# Investigation of Carathéodory Functional Boundary Value Problems by Division into Subintervals

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We study the problem

$$\frac{du(t)}{dt}=f(t,u(t)),\ \ t\in[a,b],\ \ \Phi(u)=d, \tag{1}$$

where $\Phi : C([a,b],\mathbb{R}^n)$ is a vector functional (possibly non-linear), $f:[a,b]\times\mathbb{R}^n\to\mathbb{R}^n$ is a function satisfying the Carathéeodory conditions in a certain bounded set, which will be specified below, and $d$ is a given vector.

Note that investigation of solutions of problem (1) in the paper [4] is based on reduction it to a certain simpler parametrized "model-type" problem

$$\frac{du(t)}{dt}=f(t,u(t)),\ \ t\in[a,b],\ \ u(a)=z,\ \ u(b)=\eta, \tag{2}$$

where $z:=col(z_1,\dots,z_n)$, $\eta:=col(\eta_1,\dots,\eta_n)$ are unknown parameters. Investigation of solutions of problems (2) was connected with the properties of the special sequence of functions $\{u_m(t,z,\eta)\}_{m=0}^{\infty}$ well posed on the interval $t\in[a,b]$. We note that the sufficient condition for the uniform convergence of sequence $\{u_m(t,z,\eta)\}_{m=0}^{\infty}$ consists in the assumption that the maximal in modulus eigenvalue of the matrix $Q=\frac{3(b-a)}{10}K$ is smaller than one, $r(Q)<1$, where $|f(t,u_1)-f(t,u_2)|\le K|u_1-u_2|$, a.e. $t\in[a,b]$, $u_1,u_2\in D$, $D$ is some closed bounded set. To improve twice this sufficient convergence condition, in [1–3, 6] a special interval halving and parametrization technique were suggested.

Following to the idea used in numerical methods for approximate solution of initial value problems for ordinary differential equations, let us fix a natural $N$ and choose $N+1$ grid points

$$t_k=t_{k-1}+h_k,\ \ k=1,\dots,N,\ \ t_0=a,\ \ t_N=b, \tag{3}$$

where $h_k$, $k=1,\dots,N$, are the corresponding step sizes. Thus, $[a,b]$ is divided into $N$ subintervals $[t_0,t_1],[t_1,t_2],[t_2,t_3],\dots,[t_{N-1},t_{1N}]$.

The aim of this note is to show that by using an $N$ subintervals divisions of type (3) and an appropriate parametrization technique one can $N$ times improve the sufficient convergence condition. It seems that in the case of boundary value problems interval division for approximations in analytic form was first used in [5].

Let us fix certain closed bounded sets $D^k \subset \mathbb{R}^n$, $k = 0, 1, 2, \dots, N$, and focus on the absolutely continuous solutions $u$ of problem (1) whose values at the nodes (3) lie in the corresponding sets $D^k$, i.e. $u(t_k) \in D^k$, $k = 0, 1, 2, \dots, N$.

Based on $D^k$ we introduce the sets

$$D_{k-1,k} := (1-\theta)z^{(k-1)} + \theta z^{(k)}, \quad z^{(k-1)} \in D^{k-1}, \ \ z^{(k)} \in D^k, \ \ \theta \in [0,1], \ \ k = 1, 2, \dots, N,$$

and its some componentwise $\rho^{(k)}$-vector neighbourhoods $D^{[k]} := B(D_{k-1,k}, \rho^{(k)})$, $k = 1, 2, \dots, N$, where $B(D_{k-1,k}, \rho^{(k)}) := \bigcup\limits_{\xi \in D_{k-1,k}} B(\xi, \rho^{(k)})$ and $B(\xi, \rho^{(k)}) := \{\nu \in \mathbb{R}^n : \ |\nu - \xi| \le \rho^{(k)}\}$. Recall that $D_{k-1,k}$ is the set of all possible straight line segments joining points of $D^{k-1}$ with points of $D^k$.

Let us "freeze" the values of $u$ at the nodes (3) by formally putting

$$u(t_k) = z^{(k)} = col(z_1^{(k)}, z_2^{(k)}, \dots, z_n^{(k)}), \ \ k = 0, 1, 2, \dots, N,$$

and consider the restrictions of equation (1) to each of the subintervals of the division (3).

Instead of (1) we introduce $N$ "model-type" problems

$$\frac{dx^{(k)}}{dt} = f(t, x^{(k)}), \ \ t \in [t_{k-1}, t_k], \ \ x(t_{k-1}) = z^{(k-1)}, \ \ x(t_k) = z^{(k)}, \ \ k = 1, 2, \dots, N, \tag{4}$$

where the vectors $z^{(0)}$, $z^{(1)}, \dots, z^{(N)} \in \mathbb{R}^n$ will be regarded as unknown parameters whose values are to be determined. Note that the length of the intervals in problems (4), which will be studied independently, are equal to step-size $h_k$ in opposition to $b - a$ in the case of the original BVP (1).

To study the solutions of (4) we will use the special parametrized successive approximations $x_m^{(k)}(t, z^{(k-1)}, z^{(k)})$ constructed in analytic form and well defined on the intervals $t \in [t_{k-1}, t_k]$, $k = 1, 2, \dots, N$, respectively.

**Assumption 1.** *There exist non-negative vectors $\rho^{(1)}, \rho^{(2)}, \dots, \rho^{(N)}$ such that*

$$\rho^{(k)} \ge \frac{h_k}{2}\, \delta_{[t_{k-1},t_k],D^{[k]}}(f) \ \ \textit{for all} \ \ k = 1, 2, \dots, N,$$

*where*

$$\delta_{[t_{k-1},t_k],D^{[k]}}(f) := \frac{1}{2}\Big[\operatorname*{ess\,sup}_{(t,x)\in[t_{k-1},t_k]\times D^{[k]}} f(t,x) - \operatorname*{ess\,inf}_{(t,x)\in[t_{k-1},t_k]\times D^{[k]}} f(t,x)\Big]. \tag{5}$$

**Assumption 2.** *There exist non-negative matrices $K_1, K_2, \dots, K_N$ such that*

$$\big|f(t,u_1) - f(t,u_2)\big| \le K_k|u_1 - u_2|, \ \ \textit{a.e.} \ \ t \in [t_{k-1}, t_k], \ \ u_1, u_2 \in D^{[k]}. \tag{6}$$

**Assumption 3.** *The maximal in modulus eigenvalue of the matrix $Q_k = \frac{3h_k}{10} K_k$, $k = 1, 2, \dots, N$, is smaller than one, $r(Q_k) < 1$.*

Let us define for problems (4) the recurrence parametrized sequences of functions

$$x_0^{(k)}(t, z^{(k-1)}, z^{(k)}) := z^{(k-1)} + \frac{(t - t_{k-1})}{h_k}\big[z^{(k)} - z^{(k-1)}\big] = \Big[1 - \frac{t - t_{k-1}}{h_k}\Big]z + \frac{t - t_{k-1}}{h_k}\, z^{(k)}, \tag{7}$$

$$t \in [t_{k-1}, t_k], \ \ k = 1, 2, \dots, N,$$

$$\begin{aligned} x_m^{(k)}(t, z^{(k-1)}, z^{(k)}) := z^{(k-1)} &+ \int\limits_{t_{k-1}}^{t} f\big(s, x_{m-1}^{(k)}(s, z^{(k-1)}, z^{(k)})\big)\, ds \\ &- \frac{t - t_{k-1}}{h_k} \int\limits_{t_{k-1}}^{t_k} f\big(s, x_{m-1}^{(k)}(s, z^{(k-1)}, z^{(k)})\big)\, ds + \frac{t - t_{k-1}}{h_k}\big[z^{(k)} - z^{(k-1)}\big], \end{aligned} \tag{8}$$

for all $m = 1, 2, \dots, z^{(k-1)} \in \mathbb{R}^n$, $z^{(k)} \in \mathbb{R}^n$ and $t \in [t_{k-1}, t_k]$, $k = 1, 2, \dots, N$.

**Theorem 1.** *Let Assumptions* 1–3 *hold. Then, for any fixed vectors* $(z^{(0)}, z^{(1)}, \dots, z^{(N)}) \in D^0 \times D^1 \times \dots \times D^N$ *and* $k = 1, 2, \dots, N$*:*

1. *The limit:* $\lim\limits_{m\to\infty} x_m^{(k)}(t, z^{(k-1)}, z^{(k)}) = x_\infty^{(k)}(t, z^{(k-1)}, z^{(k)})$, *exists uniformly in* $t \in [t_{k-1}, t_k]$.
2. *The limit function satisfies the conditions*

$$x_\infty^{(k)}(t_{k-1}, z^{(k-1)}, z^{(k)}) = z^{(k-1)}, \quad x_\infty^{(k)}(t_k, z^{(k-1)}, z^{(k)}) = z^{(k)}.$$

3. *The function* $x_\infty^{(k)}(t, z^{(k-1)}, z^{(k)})$ *is the unique absolutely continuous solution of the integral equation*

$$\begin{aligned} x^{(k)}(t) = z^{(k-1)} &+ \int\limits_{t_{k-1}}^{t} f(s, x^{(k)}(s))\, ds - \frac{t - t_{k-1}}{h_k} \int\limits_{t_{k-1}}^{t_k} f(s, x^{(k)}(s))\, ds \\ &+ \frac{t - t_{k-1}}{h_k} \big[z^{(k)} - z^{(k-1)}\big], \ \ t \in [t_{k-1}, t_k], \end{aligned}$$

*in the domain* $D^{[k]}$.

*In other words,* $x_\infty^{(k)}(t, z^{(k-1)}, z^{(k)})$ *is the unique solution of the following Cauchy problem for the modified system of integro-differential equations:*

$$\frac{dx^{(k)}}{dt} = f(t, x^{(k)}) + \frac{1}{h_k}\, \Delta^{(k)}(z^{(k-1)}, z^{(k)}), \ \ t \in [t_{k-1}, t_k], \ \ x(t_{k-1}) = z^{(k-1)},$$

*where* $\Delta^{(k)}(z^{(k-1)}, z^{(k)}) : D^{k-1} \times D^k \to \mathbb{R}^n$ *are the mapping given by formula*

$$\Delta^{(k)}(z^{(k-1)}, z^{(k)}) = z^{(k)} - z^{(k-1)} - \int\limits_{t_{k-1}}^{t_k} f(s, x^{(k)}(s))\, ds.$$

4. *The following estimates hold for* $m \ge 0$*:*

$$\begin{aligned} &\big|x_\infty^{(k)}(\,\cdot\,, z^{(k-1)}, z^{(k)}) - x_m^{(k)}(\,\cdot\,, z^{(k-1)}, z^{(k)})\big| \\ &\qquad \leqslant \frac{10}{9}\, \alpha_1(t, t_{k-1}, h_k) Q_k^m (1_n - Q_k)^{-1} \delta_{[t_{k-1}, t_k], D^{[k]}}(f), \ \ t \in [t_{k-1}, t_k], \end{aligned}$$

*where* $\delta_{[t_{k-1}, t_k], D^{[k]}}(f))$ *is given in* (5) *and*

$$|\alpha_1(t, t_{k-1}, h_k)| \le \frac{h_k}{2}\,, \ \ t \in [t_{k-1}, t_k].$$

Theorem 1 guarantees that under the assumed conditions, the functions $x_\infty^{(k)}(t, z^{(k-1)}, z^{(k)}) : [t_{k-1}, t_k] \to \mathbb{R}^n$, $k = 1, 2, \dots, N$, are well defined for all $(z^{(k-1)}, z^{(k)}) \in D^{k-1} \times D^k$. Therefore, by putting

$$u_\infty(t, z^{(0)}, z^{(1)}, \dots, z^{(N)}) := \begin{cases} x_\infty^{(1)}(t, z^{(0)}, z^{(1)}), & \text{if } \ t \in [t_0, t_1], \\ x_\infty^{(2)}(t, z^{(1)}, z^{(2)}), & \text{if } \ t \in [t_1, t_2], \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ x_\infty^{(N)}(t, z^{(N-1)}, z^{(N)}), & \text{if } \ t \in [t_{N-1}, t_N] \end{cases} \tag{9}$$

we obtain a function $u_\infty(\,\cdot\,, z^{(0)}, z^{(1)}, \dots, z^{(N)}) : [a,b] \to \mathbb{R}^n$, which is well defined for the values $z^{(k)} \in D^k$, $k = 0, 1, 2, \dots, N$. This function is obviously continuous since at the points $t = t_k$ we have

$$x_\infty^{(k)}(t_k, z^{(k-1)}, z^{(k)}) = x_\infty^{(k)}(t_k, z^{(k)}, z^{(k+1)}), \ \ k = 1, 2, \dots, N.$$

**Theorem 2.** *Let the conditions of Theorem* 1 *hold. Then:*

1. *The function* $u_\infty(t, z^{(k-1)}, z^{(k)}) : [a,b] \to \mathbb{R}^n$ *defined by* (9) *is an absolutely continuous solution of problem* (1) *if and only if the vectors* $z^{(k)}$, $k = 0, 1, 2, \dots, N$, *satisfy the system of* $n(N+1)$ *numerical equations*

$$\Delta^{(k)}(z^{(k-1)}, z^{(k)}) = z^{(k)} - z^{(k-1)} - \int\limits_{t_{k-1}}^{t_k} f\big(s, x_\infty^{(k)}(s, z^{(k-1)}, z^{(k)})\big)\, ds = 0, \ \ k = 1, 2, \dots, N,$$
$$\Delta^{(N+1)}(z^{(0)}, z^{(1)}, \dots, z^{(N)}) = \Phi\big(u_\infty(\,\cdot\,, z^{(0)}, z^{(1)}, \dots, z^{(N)})\big) - d = 0. \tag{10}$$

2. *For every solution* $U(\,\cdot\,)$ *of problem* (1) *with* $U(t_k) \in D^k$, $k = 0, 1, 2, \dots, N$, *there exist vectors* $z^{(k)}$, $k = 0, 1, \dots, N$, *such that* $U(\,\cdot\,) = u_\infty(\,\cdot\,, z^{(0)}, z^{(1)}, \dots, z^{(N)})$, *where the function* $u_\infty(\,\cdot\,, z^{(0)}, z^{(1)}, \dots, z^{(N)})$ *is given in* (9).

Although Theorem 2 provides a theoretical answer to the question on the construction of a solution of the BVP (1), its application faces difficulties due to the fact that the explicit form of $x_\infty^{(j)}(s, z^{(j-1)}, z^{(j)})$ and the functions $\Delta^{(k)}(z^{(k-1)}, z^{(k)}) : D^{k-1} \times D^k \to \mathbb{R}^n$, $k = 1, 2, \dots, N$, $\Delta^{(N+1)}(z^{(0)}, z^{(1)}, \dots, z^{(N)}) : D^0 \times D^1 \times \cdots \times D^N \to \mathbb{R}^n$, appearing in (10) is usually unknown. This complication can be overcome by using $x_m^{(k)}(s, z^{(k-1)}, z^{(k)})$ of form (8) for a fixed $m$, which will lead one to the so-called *approximate determining equations:*

$$\Delta_m^{(k)}(z^{(k-1)}, z^{(k)}) = z^{(k)} - z^{(k-1)} - \int\limits_{t_{k-1}}^{t_k} f\big(s, x_m^{(k)}(s, z^{(k-1)}, z^{(k)})\big)\, ds = 0, \ \ k = 1, 2, \dots, N,$$
$$\Delta_m^{(N+1)}(z^{(0)}, z^{(1)}, \dots, z^{(N)}) = \Phi\big(u_m(\,\cdot\,, z^{(0)}, z^{(1)}, \dots, z^{(N)})\big) - d = 0. \tag{11}$$

Note that, unlike system (10), the $m$-th approximate determining system (11) contains only terms involving the functions $x_m^{(j)}(\,\cdot\,, z^{(j-1)}, z^{(j)})$ which are explicitly known.

It is natural to expect that approximations to the unknown solution of problem (1) can be obtained by using the function

$$u_m(t, \widetilde{z}^{(0)}, \widetilde{z}^{(1)}, \dots, \widetilde{z}^{(N)}) := \begin{cases} x_m^{(1)}(t, \widetilde{z}^{(0)}, \widetilde{z}^{(1)}), & \text{if } \ t \in [t_0, t_1], \\ x_m^{(2)}(t, \widetilde{z}^{(1)}, \widetilde{z}^{(2)}), & \text{if } \ t \in [t_1, t_2], \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ x_m^{(N)}(t, \widetilde{z}^{(N-1)}, \widetilde{z}^{(N)}), & \text{if } \ t \in [t_{N-1}, t_N] \end{cases}$$

where $\widetilde{z}^{(k)} \in D^k$, $k = 0, 1, 2, \dots, N$, are solutions of the numerical system (11).

The constructivity of a suggested technique is shown on the following example with four absolute continuous solutions:

$$\frac{du_1(t)}{dt}=\begin{cases} u_1u_2-\dfrac{48}{25}\,t^3+\dfrac{44}{25}\,t^2-\dfrac{17}{100}\,t-\dfrac{7}{10}\,, & t\in\Big[0,\dfrac{1}{4}\Big],\\ u_1u_2+\dfrac{48}{25}\,t^3-\dfrac{28}{25}\,t^2-\dfrac{131}{20}\,t+\dfrac{483}{200}\,, & t\in\Big[\dfrac{1}{4},\dfrac{1}{2}\Big],\end{cases}$$

$$\frac{du_2(t)}{dt}=\begin{cases} t(u_1-u_2)-\dfrac{16}{5}\,t^3+\dfrac{7}{5}\,t^2-\dfrac{131}{20}\,t+\dfrac{4}{5}\,, & t\in\Big[0,\dfrac{1}{4}\Big],\\ t(u_1-u_2)+\dfrac{16}{5}\,t^3-\dfrac{9}{5}\,t^2+\dfrac{1}{4}\,t+\dfrac{3}{5}\,, & t\in\Big[\dfrac{1}{4},\dfrac{1}{2}\Big],\end{cases}$$

$$\begin{cases} \displaystyle\int_0^{\frac{1}{2}} u_1^2(s)\,ds=\frac{47}{1000}\,,\\ \displaystyle\int_0^{\frac{1}{2}} u_2^2(s)\,ds=\frac{47}{1000}\,.\end{cases}$$

For $N=2$, $t_0=0$, $t_1=\frac{1}{4}$, $t_2=\frac{1}{2}$, $m=5$ these four solutions are defined by the approximate values of parameters $z^{(0)}$, $z^{(1)}$, $z^{(2)}$ given in table.

| | 1-solution | 2-solution | 3-solution | 4-solution |
|---|---|---|---|---|
| $\widetilde{z}_1^{(0)}$ | 0.3999999998 | 0.4469892219 | −0.1615332331 | −0.2084976508 |
| $\widetilde{z}_2^{(0)}$ | 0.25 | −0.3803603881 | 0.2769448823 | −0.3583253898 |
| $\widetilde{z}_1^{(1)}$ | 0.2499999998 | 0.2446667248 | −0.3540518758 | −0.3583375962 |
| $\widetilde{z}_2^{(1)}$ | 0.2500000001 | −0.3606725966 | 0.2579658912 | −0.3583910008 |
| $\widetilde{z}_1^{(2)}$ | 0.2499999998 | 0.2046115983 | −0.4035821965 | −0.3584724797 |
| $\widetilde{z}_2^{(2)}$ | 0.4000000003 | −0.1585615166 | 0.3508654384 | −0.2082301147 |

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