \varepsilon(t), \cos \varphi_{jT} t \rangle + \langle g(t, \theta^0) m \cos \varphi_{jT} t \rangle = d_j^{(1)}(T) + d_j^{(2)}(T),$$

and moreover, $d_j^{(1)}(T) = o(1)$ by Lemma. Then

$$d_j^{(2)}(T) = A_j^0\langle\cos\varphi_j^0 t, \cos\varphi_{jT}t\rangle + B_j^0\langle\sin\varphi_j^0 t, \cos\varphi_{jT}t\rangle + o(1) \tag{27}$$

$$= \frac{1}{2}[A_j^0 y_{jT} - B_j^0 x_{jT}] + o(1), \qquad j = \overline{1, N}.$$

Similarly

$$c_j^{(2)}(T) = \frac{1}{2}[A_j^0 y_{jT} + B_j^0 x_{jT}] + o(1), \qquad j = \overline{1, N}. \tag{28}$$

Now relations (23) follow from (24) –(28).

Since $|x_{kT}|, |y_{kT}| \leq 1$, relations (23) imply that

$$|A_{kT}|, |B_{kT}| \leq |A_{kT}| + |B_{kT}| + o(1), \qquad k = \overline{1, N}. \tag{29}$$

Let $\mathit{\Delta} g(t;\theta_1,\theta_2) = g(t;\theta_1) - g(t;\theta_2)$ and $G_T(\theta_1,\theta_2) = \langle \mathit{\Delta} g(t;\theta_1,\theta_2), \mathit{\Delta} g(t;\theta_1,\theta_2)\rangle$. By the definition of the LSE

$$Q_T(\theta_T) \leq Q_T(\theta^0). \tag{30}$$

On the other hand,

$$Q_T(\theta_T) - Q_T(\theta^0) = G_T(\theta_T, \theta^0) + 2\langle\varepsilon(t), \mathit{\Delta} g(t;\theta^0,\theta_T)\rangle, \tag{31}$$

where

$$\langle\varepsilon(t), \mathit{\Delta} g(t;\theta^0,\theta_T)\rangle = o(1) \tag{32}$$

in view of Lemma and bounds (29). Taking into account inequality (30), we obtain from (31) and (32) that

$$G_T(\theta_T, \theta^0) \to 0 \quad a.s., \quad as \quad T \to \infty. \tag{33}$$

Put

$$g_{kT}(t) = A_{kT}\cos\varphi_{kT}t + B_{kT}\sin\varphi_{kT}t - A_k^0\cos\varphi_k^0 t - B_k^0\sin\varphi_k^0 t.$$

Then

$$G_T(\theta_T, \theta^0) = \sum_{k=1}^{N}\langle g_{kT}(T), g_{kT}(T)\rangle + 2\sum_{k<j}\langle g_{kT}(T), g_{jT}(T)\rangle.$$

Using the above reasoning and bounds (29), we find that

$$\langle g_{kT}(t), g_{jT}(t)\rangle = o(1), \qquad k \neq j, \tag{34}$$

$$\langle g_{kT}(t), g_{kT}(t)\rangle = \frac{1}{2}[A_{kT}^2 + B_{kT}^2 + (A_k^0)^2 + (B_k^0)^2] - (A_{kT}A_k^0 + B_{kT}B_k^0)x_{kT} + \tag{35}$$

$$+(A_{kT}B_k^0 - A_k^0 B_{kT})y_{kT} + o(1), \qquad k = \overline{1, N}.$$

Substituting relations (23) into (35) and considering (34), we deduce that

$$G_T(\theta_T, \theta^0) = \frac{1}{2}\sum_{k=1}^{N}((A_k^0)^2 + (B_k^0)^2)(1 - x_{kT}^2 - y_{kT}^2) + o(1)$$

$$= \frac{1}{2}\sum_{k=1}^{N}((A_k^0)^2 + (B_k^0)^2)\left(1 - \left(\frac{\sin\frac{1}{2}T(\varphi_{kT} - \varphi_k^0)}{\frac{1}{2}T(\varphi_{kT} - \varphi_k^0}\right)^2\right) + o(1).$$

Thus relation (33) holds if and only if

$$T(\varphi_{kT} - \varphi_k^0) \to 0 \quad a.s., \quad as \quad T \to \infty,\ k = \overline{1,N}. \tag{36}$$

Relation (36) obviously imply that

$$x_{kT} \to 1, \quad y_{kT} \to 0 \quad a.s., \quad as \quad T \to \infty,\ k = \overline{1,N}.$$

The strong consistency of the estimators A_{kT} and B_{kT} follows from equalities (23).

5. Conclusions. In the paper strong consistency of the least squares estimators in Walker sense of the trigonometric regression parameters provided that the parametric set contains the true value of the vector parameter and in which the least squares estimate is sought, separates the angular frequencies in some way, and random noise is a Lévy-driven stochastic process with zero mean that satisfies some regularity conditions. An important feature of this work is rejection of any assumptions related to the Gaussianity of the random noise.

The natural direction of further research is to obtain conditions for the asymptotic normality of the least squares estimators in trigonometric regression model under our assumptions about random noise. A very important task is also to obtain the properties of strong consistency and asymptotic normality of periodogram estimates of the parameters in the model considered in the paper.

References

1. Serebrennikov, M. G., & Pervozvanskii, A. A. (1965). Vyiyavlenie skryityih periodichnostey [The detection of hidden periodicities]. *Mosow:Nauka.* [in Russian]
2. Artis, M., Hoffman, M., Nachane, D., & Toro J. (2004). The detection of hidden periodicities: a comparison of alternative methods. *EUI Working paper ECO, 10*, 26.
3. Quinn, B. G., & Hannan, E. J. (2001). The estimation and tracking of frequency. *Cambridge Univ. Press.*
4. Whitle, P. (1952). On the estimation of a time series harmonic components and covariance structure. *Trabajos Estadistica, 3*, 43–57.
5. Walker, A. M. (1973). On the estimation of a harmonic component in a time series with stationary dependent residuals. *Adv. Appl. Probab., 5, 2*, 217–241.
6. Hannan, E. J. (1973). The estimation of frequency. *J. Appl. Probab., 10, 3*, 510–519.
7. Ivanov, A. V. (1980). A solution of the problem of detecting hidden periodicities. *Theor. Probab. Math. Statist., 20*, 51–68.
8. Ivanov, A. V. (2010). Consistency of the least squares estimator of the amplitudes and angular frequencies of a sum of harmonic oscillations i models with long-range dependence. *Theor. Probab. Math. Statist., 80*, 61–69.
9. Ivanov, A. V., Leonenko, N. N., Ruiz-Medina, M. D., & Zhurakovsky, V. M. (2015). Estimation of harmonic component in regression with cyclically dependent errors. *Statistics: A Journal of Theoretical and Applied Statistics, 49, 1*, 156–186.
10. Ivanov, A. V., Leonenko, N. N., & Orlovsky, I. V. (2020). On the Whittle estimation for linear random noise spectral density parameter in continuous-time nonlinear regression models. *Stat. Inference Stoch. Process, 23*, 129–169.

11. Sato, K. (1999). Lévy processes and infinitely divisible distributions. *Cambrige: Cambridge University Press.*
12. Anh, V. V., Heyde, C. C., & Leonenko, N. N. (2002). Dynamic models of long-memory processes driven by Lévy nose. *J. Appl. Prob., 39, 4*, 730–747.
13. Rajput, B., & Rosinski, J. (1989). Spectral representations of infinitely divisible processes. *Prob. Theory Rel. Fields, 82*, 451–487.

Іванов О. В., Митрофанова О. В. Консистентність оцінки найменших квадратів параметрів тригонометричної моделі регресії у присутності лінійного випадкового шуму.

Регресійний аналіз є істотною частиною математичної та прикладної статистики. Нелінійний регресійний аналіз є значним розширенням та ускладненням класичного лінійного регресійного аналізу, завдяки використанню нелінійних або частково нелінійних за параметрами моделей, які адекватніше описують, ніж лінійні моделі, явища, що потребують статистичного аналізу. Велика кількість прикладних проблем у численних наукових, технічних та гуманітарних галузях знань дають поштовх розвитку нелінійного регресійного аналізу.

Задача оцінювання векторного параметра сигналу в моделях спостереження «сигнал + шум» є добре відомою проблемою статистики випадкових процесів, та у випадку нелінійного параметра сигналу – задачею нелінійного регресійного аналізу.

Серед різноманітності задач нелінійного регресійного аналізу оцінювання амплітуд та кутових частот суми гармонічних коливань, що спостерігається на фоні випадкового шуму, займає значне місце, завдяки її численним застосуванням. Статистичні моделі такого типу називаються тригонометричними моделями регресії, а проблема статистичного оцінювання її параметрів називається задачею виявлення прихованих періодичностей.

Статтю присвячено вивченню тригонометричної моделі регресії, в якій випадковий шум є лінійним Леві-керованим стаціонарним четвертого порядку випадковим процесом із нульовим середнім, інтегровную та інтегровную з квадратом імпульсною перехідною функцією. Це припущення призводить до інтегровності коваріаційної функції та кумулянтної функції 4-го порядку.

Для оцінювання амплітуд та кутових частот такої тригонометричної моделі ми використовуємо оцінки найменших квадратів у сенсі Уолкера, тобто розглянуто спеціальну множину параметрів, щоб розрізнити належним чином різні кутові частоти в сумі гармонічних коливань.

У статті доведено теорему про сильну консистентність оцінки найменших квадратів за описаними вище припущеннями щодо випадкового шуму.

Для отримання такого результату було доведено дуже важливу лему про рівномірну збіжність майже напевно середнього значення перетворення Фурьє лінійного Леві-керованого випадкового процеса.

Ця лема є головним інструментом доведення теореми про сильну консистентність. Для доведення теореми, по-перше, знаходимо деякі представлення оцінок найменших квадратів амплітуд через відповідні оцінки кутових частот. По-друге, ми підставляємо ці формули у функціонал методу найменших квадратів. Останній крок доведення полягає у перетворенні L_2-норми різниці між емпіричною тригонометричною функцією регресії та істиною функцією регресії таким чином, що ця норма прямує до нуля майже напевно тоді і тільки тоді, коли оцінки є сильно консистентними.

Ключові слова: Виявлення прихованих періодичностей, оцінка найменших квадратів, консистентність, Леві-керований лінійний випадковий процес.

Список використаної літератури

1. Серебренников М. Г., Первозванский А. А. Выявление скрытых периодичностей. Москва: Наука, 1965. 244 с.
2. Arti M., Hoffman M., Nachane D., Toro J. The detection of hidden periodicities: a comparison of alternative methods. EUI Working paper ECO, 10, 2004. 26 p.

3. Quinn B. G., Hannan, E. J. The estimation and tracking of frequency. Cambridge: Cambridge Univ. Press, 2001. 280 p.
4. Whitle P. On the estimation of a time series harmonic components and covariance structure. *Trabajos Estadistica*. 1952. Vol. 3. P. 43–57.
5. Walker A. M. On the estimation of a harmonic component in a time series with stationary dependent residuals. *Adv. Appl. Probab.* 1973. Vol. 5, Iss. 2. 217–241.
6. Hannan E. J. The estimation of frequency. *J. Appl. Probab.*. 1973. Vol. 10, No. 3. P. 510–519.
7. Ivanov A. V. A solution of the problem of detecting hidden periodicities. *Theor. Probab. Math. Statist.* 1980. Vol. 20. P. 51–68.
8. Ivanov A. V. Consistency of the least squares estimator of the amplitudes and angular frequencies of a sum of harmonic oscillations i models with long-range dependence. *Theor. Probab. Math. Statist.* 2010. Vol. 80. P. 61–69.
9. Ivanov A. V., Leonenko N. N., Ruiz-Medina M. D., Zhurakovsky V. M. Estimation of harmonic component in regression with cyclically dependent errors. *Statistics: A Journal of Theoretical and Applied Statistics*. 2015. Vol. 49, Iss. 1. P. 156–186.
10. Ivanov A. V., Leonenko N. N., Orlovsky I. V. On the Whittle estimation for linear random noise spectral density parameter in continuous-time nonlinear regression models. *Stat. Inference Stoch. Process.* 2020. Vol. 23. P. 129–169.
11. Sato K. Lévy processes and infinitely divisible distributions. Cambrige: Cambridge University Press. 1999. 500 p.
12. Anh V. V., Heyde C. C., Leonenko N. N. Dynamic models of long-memory processes driven by Lévy nose. *J. Appl. Prob.* 2002. Vol. 39, No. 4. P. 730–747.
13. Rajput B., Rosinski J. Spectral representations of infinitely divisible processes. *Prob. Theory Rel. Fields.* 1989. Vol. 82. P. 451–487.

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