UDC 512.53, 512.64
DOI https://doi.org/10.24144/2616-7700.2021.38(1).7-15

V. M. Bondarenko[1], M. V. Styopochkina[2]

[1] Institute of Mathematics of NAS of Ukraine,
leading researcher of the department of algebra and topology,
doctor of physical and mathematical sciences
vitalij.bond@gmail.com
ORCID: https://orcid.org/0000-0002-5064-9452

[2] Polissia National University,
associate professor of higher and applied mathematics,
candidate of physical and mathematical sciences
stmar@ukr.net
ORCID: https://orcid.org/0000-0002-7270-9874

ON POSETS OF SIXTH ORDER HAVING OVERSUPERCRITICAL *MM*-TYPE

Representations of posets (partially ordered sets) were introduced by L. A. Nazarova and A. V. Roiter in 1972. In the same year M. M. Kleiner proved that a posets S is of finite representation type if and only if it does not contain subposets of the form $K_1 = (1,1,1,1), K_2 = (2,2,2), K_3 = (1,3,3), K_4 = (1,2,5)$ and $K_5 = (\mathrm{N},4)$. These posets are called critical posets relative to the finiteness of type (in the sense that they are minimal posets with an infinite number, up to equivalence, of indecomposable representations) or the Kleiner's posets. In 1974 Yu. A. Drozd proved that a poset S has finite representation type if and only if its Tits quadratic form

$$q_S(z) =: z_0^2 + \sum_{i\in S} z_i^2 + \sum_{i<j, i,j\in S} z_i z_j - z_0 \sum_{i\in S} z_i$$

is weakly positive (i.e., positive on the set of non-negative vectors). Consequently, the Kleiner's posets are critical relative to weak positivity of the Tits quadratic form, and there are no (up to isomorphism) other such posets. In 2005 the authors proved that a poset is critical relative to the positivity of the Tits quadratic form if and only if it is minimax isomorphic to a Kleiner's poset.

A similar situation takes place for posets of tame representation type. In 1975 L. A. Nazarova proved that a poset S is tame if and only if it does not contain subsets of the form $N_1 = (1,1,1,1,1)$, $N_2 = (1,1,1,2)$, $N_3 = (2,2,3)$, $N_4 = (1,3,4)$, $N_5 = (1,2,6)$ and $(\mathrm{N},5)$. She called these posets supercritical; they are critical relative to weak non-negativity of the Tits quadratic form and there are no other such posets. In 2009 the authors proved that a poset is critical relative to non-negativity of the Tits quadratic form if and only if it is minimax isomorphic to a supercritical poset.

The first author suggested to introduce posets (called oversupercritical), which differ from supercritical sets in the same degree as the latter differ from critical ones. Among these posets, there are four of the smallest order, namely 6. In this article, we describe all posets that are minimax isomorphic to them and study some of their combinatorial properties. The importance of studying minimax isomorphic posets is determined by the fact that their Tits quadratic forms are $\mathbb{Z}$-equivalent, and minimax isomorphism itself is a fairly general constructively defined $\mathbb{Z}$-equivalence of the Tits quadratic forms for posets.

Keywords: representation, critical and supercritical poset, oversupercritical poset, Tits quadratic form, finite and tame representation type, positivity and weak positivity, non-negativity and weak non-negativity.

1. Introduction. M. M. Kleiner [1] proved that a poset S is of finite representation type (i.e. has only a finite number of equivalence classes of indecomposable representations) if and only if it does not contain (full) subposets of the form $K_1 = (1,1,1,1), K_2 = (2,2,2), K_3 = (1,3,3), K_4 = (1,2,5)$ and $K_5 = (\mathrm{N},4)$, which are called *critical posets*. Now they are called (critical) *Kleiner's posets*. On the other hand, Yu. A. Drozd [2] showed that a poset has finite representation type if and only if its Tits quadratic form is weakly positive (i.e., positive on the set of nonnegative vectors). Consequently, the critical posets are also critical relatively to the weak positivity of the Tits quadratic form. In [3] the authors proved that a poset is critical relatively to the positivity of the Tits quadratic form if and only if it is minimax isomorphic to a Kleiner's poset (such isomorphism was introduced by the first author in [4]); in this paper all such posets are fully described (they are named by the authors as P-critical).

A similar situation takes place for tame posets. A poset S is tame if and only if it does not contain subsets of the form $N_1 = (1,1,1,1,1)$, $N_2 = (1,1,1,2)$, $N_3 = (2,2,3)$, $N_4 = (1,3,4)$, $N_5 = (1,2,6)$, $(\mathrm{N},5)$ [5], and this is equivalent to the weak non-negativity of the Tits quadratic form of S; these posets are called *supercritical*. In [6] the authors proved that a poset is critical relatively to the non-negativity of the Tits quadratic form if and only if it is minimax isomorphic to a supercritical poset. In [7] all such posets are fully described (they are named by the authors as NP-critical).

The importance of studying minimax isomorphic posets (introduced in [4]) is determined by the fact that their Tits quadratic forms are $\mathbb{Z}$-equivalent, and minimax isomorphism itself is a fairly general constructively defined $\mathbb{Z}$-equivalence for posets.

In [8] were introduced *1-oversupercritical posets* which differ from supercritical sets in the same degree as the latter differ from critical ones; often, including in this article, they are simply called *oversupercritical*. Among these posets, namely the posets of the forms

1) $(1,1,1,1,1,1)$, 2) $(1,1,1,1,2)$, 3) $(1,1,2,2)$,

4) $(1,1,1,3)$, 5) $(2,3,3)$, 6) $(2,2,4)$, 7) $(1,4,4)$,

8) $(1,3,5)$, 9) $(1,2,7)$, 10) $(\mathrm{N},6)$,

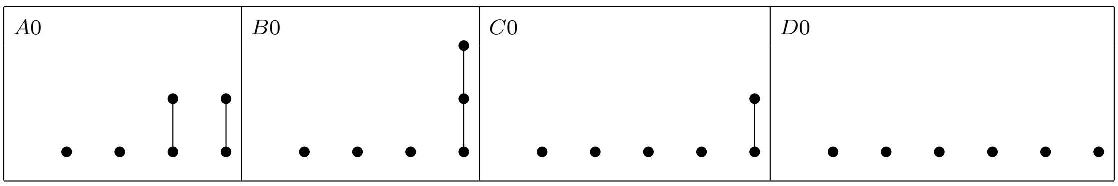
here are four of the smallest order, which is equal to 6 (for posets X, Y, Z=(X,Y) denotes their direct sum, i.e. $Z = X \cup Y$ and any elements $x \in X$ and $y \in Y$ are incomparable; (m) denotes the linearly ordered set of order m). In this article, we describe all posets that are minimax isomorphic to them and study some of their combinatorial properties.

2. The main result. Throught the paper, all posets are finite.

Let S be a poset. For a minimal (resp. maximal) element a of S, denote by $T = S_a^{\uparrow}$ (respect. $T = S_a^{\downarrow}$) the following poset: $T = S$ as usual sets, $T \setminus a = S \setminus a$ as posets, the element a is maximal (resp. minimal) in T, and a is comparable with x in T if and only if they are incomparable in S. Two posets S and T are called (min, max)-*equivalent* if there are posets $S_1, \ldots, S_p$ $(p \geq 0)$ such that, if we put $S = S_0$ and $T = S_{p+1}$, then, for every $i = 0, 1, \ldots, p$, either $S_{i+1} = (S_i)_{x_i}^{\uparrow}$ or $S_{i+1} = (S_i)_{y_i}^{\downarrow}$ [4]. Obviously, any poset is (min, max)-equivalent to itself. Since some time we also use the term *minimax equivalence*.

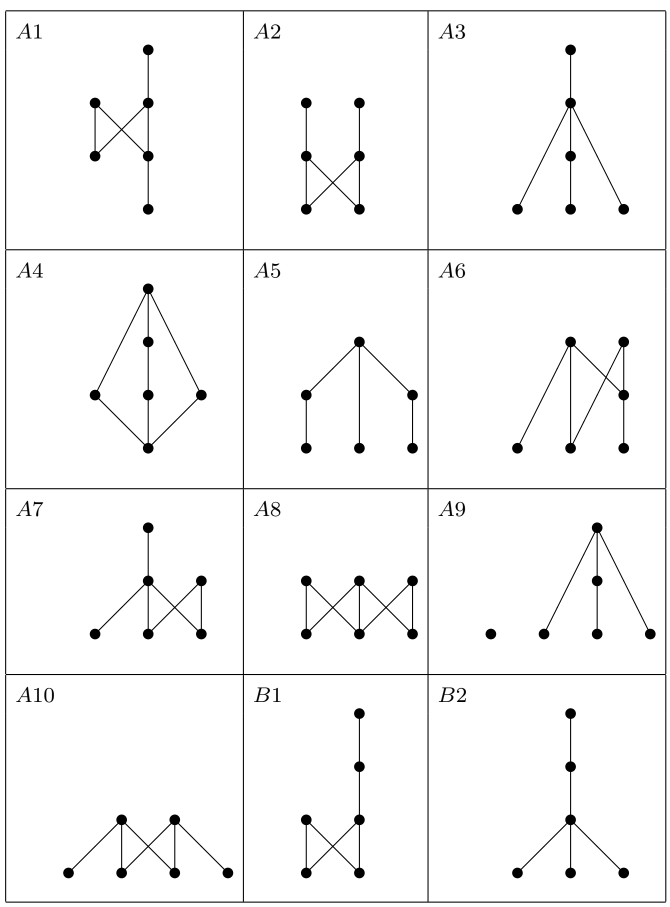
The notion of minimax equivalence can be naturally continued to the notion of *minimax isomorphism*: posets S and S' are minimax isomorphic if there exists a poset T, which is minimax equivalent to S and isomorphic to S'.

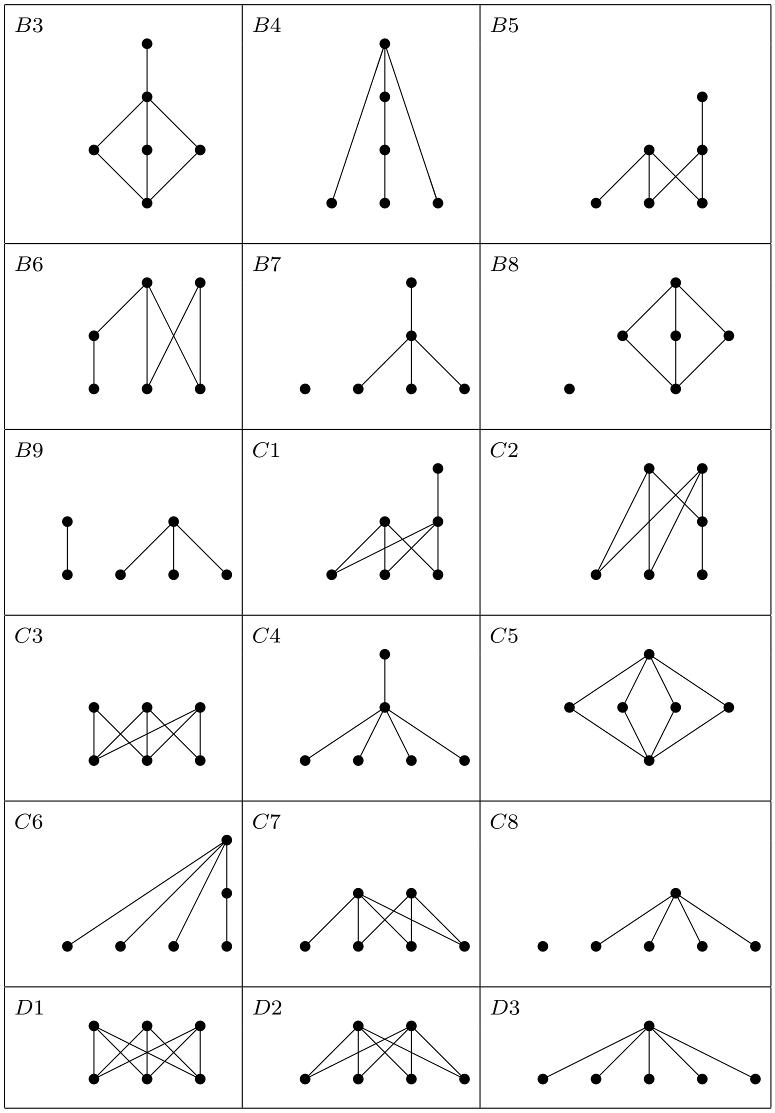
Let P be a fix poset. A poset S is called of *MM-type* P if S is minimax isomorphic to P [12]. In the case when the poset P is an oversupercritical one we say that S is of *oversupercritical MM-type*. Posets of such type were studied in many papers (see [8] – [14]). In particular, the work [8] describes all posets minimax isomorphic to $(1, 3, 5)$ and $(1, 2, 7)$. The main result of this paper describes all posets of oversupercritical MM-type in the case when P runs through all oversupercritical posets of order 6, i.e. it has (up to isomorphism) one of the following forms $A0$–$D0$:



Recall that a poset T is called *dual* to a poset S and is denoted by S^{op} if $T = S$ as usual sets and $x < y$ in T if and only if $x > y$ in S.

Theorem 1. *Up to isomorphism and duality, the complete set of posets minimax isomorphic to $A0$,$B0$, $C0$, $D0$ consists, in addition to $A0$-$D0$ themselves, of the posets indicated in the following table (Ai,Bj, Ck, Ds are, respectively, minimax isomorphic to $A0$,$B0$, $C0$, $D0$):*





3. Proof of Theorem 1. The definition of posets of the form $T = S_a^{\uparrow}$ can be extended to posets of the form $T = S_A^{\uparrow}$, where A is a lower subposet of S, i.e. $x \in A$ whenever $x < y$ and $y \in A$. Namely, $T = S_A^{\uparrow}$ is defined as follows: $T = S$ as usual sets, partial orders on A and $S \setminus A$ are the same as before, but comparability and incomparability between elements of $x \in A$ and $y \in S \setminus A$ are interchanged and the new comparability can only be of the form $x > y$. In the special case, when $A = \{a\}$ is a one-element subposet, we identify A with a. Instead of $(S_A^{\uparrow})_B^{\uparrow}$ write write $S_{AB}^{\uparrow\uparrow}$.

Let S be a poset. For subposets X, Y of S, $X < Y$ means that $x < y$ for any $x \in X, y \in Y$. We call subposets X and X' of S *strongly isomorphic* if there exists an automorphism $\varphi : S \to S$ such that $\varphi(X) = X'$ (as equality of subposets). Similarly, pairs (Y, X) and (Y', X') of subposets of S are called *strongly isomorphic* if there exists an automorphism $\varphi : S \to S$ such that $\varphi(Y) = Y'$ and $\varphi(X) = X'$.

In [3], the authors propose the following algorithm for finding (up to isomorphism) all posets that are minimax isomorphic to a given one.

I. Describe, up to strongly isomorphic, all lower subposets of $P \neq S$ in S, and, for every of them, build the poset $S_P^{\uparrow}$ ($P = \varnothing$ is not excluded).

II. Describe. up to strongly isomorphic, all pairs (Q, P) consisting of a proper lower subposet Q in S and a nonempty lower subposet P in Q such that $P < S \setminus Q$; for every such pair, build the poset $S^{\uparrow\uparrow}_{QP}$.

III. Among the posets obtained in I and II, choose one from each class of isomorphic posets.

For each poset $A0$,$B0$, $C0$, $D0$ (see the first table), we denote the partial order by $\prec$ and number the points with numbers $1, 2, 3, \ldots$ in such a way that $i < j$ whenever $i \prec j$ or i is (in the picture) to the left of j. Then the posets $A0$,$B0$, $C0$, $D0$ consist of the numbers $1, 2, 3, 4, 5, 6$ and we have $3 \prec 4, 5 \prec 6$ for $A0$, $4 \prec 5 \prec 6$ for $B0$, $5 \prec 6$ for $C0$, and the empty set of relations for $D0$.

Now we apply our algorithm to the proof of the theorem.

Step I. Describe (up to strongly isomorphic) all lower subposets. They are:

for $A0$ — $X_0 = \varnothing$, $X_1 = \{1\}$, $X_2 = \{3\}$, $X_3 = \{1, 2\}$, $X_4 = \{1, 3\}$, $X_5 = \{3, 4\}$, $X_6 = \{3, 5\}$, $X_7 = \{1, 2, 3\}$, $X_8 = \{1, 3, 4\}$, $X_9 = \{1, 3, 5\}$, $X_{10} = \{3, 4, 5\}$, $X_{11} = \{1, 2, 3, 4\}$, $X_{12} = \{1, 2, 3, 5\}$, $X_{13} = \{1, 3, 4, 5\}$, $X_{14} = \{3, 4, 5, 6\}$, $X_{15} = \{1, 2, 3, 4, 5\}$, $X_{16} = \{1, 3, 4, 5, 6\}$;

for $B0$ — $Y_0 = \varnothing$, $Y_1 = \{1\}$, $Y_2 = \{4\}$, $Y_3 = \{1, 2\}$, $Y_4 = \{1, 4\}$, $Y_5 = \{4, 5\}$, $Y_6 = \{1, 2, 3\}$, $Y_7 = \{1, 2, 4\}$, $Y_8 = \{1, 4, 5\}$, $Y_9 = \{4, 5, 6\}$, $Y_{10} = \{1, 2, 3, 4\}$, $Y_{11} = \{1, 2, 4, 5\}$, $Y_{12} = \{1, 4, 5, 6\}$, $Y_{13} = \{1, 2, 3, 4, 5\}$, $Y_{14} = \{1, 2, 4, 5, 6\}$;

for $C0$ — $Z_0 = \varnothing$, $Z_1 = \{1\}$, $Z_2 = \{5\}$, $Z_3 = \{1, 2\}$, $Z_4 = \{1, 5\}$, $Z_5 = \{5, 6\}$, $Z_6 = \{1, 2, 3\}$, $Z_7 = \{1, 2, 5\}$, $Z_8 = \{1, 5, 6\}$, $Z_9 = \{1, 2, 3, 4\}$, $Z_{10} = \{1, 2, 3, 5\}$, $Z_{11} = \{1, 2, 5, 6\}$, $Z_{12} = \{1, 2, 3, 4, 5\}$, $Z_{13} = \{1, 2, 3, 5, 6\}$;

for $D0$ — $T_0 = \varnothing$, $T_1 = \{1\}$, $T_2 = \{1, 2\}$, $T_3 = \{1, 2, 3\}$, $T_4 = \{1, 2, 3, 4\}$, $T_5 = \{1, 2, 3, 4, 5\}$.

Denote by $K_{i,j}$ the poset $S^{\uparrow}_V$ for $i = 1$, $S = A_0$ and $V = X_j$, $i = 2$, $S = B_0$ and $V = Y_j$, $i = 3$, $S = C_0$ and $V = Z_j$, $i = 4$, $S = D_0$ and $V = T_j$. Then it is easy to see that

$K_{1,0} \cong A0$, $K_{1,1} \cong A5$, $K_{1,2} \cong A9$, $K_{1,3} \cong A2^{\rm op}$, $K_{1,4} \cong A6$, $K_{1,5} \cong A3$, $K_{1,6} \cong A10$, $K_{1,7} \cong A7^{\rm op}$, $K_{1,8} \cong A1$, $K_{1,9} \cong A8$, $K_{1,10} \cong A7$, $K_{1,11} \cong A10^{\rm op}$, $K_{1,12} \cong A3^{\rm op}$, $K_{1,13} \cong A6^{\rm op}$, $K_{1,14} \cong A2$, $K_{1,15} \cong A9^{\rm op}$, $K_{1,16} \cong A5^{\rm op}$;

$K_{2,0} \cong B0$, $K_{2,1} \cong B4$, $K_{2,2} \cong B9$, $K_{2,3} \cong B1^{\rm op}$, $K_{2,4} \cong B6$, $K_{2,5} \cong B7$, $K_{2,6} \cong B2^{\rm op}$, $K_{2,7} \cong B5^{\rm op}$, $K_{2,8} \cong B5$, $K_{2,9} \cong B2$, $K_{2,10} \cong B7^{\rm op}$, $K_{2,11} \cong B6^{\rm op}$, $K_{2,12} \cong B1$, $K_{2,13} \cong B9^{\rm op}$, $K_{2,14} \cong B4^{\rm op}$;

$K_{3,0} \cong C0$, $K_{3,1} \cong C6$, $K_{3,2} \cong C8$, $K_{3,3} \cong C2$, $K_{3,4} \cong C7$, $K_{3,5} \cong C4$, $K_{3,6} \cong C1^{\rm op}$, $K_{3,7} \cong C3$, $K_{3,8} \cong C1$, $K_{3,9} \cong C4^{\rm op}$, $K_{3,10} \cong C7^{\rm op}$, $K_{3,11} \cong C2^{\rm op}$, $K_{3,12} \cong C8^{\rm op}$, $K_{3,13} \cong C6^{\rm op}$;

$K_{4,0} \cong D0$, $K_{4,1} \cong D3$, $K_{4,2} \cong D2$, $K_{4,3} \cong D1$, $K_{4,4} \cong D2^{\rm op}$, $K_{4,5} \cong D3^{\rm op}$.

Step II. Describe (up to strongly isomorphic) all pairs of lower subposets (see the algorithm). They are:

for $A0$ — $X'_1 = (X_{15}, \{5\})$;

for $B0$ — $Y'_1 = (Y_{10}, \{4\})$, $Y'_2 = (Y_{13}, \{4\})$, $Y'_3 = (Y_{13}, \{4, 5\})$;

for $C0$ — $Z'_1 = (Z_{12}, \{5\})$;

for $D0$, there are no such pairs.

Denote by $K'_{i,j}$ the poset $(S^{\uparrow}_V)^{\uparrow}_W$ for $i = 1$, $S = A_0$ and $(V, W) = X'_j$, $i = 2$, $S = B_0$ and $(V, W) = Y'_j$, $i = 3$ $S = C_0$ and $(V, W) = Z'_j$. Then it is easy to see that $K'_{1,1} \cong A4$; $K'_{2,1} \cong B3^{\rm op}$, $K'_{2,2} \cong B8$, $K'_{2,3} \cong B3$; $K'_{3,1} \cong C5$.

Step III. It is easy to verify that in I and II each of the posets Ai, Bl, Ck, Ds, indicated in the condition of the theorem, and dual to them (in the non-dual cases) occurs only once. And hence the theorem is proved.

4. Coefficientts of transitivity. Let S be a (finite) poset and $S^2_< := \{(x,y) \,|\, x,y \in S, x < y\}$. If $(x,y) \in S^2_<$ and there is no z satisfying $x < z < y$, then we say that x and y are *neighboring*. We put $n_w = n_w(S) := |S^2_<|$ and denote by $n_e = n_e(S)$ the number of pairs of neighboring elements. The ratio $k_t = k_t(S)$ of the numbers $n_w - n_e$ and n_w are called the *coefficient of transitiveness of* S; if $n_w = 0$ (then $n_e = 0$), we assume $k_t = 0$ (the coefficient of transitivity is introduced in [15]).

In this part of the paper we calculate k_t for the posets of MM-type to be $A0, B0, C0, D0$.

Theorem 2. *The following holds for posets Ai, Bj, Ck, Ds:*

N	n_e	n_w	k_t
$A0$	2	2	0
$B0$	2	3	0,33333
$C0$	1	1	0
$D0$	0	0	0

N	n_e	n_w	k_t	N	n_e	n_w	k_t	N	n_e	n_w	k_t
$A1$	6	11	0,45455	$B1$	6	11	0,45455	$C2$	7	9	0,22222
$A2$	6	10	0,4	$B2$	5	12	0,58333	$C3$	8	8	0
$A3$	5	10	0,5	$B3$	7	12	0,41667	$C4$	5	9	0,44444
$A4$	7	10	0,3	$B4$	5	8	0,375	$C5$	8	9	0,11111
$A5$	5	7	0,28571	$B5$	6	8	0,25	$C6$	5	6	0,16667
$A6$	6	8	0,25	$B6$	6	7	0,14286	$C7$	7	7	0
$A7$	6	9	0,33333	$B7$	4	7	0,42857	$C8$	4	4	0
$A8$	7	7	0	$B8$	6	7	0,14286	$D1$	9	9	0
$A9$	4	5	0,2	$B9$	4	4	0	$D2$	8	8	0
$A10$	6	6	0	$C1$	7	10	0,3	$D3$	5	5	0

The transitivity coefficients are written out with an accuracy of five decimal places. The value is exact if and only if the number of decimal places is less than five, and two values equal to exactly five digits are equal at all.

The proof is carried out by direct calculations.

Recall that the greatest length among the lengths of all linear ordered subsets of a poset S is called its *height* and the greatest number of pairwise incomparable elements of S is called its *weight*, An element of a poset is called *nodal*, if it is comparable with all the others elements. A subposet X of T is said to be *dense* if there is not $x_1, x_2 \in X$, $y \in T \setminus X$ such that $x_1 < y < x_2$.

Note that a poset of MM-type $A0$–$D0$ can have at most three nodal elements.

Corollary 1. *The coefficient $k_t(S)$ of a poset S is the largest among all the posets of MM-type $A0$–$D0$ if and only if S contains a dense subposet with three nodal elements.*

Corollary 2. *The coefficient $k_t(S)$ of a poset S is the smallest among all the posets of MM-type $A0$–$D0$ if and only if S is a self-dual connected poset of height two.*

Corollary 3. *For a posets S of MM-type A0–D0, the following conditions are equivalent:*

(a) $k_t(S) = \frac{1}{2}$;

(b) S is a non-self-dual poset of width three with two nodal elements.

5. Conclusions. In this article we study combinatorial aspects of oversupercritical posets which differ from supercritical sets in the same degree as the latter differ from critical ones. Namely, we describe, up to isomorphism, all the posets that are minimax isomorphic to oversupercritical posets of the order 6.

The importance of studying minimax isomorphic posets is determined by the fact that their Tits quadratic forms are $\mathbb{Z}$-equivalent. This allowed the authors (earlier)

(1) to prove that a poset is critical relative to the positivity of the Tits quadratic form if and only if it is minimax isomorphic to a Kleiner's poset;

(2) to prove that a poset is critical relatively to the non-negativity of the Tits quadratic form if and only if it is minimax isomorphic to a supercritical poset;

(3) to describe all the posets mentioned in (1) and (2),

and also to solve a number of other problems, which were not mentioned in this article and, in particular,

(4) to describe, up to isomorphism, all posets with the Tits quadratic form to be positive.

We also describe the transitivity coefficients for all posets minimax isomorphic to oversupercritical posets of the order 6.

The obtained results (together with the corresponding research methods) will be used in the study of combinatorial aspects of other classes of posets.

References

1. Kleiner, M.M. (1972). Partially ordered sets of finite type. *Zap. Nauchn. Sem. LOMI, 28*, 32–41 [in Russian].
2. Drozd, Yu.A. (1974). Coxeter transformations and representations of partially ordered sets. *Funktsional. Anal. i Prilozhen., 8, 3*, 34-42 [in Russian].
3. Bondarenko, V.M., & Styopochkina, M.V. (2005). (Min, max)-equivalence of partially ordered sets and the Tits quadratic form. *Problems of Analysis and Algebra: Zb. Pr. Inst. Mat. NAN Ukr., 2, 3*, 18-58 [in Russian].
4. Bondarenko, V.M. (2005). On (min, max)-equivalence of posets and applications to the Tits forms. *Bull. of Taras Shevchenko University of Kyiv. (series: Physics & Mathematics), 1*, 24-25 [in Russian].
5. Nazarova, L.A. (1975). Partially ordered sets of infinite type. *Izv. Akad. Nauk SSSR Ser. Mat., 39, 5*, 963-991 [in Russian].
6. Bondarenko, V.M., & Styopochkina, M.V. (2008). (Min, max)-equivalence of posets and nonnegative Tits forms. *Ukr. Math. J. 60(9)*, 1157-1167 [in Russian].
7. Bondarenko, V.M., & Styopochkina, M.V. (2009). Description of posets critical with respect to the nonnegativity of the quadratic Tits form. *Ukr. Math. J., 61(5)*, 611-624 [in Russian].
8. Bondarenko, V.V., Bondarenko, V.M., Styopochkina, M.V., & Chervyakov, I.V. (2011). 1-oversupercritical partially ordered sets with trivial group of automorphisms and min-equivalence. I. *Scien. Bull. of Uzhhorod Univ. Series of Math. and Inform.*, 22, 2, 17-25 [in Russian].
9. Bondarenko, V.V., & Styopochkina, M.V. (2013). Non-primitive 1-oversupercritical partially ordered set and min-equivalence. *Scien. J. of NPU named after Dragomanov. Series 1. Phys.-Math. sciences, 14*, 55-61 [in Russian].
10. Styopochkina, M.V., & Chervyakov, I.V. (2015) The number of partially ordered sets, (min, max)-equivalent to the set (1, 2, 7). *Applied problems of mech. and math., 13*, 18-21

[in Ukrainian].
11. Styopochkina, M.V., & Chervyakov, I. V. (2016). The number of partially ordered sets, (min, max)-equivalent to the 1-oversupercritical partially ordered set (1, 3, 5). *Applied problems of mech. and math., 14*, 12-15 [in Ukrainian].
12. Bondarenko, V.M., & Styopochkina, M.V. (2018). On properties of posets of MM-type (1,3,5). *Scien. Bull. of Uzhhorod Univ. Series of Math. and Inform, 1(32)*, 50-53.
13. Bondarenko, V.M., Orlovskaja, Yu.M., & Styopochkina, M.V. (2018). On Hasse diagrams connected with the 1-oversupercritical poset (1,3,5). *Applied problems of mech. and math., 16*, 30-32.
14. Bondarenko, V.M., & Styopochkina, M.V. (2019). On properties of posets of MM-type (1, 2, 7). *Applied problems of mech. and math., 17*, 7-10.
15. Bondarenko, V.M., & Styopochkina, M.V. (2017). Coefficients of transitiveness of P-critical posets. *Proc. Inst. Math. of NAS of Ukraine, 1(14)*, 46-51.

Бондаренко В. М., Стьопочкіна М. В. Про частково впорядковані множини шостого порядку, що мають надсуперкритичний MM-тип.

Зображення ч. в. множин (частково впорядкованих множин) ввели Л. А. Назарова і А. В. Ройтер в 1972 р. В тому ж році М. М. Клейнер довів, що ч. в. множина S має скінченний зображувальний тип тоді і лише тоді, коли вони не містить ч. в. підмножин вигляду $K_1 = (1,1,1,1), K_2 = (2,2,2), K_3 = (1,3,3), K_4 = (1,2,5)$ і $K_5 = (\mathrm{N},4)$. Ці ч. в. множин називаються критичними ч. в. множин щодо скінченностсті типу (в тому сенсі, що це мінімальні ч. в. множин з нескінченною кількістю нерозкладних зображень, з точністю до еквівалентності) або ч. в. множинами Клейнера. У 1974 році Ю. А. Дрозд довів, що ч. в. множина S має скінченний зображувальний тип тоді і лише тоді, коли її квадратична форма Тітса

$$q_S(z) =: z_0^2 + \sum_{i \ inS} z_i^2 + \sum_{i<j,i,j \ inS} z_i z_j - z_0 \sum_{i \ inS} z_i$$

є слабко додатною (тобто додатною на множині невід'ємних векторів). Отже, ч. в. множини Клейнера є критичними щодо слабкої додатності квадратичної форми Тітса, і інших таких ч. в. множин немає (з точністю до ізоморфізму). У 2005 році автори довели що ч. в. множин є критичною щодо додатності квадратичної форми Титса тоді і лише тоді, коли вона є мінімаксно ізоморфна деякій ч. в. множині Клейнера.

Подібну ситуацію маємо з ч. в. множинами ручного зображувального типу. У 1975 р. Л. А. Назарова довела, що ч. в. множина S є ручною тоді і лише тоді, коли вона не містить ч. в. підмножин вигляду $N_1 = (1,1,1,1,1)$, $N_2 = (1,1,1,2)$, $N_3 = (2,2,3)$, $N_4 = (1,3,4)$, $N_5 = (1,2,6)$ і $(\mathrm{N},5)$. Вона назвала ці ч. в. множини суперкритичними; вони є критичними щодо слабкої невід'ємності квадратичної форми Тітса, і інших таких ч. в. множин немає. У 2009 році автори довели, що ч. в. множина є критичною щодо невід'ємності квадратичної форми Тітса тоді і лише тоді, коли вона мінімаксно ізоморфна деякій суперкритичній ч. в. множині.

Перший автор запропонував ввести ч. в. множини (названі надсуперкритичними), які відрізняються від суперкритичних ч. в. множин в тій самій мірі, що і останні відрізняються від критичних. Серед цих ч. в. множин є чотири найменшого порядку, а саме 6. У цій статті ми описуємо всі ч. в. множини мінімаксно еквівалентні їм, і вивчаємо деякі їхні комбінаторні властивості. Важливість вивчення мінімаксно ізоморфних ч. в. множин визначається тим фактом, що їх квадратичні форми Тітса $\mathbb{Z}$-еквівалентні, а сам мінімаксний ізоморфізм є досить загальною конструктивно визначеною $\mathbb{Z}$-еквівалентністю для квадратичних форм Тітса ч. в. множин.

Ключові слова: зображення, критична та суперкритична ч. в. множина, надсуперкритична ч. в. множина, квадратична форма Тітса, скінченний і ручний зображувальний тип, додатність і слабка додатність, негативність і слабка негативність.

Список використаної літератури

1. Клейнер М. М. Частично упорядоченные множества конечного типа. *Зап. науч. семинаров ЛОМИ.* 1972. Т. 28. С. 32-41.
2. Дрозд Ю. А. Преобразования Кокстера и представления частично упорядоченных множеств. *Функц. анализ и его прил.* 1974. Т. 8, Вып. 3. С. 34-42.
3. Бондаренко В. М., Степочкина М. В. (Min, max)-эквивалентность частично упорядоченных множеств и квадратичная форма Титса. *Проблеми аналізу і алгебри: Зб. праць Ін-ту математики НАН України.* 2005. Т. 2, №3. С. 18-58.
4. Bondarenko V. M. On (min, max)-equivalence of posets and applications to the Tits forms. *Bull. of Taras Shevchenko University of Kyiv. (series: Physics & Mathematics).* 2005. No.1. С. 24-25.
5. Назарова Л. А. Частично упорядоченные множества бесконечного типа. *Изв. АН СССР.* 1975. Т. 39, Вып. 5. С. 963–991.
6. Бондаренко В. М., Степочкина М. В. (Min, max)-эквивалентность частично упорядоченных множеств и неотрицательные формы Титса. *Укр. мат. журнал.* 2008. Т. 60, №9. С. 1157-1167.
7. Бондаренко В. М., Степочкина М. В. Описание частично упорядоченных множеств, критических относительно неотрицательности квадратичной формы Титса. *Ukr. Math. J.*, 2009. Т. 61, №5. С. 734–746.
8. Бондаренко В. В., Бондаренко В. М., Степочкина М. В., Червяков И. В. 1-надсуперкритические частично упорядоченные множества с тривиальной группой автоморфизмов и min- эквивалентность. *Наук. вісн. Ужгород. ун-ту. Сер. матем. і інформ.* 2011. Вип. 22, №2. С. 17-25.
9. Бондаренко В. В., Степочкина М. В. Непримитивное 1-надсуперкритическое частично упорядоченное множество и min-эквивалентность. *Науковий часопис НПУ ім. Драгоманова. Серія 1. фіз.-мат. науки.* 2013. №14. С. 55-61.
10. Стьопочкіна М. В., Черв'яков І. В. Кількість частково впорядкованих множин, (min, max)-еквівалентних множині (1, 2, 7). *Прикл. проблеми мех. і мат.* 2015. Вип. 13. С. 18-21.
11. Стьопочкіна М. В., Черв'яков І. В. Кількість частково впорядкованих множин, (min, max)-еквівалентних 1-надсуперкритичній частково впорядкованій множині (1, 3, 5). *Прикл. проблеми мех. і мат.* 2016. Вип. 14. С. 12-15.
12. Bondarenko V. M., Styopochkina M. V. On properties of posets of MM-type (1,3,5). *Scien. Bull. of Uzhhorod Univ. Series of Math. and Inform.* 2018. Issue No. 1 (32). P. 50-53.
13. Bondarenko V. M., Orlovskaja Yu. M., Styopochkina M. V. On Hasse diagrams connected with the 1-oversupercritical poset (1,3,5). *Прикл. проблеми мех.. і мат.* 2018. Вип. 16. С. 30-32.
14. Bondarenko V. M., Styopochkina M. V. On properties of posets of MM-type (1, 2, 7). *Прикл. проблеми мех.. і мат.* 2019. Вип. 17. С. 7-10.
15. Bondarenko V. M., Styopochkina M. V. Coefficients of transitiveness of P-critical posets. *Зб. праць Ін-ту математики НАН України.* 2017. Т.14, No.1. С. 46-51.

Recived 27.04.2021