On a Hybrid Non-Linear Jump Control Problem for Functional Differential Equations with State-Dependent Impulses

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We study a control problem for the hybrid non-linear functional differential boundary value problem which is, in some sense, inverse to problems investigated in [1,3]. The problem consists of two systems of differential equations

$$x'(t) = f\big(t, x(t), x(\beta_1(t)), x(\beta_2(t))\big), \quad t \in [a, \tau], \tag{1}$$

$$y'(t) = \phi(t, y(t)), \quad t \in [\tau, b] \tag{2}$$

on the intervals $[a, \tau]$ and $[\tau, b]$ respectively, where the switching time τ is such that

$$g(t, x(\tau-)) = 0, \tag{3}$$

of the non-linear two-point boundary condition

$$V(x(a), y(b)) = 0, \tag{4}$$

the jump condition at the time instant τ

$$y(\tau+) - x(\tau-) = \gamma, \tag{5}$$

and the additional two-point conditions

$$\begin{aligned} &x_i(a) = z_i, \quad i = 1, \dots, j, \\ &y_k(b) = \eta_k, \quad k = j+1, \dots, n, \end{aligned} \tag{6}$$

where $1 \le j \le n$ is fixed.

The values of the time instant τ and the size of the jump γ are not specified beforehand and remain unknown. Thus, problem (1)–(6) is to determine the unknown values of τ and γ so that the solutions of (1), (2) satisfy the non-linear boundary conditions (4), the jump condition (3), (5) and the two-point conditions (6).

We consider the single-jump case [1, 2], i.e., it is assumed that the switching time τ is unique, which means that there is only one intersection of the integral curve of system (1) with the barrier set

$$G = \big\{(t,x) \in [a,b] \times \mathbb{R}^n : \ g(t,x) = 0\big\}. \tag{7}$$

The impulse action in this problem is state-dependent since the time instant τ is determined by the intersection of the curve with the barrier. In contrast to [1,2], the jump magnitude γ here is unknown and plays the role of a control parameter. By a solution of (1)–(6), we mean the triplet (u, τ, γ), where

$$u(t) = \begin{cases} x(t) & \text{if } \ t \in [a,\tau], \\ y(t) & \text{if } \ t \in (\tau, b] \end{cases} \tag{8}$$

is left-continuous. The pre-jump evolution of the solution is described by the functional differential equation (1) and its after-jump behaviour is characterized by the ordinary differential equation (2). Equation (1), generally speaking, may contain other types of functional terms, which can be treated in a similar way [4].

In (1), (2), $f : [a,b] \times \mathbb{R}^{3n} \to \mathbb{R}^n$ and $\phi : [a,b] \times \mathbb{R}^n \to \mathbb{R}^n$ satisfy the Carathéodory conditions, $\beta_i : [a,\tau] \to [a,\tau]$, $i = 0,1$, are measurable, g is continuous.

We will use an approach similar to [1] and approximate a solution u of form (8) of problem (1)–(6) by suitable sequences of functions separately on the intervals before and after the time when the jump occurs. The jump time τ itself remains unknown and is treated as parameter the value of which is to be determined.

Let us fix an arbitrary point $\tau \in (a,b)$ and choose certain compact convex sets D_a, $D_{\tau-}$, $D_{\tau+}$, D_b, Γ and define the sets

$$\begin{aligned} D_{a,\tau} &:= \big\{(1-\theta)z + \theta\lambda : \ z \in D_a, \ \lambda \in D_{\tau-}, \ \theta \in [0,1]\big\}, \\ D_{\tau+,b} &:= \big\{(1-\theta)(\lambda+\gamma) + \theta\eta : \ \lambda \in D_{\tau-}, \ \gamma \in \Gamma, \ \eta \in D_b, \ \theta \in [0,1]\big\}. \end{aligned}$$

Our technique is based on the parametrization

$$\begin{aligned} z &= \operatorname{col}\big(x_1(a), x_2(a), \dots, x_j(a), z_{j+1}, \dots, z_n\big), \\ \eta &= \operatorname{col}\big(\eta_1, \eta_2, \dots, \eta_j, y_{j+1}(b), \dots, y_n(b)\big), \\ x(\tau-) &= \operatorname{col}(\lambda_1, \lambda_2, \dots, \lambda_n) = \lambda, \end{aligned}$$

which, together with $\gamma = \operatorname{col}(\gamma_1, \gamma_2, \dots, \gamma_n)$ and $\tau \in (a,b)$, constitutes the set of parameters $(z, \lambda, \eta, \gamma, \tau)$ with

$$z \in D_a, \ \ \lambda \in D_{\tau-}, \ \ \eta \in D_b, \ \ \gamma \in \Gamma. \tag{9}$$

Let us fix non-negative vectors $\varrho^{(i)}$, $i = 0,1$, and put

$$\Omega_0(\varrho^{(0)}) = O_{\varrho^{(0)}}(D_{a,\tau}), \quad \Omega_1(\varrho^{(1)}) = O_{\varrho^{(1)}}(D_{\tau+,b}), \tag{10}$$

where $O_\varrho(D) := \bigcup\limits_{z \in D} O_\varrho(z)$ for $D \subset \mathbb{R}^n$ and $O_\varrho(z) := \{\xi \in \mathbb{R}^n : \ |\xi - z| \le \varrho\}$ stand for the corresponding componentwise neighbourhoods of a set and a vector.

Introduce the following two auxiliary parametrized two-point boundary value problems for the investigation of the pre-jump and after-jump equations

$$\begin{gathered} x'(t) = f\big(t, x(t), x(\beta_1(t)), x(\beta_2(t))\big), \ \ t \in [a,\tau]; \quad x(a) = z, \ x(\tau) = \lambda, \\ y'(t) = \phi(t, y(t)), \ \ t \in (\tau, b]; \quad y(\tau) = \lambda + \gamma, \ \ y(b) = \eta, \end{gathered} \tag{11}$$

where the time instant τ and the vectors z, λ, γ, and η are treated as free parameters. We suppose that the functions x and y in (11) should take values in the sets Ω_0 and Ω_1, respectively.

To study problems (11), by analogy to [1, 5], we introduce two sequences of functions $\{x_m(\,\cdot\,,\tau,z,\lambda):\ m\ge 0\}$ and $\{y_m(\,\cdot\,,\tau,\lambda,\gamma,\eta):\ m\ge 0\}$ by putting

$$\begin{aligned} x_0(t,\tau,z,\lambda) &= \Big(1-\frac{t-a}{\tau-a}\Big)z+\frac{t-a}{\tau-a}\,\lambda,\ \ t\in[a,\tau],\\ y_0(t,\tau,\lambda,\gamma,\eta) &= \Big(1-\frac{t-\tau}{b-\tau}\Big)(\lambda+\gamma)+\frac{t-\tau}{b-\tau}\,\eta,\ \ t\in(\tau,b] \end{aligned}$$

and

$$\begin{aligned} x_{m+1}(t,\tau,z,\lambda) &= x_0(t,\tau,z,\lambda)\\ &\quad+\int\limits_a^t (Fx_m(\,\cdot\,,\tau,z,\lambda))(s)\,ds-\frac{t-a}{\tau-a}\int\limits_a^\tau (Fx_m(\,\cdot\,,\tau,z,\lambda))(s)\,ds,\ \ t\in[a,\tau],\ \ m\ge 0, \end{aligned} \tag{12}$$

and

$$\begin{aligned} y_{m+1}(t,\tau,\lambda,\gamma,\eta) &= y_0(t,\tau,\lambda,\gamma,\eta)\\ &\quad+\int\limits_\tau^t (\Phi y_m(\,\cdot\,,\tau,z,\lambda))(s)\,ds-\frac{t-\tau}{b-\tau}\int\limits_t^b (\Phi y_m(\,\cdot\,,\tau,z,\lambda))(s)\,ds,\ \ t\in(\tau,b],\ \ m\ge 0, \end{aligned} \tag{13}$$

where

$$\begin{aligned} &(Fx)(t) := f\big(t,x(t),x(\beta_1(t)),x(\beta_2(t))\big),\ \ t\in[a,\tau],\\ &(\Phi y)(t) := \phi(t,y(t)),\ \ t\in(\tau,b]. \end{aligned}$$

Assume that $\varrho^{(0)}$, $\varrho^{(1)}$ involved in (10) can be chosen so that

$$\varrho^{(0)}\ge\frac{\tau-a}{2}\,\delta_{[a,b],(\Omega_0(\varrho^{(0)}))^3}(f),\qquad \varrho^{(1)}\ge\frac{b-\tau}{2}\delta_{[a,b],\Omega_1(\varrho^{(1)})}(\phi), \tag{14}$$

where $\delta_{[a,b],\Omega_1(\varrho^{(1)})}(\phi)$ is 1/2 of the oscillation of ϕ over $[a,b]\times\Omega_1(\varrho^{(1)})$,

$$\delta_{[a,b],\Omega_1(\varrho^{(1)})}(\phi) := \frac12\Big(\operatorname*{ess\,sup}_{(t,x)\in[a,b]\times\Omega_1(\varrho^{(1)})}\phi(t,x)-\operatorname*{ess\,inf}_{(t,x)\in[a,b]\times\Omega_1(\varrho^{(1)})}\phi(t,x)\Big)$$

and the value $\delta_{[a,b],(\Omega_0(\varrho^{(0)}))^3}(f)$ is defined by analogy. Assume that f and ϕ satisfy the Lipschitz conditions with respect to the space variables

$$\begin{gathered} \big|f(t,x_1,y_1,z_1)-f(t,x_2,y_2,z_2)\big|\le K_0|x_1-x_2|+K_1|y_1-y_2|+K_2|z_1-z_2|,\\ |\phi(t,\xi_1)-\phi(t,\xi_2)|\le L|\xi_1-\xi_2|, \end{gathered} \tag{15}$$

respectively, on the sets $[a,b]\times(\Omega_0(\varrho^{(0)}))^3$ and $[a,b]\times\Omega_1(\varrho^{(1)})$, and put

$$Q_0=\frac{\tau-a}{2}\,(K_0+K_1+K_2),\quad Q_1=\frac{b-\tau}{2}\,L.$$

Theorem 1. *Let there exist non-negative vectors $\varrho^{(0)}$, $\varrho^{(1)}$ with properties* (14) *such that* (15) *holds on the sets $[a,b]\times(\Omega_0(\varrho^{(0)}))^3$ and $[a,b]\times\Omega_1(\varrho^{(1)})$ and*

$$r(Q_0)<1,\quad r(Q_1)<1.$$

Then, for all fixed $\tau\in(a,b)$, $z\in D_a$, $\lambda\in D_{\tau-}$, $\gamma\in\Gamma$, $\eta\in D_b$:

1. *Functions* (12), (13) *are absolutely continuous on* $[a,\tau]$ *and* $(\tau,b]$ *for* $m\geq 0$.
2. $\{x_m(t,\tau,z,\lambda):\ t\in[a,\tau],\ m\geq 0\}\subset\Omega_0(\varrho^{(0)})$, $\{y_m(t,\tau,\lambda,\gamma,\eta):\ t\in[\tau,b],\ m\geq 0\}\subset\Omega_1(\varrho^{(1)})$.
3. $\{x_m(\,\cdot\,,\tau,z,\lambda):\ m\geq 0\}$ *and* $\{y_m(t,\tau,\lambda,\gamma,\eta):\ m\geq 0\}$ *converge to the limit functions* $x_\infty(\,\cdot\,,\tau,z,\lambda)$, $y_\infty(\,\cdot\,,\tau,\lambda,\gamma,\eta)$ *uniformly on* $[a,\tau]$ *and* $[\tau,b]$.
4. $x_\infty(\,\cdot\,,\tau,z,\lambda)$ *and* $y_\infty(\,\cdot\,,\tau,\lambda,\gamma,\eta)$ *are the solutions of the boundary value problems*
$$\begin{gathered} x'(t)=(Fx)(t)+\frac{1}{\tau-a}\bigg(\lambda-z-\int_a^\tau(Fx)(s)\,ds\bigg),\\ x(a)=z,\quad x(\tau)=\lambda, \end{gathered}\tag{16}$$
$$\begin{gathered} y'(t)=\phi(t,y(t))+\frac{1}{b-\tau}\bigg(\eta-\lambda-z-\int_\tau^b\phi(s,y(s)))\,ds\bigg),\\ y(\tau)=\lambda+\gamma,\quad y(b)=\eta, \end{gathered}\tag{17}$$
and problems (16), (17) *have no other solutions with values in* $\Omega_0(\varrho^{(0)})$ *and* $\Omega_1(\varrho^{(1)})$.
5. *The following estimates hold:*
$$\begin{aligned} \big|x_\infty(t,\tau,z,\lambda)-x_m(t,\tau,z,\lambda)\big|&\leq\frac{\tau-a}{2}\,Q_0^m(1_n-Q_0)^{-1}\delta_{[a,b],(\Omega_0(\varrho^{(0)}))^3}(f),\\ \big|y_\infty(t,\tau,\lambda,\gamma,\eta)-y_m(t,\tau,\lambda,\gamma,\eta)\big|&\leq\frac{b-\tau}{2}\,Q_1^m(1_n-Q_1)^{-1}\delta_{[a,b],\Omega_1(\varrho^{(1)})}(\phi). \end{aligned}$$

Theorem 2. *If, under the above conditions, some values* $(\tau,z,\lambda,\gamma,\eta)\in(a,b)\times D_a\times D_{\tau-}\times\Gamma\times D_b$ *satisfy the system of* $3n+1$ *scalar determining equations*
$$\begin{gathered} \int_a^\tau(Fx_\infty(\,\cdot\,,\tau,z,\lambda))(s)\,ds=\lambda-z,\quad g(\tau,\lambda)=0,\\ \int_\tau^b\phi\big(s,y_\infty(s,\tau,\lambda,\gamma,\eta)\big)\,ds=\eta-\lambda-\gamma,\quad V(z,\eta)=0 \end{gathered}\tag{18}$$
and, in addition, $g(t,y_\infty(t,\tau,\lambda,\gamma,\eta))\neq 0$ *for any* $t\in(\tau,b]$, *then the function*
$$u_\infty(t,\tau,z,\lambda,\gamma,\eta):=\begin{cases} x_\infty(t,\tau,z,\lambda) & \text{if } t\leq\tau\\ y_\infty(t,\tau,\lambda\gamma,\eta) & \text{if } t>\tau\end{cases}$$
is a solution of the original problem (1)–(6) *with a single jump* γ *at the time* τ.

The solvability of (1)–(6) can be checked by studying the approximate determining system obtained from (18) after replacing ∞ by a certain m.

Let us apply the approach described above to the systems
$$\begin{aligned} x_1'(t)&=-\frac{4}{3}\,(x_2(t^2))^2,\\ x_2'(t)&=(1-t)^2\Big(1.8\,x_1\Big(\frac{t}{3}\Big)+\frac{2}{9}\,t\Big),\ \ t\in[0,\tau];\\ y_1'(t)&=y_2(t)-\frac{1}{4}\,,\\ y_2'(t)&=\frac{1}{2}\,\Big(y_2(t)-\frac{1}{4}\Big)^2-y_1(t)+1,\ \ t\in[\tau,1], \end{aligned}$$

under the boundary conditions

$$x_1^2(a) + y_2(b) = 0.6, \quad x_2^2(a) + x_1(a)y_2(b) = 0.25, \tag{19}$$

$$x_1(a) = 0.43, \quad y_2(b) = 0.42 \tag{20}$$

and the jump condition (3), (5) with the barrier set (7) of the form $\{(t, x_1, x_2) : \ (x_1+\frac{1}{2})^2 - t^2 = 0.1\}$. Clearly, (19) is a particular case of (4) and (20) has form (6) with $j = 1$, $z_1 = 0.43$, $\eta_2 = 0.42$.

The admissible sets for the parameter values were choosen as

$$\begin{aligned} D_a = D_{\tau-} = D_{a,\tau-} &= \big\{(x_1, x_2) : \ 0.27 \le x_1 \le 0.45, \ 0.2 \le x_2 \le 0.56\big\}, \\ \Gamma &= \big\{(x_1, x_2) : \ 0.15 \le x_1 \le 0.25, \ -0.3 \le x_2 \le -0.2\big\}, \\ D_{\tau+} = D_b = D_{\tau+,b} &= \big\{(x_1, x_2) : \ 0.495 \le x_1 \le 0.58, \ -0.2 \le x_2 \le -0.45\big\}. \end{aligned}$$

Putting in (14) $\varrho^{(0)} = \operatorname{col}(0.8, 0.9)$, $\varrho^{(1)} = \operatorname{col}(0.4, 0.5)$, for the pre-jump and after-jump curves we obtain the domains of form (10)

$$\begin{aligned} \Omega_0(\varrho^{(0)}) &= \big\{(x_1, x_2) : \ -0.53 \le x_1 \le 1.25, \ -0.7 \le x_2 \le 1.46\big\}, \\ \Omega_1(\varrho^{(1)}) &= \big\{(x_1, x_2) : \ 0.095 \le x_1 \le 0.98, \ -0.3 \le x_2 \le 0.95\big\}. \end{aligned}$$

Applying Maple 14, we carried out computations according to (12), (13) and, for several values of m, obtained from the corresponding approximate determining systems the numerical values for the parameters which are presented in the table below. Note that the third and the fourth columns contain the numerical values of the jump magnitude $\gamma = \operatorname{col}(\gamma_1, \gamma_2)$ and the seventh one shows approximate values of the jump time τ.

m	η_1	γ_1	γ_2	λ_1	λ_2	τ	z_2
0	0.54721759	0.2025730	−0.2247659	0.31711742	0.52806054	0.75344599	0.26343879
1	0.54721759	0.2238353	−0.2364152	0.29470392	0.52737158	0.72907772	0.26343879
2	0.54721759	0.2240919	−0.2382983	0.29444608	0.52911668	0.72879666	0.26343879
3	0.54721759	0.2240736	−0.2383145	0.29446539	0.52914210	0.72881771	0.26343879
4	0.54721759	0.2240731	−0.2383131	0.29446596	0.52914096	0.72881833	0.26343879
5	0.54721759	0.2240731	−0.2383131	0.29446598	0.52914093	0.72881835	0.26343879

The residual obtained by substituting the approximate solution of the fifth approximation into the pre-jump and after-jump equations is of order 10^{-7}.

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