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STABILITY OF LIMIT REGIMES IN GENERAL REACTION-DIFFUSION TYPE SYSTEMS

In this paper, we consider the stability of limit regimes for a general class of nonlinear distributed mathematical models named Reaction-Diffusion models. RD systems naturally arise in many applications. For instance, in the biological mathematical modeling and in the signal transmission theory the FitzHugh–Nagumo model, whose distributed variant is a particular case of general RD system, is widely used. We investigate the problem of stability of attracting sets for an infinite-dimensional RD system with respect to bounded external signals (disturbances). The interaction functions as well as nonlinear perturbations do not assume to be Lipschitz continuous. Therefore, we cannot expect the uniqueness of solution for the corresponding initial-value problem and we have to use a multi-valued semigroup approach. An undisturbed system is considered to have a global attractor, i.e., the minimal compact uniformly attracting set. The main purpose is to estimate the deviation of the trajectory of the disturbed system from the global attractor of the undisturbed one as a function of the magnitude of external signals. Such an estimate can be obtained in the framework of the theory of input-to-state stability (ISS). The paper proposes a new approach to obtaining estimates of robust stability of the attractor in the case of a multivalued evolutionary operator. In particular, it is proved that the multivalued semigroup generated by weak solutions of a nonlinear reaction-diffusion system has the property of local ISS with respect to the attractor of the undisturbed system.

Keywords: reaction-diffusion system, system without uniqueness, input-to-state stability, robust stability, global attractor.

1. Introduction. Conditions of practical stabilization of differential inclusions and properties of optimal sets of practical stability of differential inclusions with a spatial component were studied in [1], [2]. Important results of practical stability, conditions of practical stability were obtained in works [3], [4].

In the present paper we investigate stability of limit regimes in general reaction-diffusion type systems. In dissipative evolutionary systems it is a common view to characterize such regimes in terms of the global attractor theory [5]–[8]. For ill-posed problems when there are no results about uniqueness or regularity of solutions, and for the control problems with singular perturbations the corresponding theory was developed in [9]–[15]. If the considered autonomous system with global attractor undergoes external signals (disturbances) then the natural problem is to estimate the deviation of the trajectory of the disturbed system from the global attractor of the undisturbed one as a function of the magnitude of external signals. Such an estimate can be obtained in the framework of the theory of input-to-state stability (ISS) [16]–[19]. Recently this theory has been developed to infinite-dimensional systems with non-trivial attractors in [13]– [21]. In particular, the local input-to-state stability and asymptotic gain properties were obtained for well-posed semilinear parabolic and hyperbolic equations. The ISS property for the attractor of a PDE-ODE type system (which consisting of a parabolic system of the reaction-diffusion type and a system of ordinary differential equations) undergoing additive bounded perturbations was explored in paper [22].

In the present paper we generalize these results for more general classes of PDEs like reaction-diffusion systems with non-smooth interaction functions which do not guarantee the uniqueness of solutions of the corresponding initial value problem in the natural infinite-dimensional phase spaces.

2. Setting of the problem. In bounded domain $\Omega \subset \mathbb{R}^n$ we consider the following general reaction-diffusion system

$$\begin{cases} u_t = a\Delta u - f(u) + h(x) + g(u)d(t), \ x \in \Omega, \ t > 0, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{1}$$

where $u = u(t,x) = (u^1(t,x), \dots, u^N(t,x))$ is an unknown vector-function, $h = (h^1, \dots, h^N)$, $f = (f^1, \dots, f^N)$ are given vector-functions, $g = ((g^{ij}))_{i,j=1}^N$ is a given matrix-valued function, a is a real $N \times N$ matrix such that $\frac{1}{2}(a + a^*) \geq \mu I$, $\mu > 0$, $d = (d^1, \dots, d^N)$ is an external signal (disturbances).

Under rather general assumption (see the last section) we can claim that this problem is globally resolvable in weak sense in the phase space $H = (L^2(\Omega))^N$, i.e., for any disturbances $d \in L^\infty(\mathbb{R}_+; H)$ and for any $u_0 \in H$ there exists (maybe not unique) solution of the problem (1) $u \in C([0, +\infty); H)$ with $u(0) = u_0$.

Let us consider undisturbed system ($d \equiv 0$)

$$\begin{cases} u_t = a\Delta u - f(u) + h(x), \ x \in \Omega, \ t > 0, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{2}$$

It is known [7], that the corresponding multi-valued semiflow (m-semiflow for short)

$$S(t, u_0) = \{u(t) | \ u(\cdot) \text{ is a weak solution (2)}, \ u(0) = u_0\} \tag{3}$$

has a global attractor Θ in H, i.e., there exists a compact set $\Theta \subset H$ such that

(i) $\Theta = S(t, \Theta), \ t \geq 0$,

(ii) for any bounded set $B \subset H$

$$dist(S(t,B), \Theta) \to 0 \text{ as } t \to \infty,$$

where for $A, B \subset H$ we denote

$$S(t, B) = \bigcup_{b \in B} S(t, b),$$

$$dist(A, \Theta) = \sup_{x \in A} \inf_{y \in \Theta} \|x - y\|_H,$$

$$\|A\|_\Theta = dist(A, \Theta).$$

Property (ii) means that all trajectories of the system ultimately belong to a given neighborhood of the global attractor is stable in Liapunov sense, i.e.,

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall t \geq 0 \ S(t, O_\delta(\Theta)) \subset O_\varepsilon(\Theta), \tag{4}$$

where here and after

$$O_\delta(A) = \{x \in H | \ dist(x, A) < \delta\}, \text{ for } A \subset H.$$

Thus, the limit behavior of undisturbed system is completely determined by the global attractor Θ. The main question is how external disturbances d affect such dynamics? Let the initial point $u_0 := u(0) \in H$ and disturbances d are given. Let us denote by $S_d(t, 0, u_0)$ the set of all solutions of (1) with $u(0) = u_0$. It is not true in general that those solutions will converge to Θ as $t \to \infty$. But it turns out that under additional assumptions it possible to show that we can estimate the deviation of the trajectory of the disturbed system from the global attractor of the undisturbed one as a function of the magnitude of external signals. For that purpose, we need the following classes of comparison functions [21]:

$$\mathcal{K} := \{\gamma : [0, +\infty) \mapsto [0, +\infty) \,|\, \gamma \text{ is continuous, strictly increasing, } \gamma(0) = 0\},$$

$$\mathcal{K}_\infty := \{\gamma \in \mathcal{K} \,|\, \gamma \text{ is unbounded}\},$$

$$\mathcal{L} := \{\gamma : [0, +\infty) \mapsto [0, +\infty) \,|\, \gamma \text{ is continuous, strictly decreasing, } \ \gamma(t) \to 0, \ t \to \infty\},$$

$$\mathcal{KL} := \{\beta : [0, +\infty) \times [0, +\infty) \mapsto [0, +\infty) \,|\, \beta \text{ is continuous, } \beta(\cdot, t) \in \mathcal{K}, \ \forall t \geq 0, \ \beta(s, \cdot) \in \mathcal{L}, \ \forall s > 0\}.$$

First, under rather general assumptions, we will show that for the m-semiflow S, generated by undisturbed system (2), with global attractor Θ $\exists \beta \in \mathcal{KL}$ $\forall u_0 \in H, \ \forall t \geq 0$

$$\|S(t, u_0)\|_\Theta \leq \beta(\|u_0\|_\Theta, t).$$

This property helps us to prove the main result about robust stability of our disturbed system (1): $\exists r > 0, \ \exists \beta \in \mathcal{KL}, \ \exists \gamma \in \mathcal{K}$ such that

$$\|u_0\|_\Theta \leq r, \ \|d\|_\infty \leq r \ \Rightarrow \ \forall t \geq 0$$

$$\|S_d(t, 0, u_0)\|_\Theta \leq \beta(\|u_0\|_\Theta, t) + \gamma(\|d\|_\infty), \tag{5}$$

where $\|d\|_\infty = \operatorname*{ess\,sup}_{t \geq 0} |d(t)|$.

3. Stability of attractors for multi-valued semiflows in abstract spaces. We consider an abstract evolutionary (autonomous) system, which is characterized by a normed phase space $(X, \|\cdot\|)$ and a family of maps (solutions) $K \subset \subset \mathbb{C}\,([0, +\infty); X)$ such that the following conditions hold:

(K1) $\forall x \in X \ \exists \varphi \in K$ such that $\varphi(0) = x$;

(K2) $\varphi_\tau(\cdot) := \varphi(\cdot + \tau) \in K, \forall \tau \geq 0, \forall \varphi \in K$.

Than the multi-valued map $S : \mathbb{R}_+ \times X \mapsto 2^X$

$$S(t, x) = \{\varphi(t) | \ \varphi \in K, \ \varphi(0) = x\},$$

is semiflow. Moreover,

$$\varphi(t+s) \in S(t, \varphi(s)),\ \forall \varphi \in K,\ \forall t, s \geq 0.$$

Assume, that S is strict, i.e., $S(t+s, x) = S(t, S(s, x))$. The last equality allows us to state existence of invariant global attractor.

Lemma 1. *Assume that* (K1), (K2) *hold, S is strict, and*

(G1) *exists bounded $B_0 \subset X$ such that for all bounded $B \subset X$ $\exists T = T(B)$ $\forall t \geq T$ $S(t, B) \subset B_0$ (dissipativity),*

(G2) *$\forall t_n \nearrow \infty$, for all bounded $B \subset X$, $\forall \xi_n \in S(t_n, B)$ the sequence $\{\xi_n\}$ is precompact (asymptotic compactness),*

(G3) *$\forall t > 0$, $\forall x_n \to x_0$, $\forall \xi_n \in S(t, x_n)$, $\xi_n \to \xi_0$ we have: $\xi_0 \in S(t, x_0)$ (closed graph).*

Then m-semiflow S possesses global attractor Θ.

Moreover, if

(G4) *$\forall t_n \to t_0 \geq 0$, $\forall x_n \to x_0$, $\forall \xi_n \in S(t_n, x_n)$ up to sequence $\xi_n \to \xi_0 \in S(t_0, x_0)$*

then Θ is stable in the sense of (4).

Lemma 2. *Assume that $S : \mathbb{R}_+ \times X \mapsto 2^X$ is a strict m-semiflow, which has a stable global attractor Θ. Also, assume that*

$$\text{for all bounded } B \subset X \text{ the set } \bigcup_{t \geq 0} S(t, B) \text{ is bounded.} \tag{6}$$

Then $\exists \beta \in \mathcal{KL}$ $\forall x \in X$, $\forall t \geq 0$

$$\|S(t, x)\|_\Theta \leq \beta(\|x\|_\Theta, t). \tag{7}$$

Now assume that our evolutionary system undergoes disturbances $d \in U$, where the set U satisfies

(U) $U \subset L^\infty(\mathbb{R}_+)$, $0 \in U$, U is translation-invariant, i.e.,

$$d_h(\cdot) = d(\cdot + h) \in U,\ \forall h \geq 0,\ \forall d(\cdot) \in U.$$

Denote by $K_d^\tau \subset \mathbb{C}([\tau, +\infty); X)$ the family of maps satisfying the following properties:

(S1) $\forall x \in X$, $\forall \tau \geq 0$, $\forall d \in U$ $\exists \varphi \in K_d^\tau :$ $\varphi(\tau) = x$,

(S2) $\varphi|_{[s,+\infty)} \in K_d^s$, $\forall \varphi \in K_d^\tau$, $\forall s \geq \tau$,

(S3) $\varphi(\cdot + h) \in K_{d(\cdot+h)}^\tau$, $\forall \varphi \in K_d^{\tau+h}$, $\forall h \geq 0$.

Let us put

$$S_d(t, \tau, x) := \{\varphi(t) |\ \varphi \in K_d^\tau,\ \varphi(\tau) = x\}.$$

Then $\{S_d\}_{d \in U}$ generates the family of m-semiprocesses, i.e., $\forall d \in U$, $\forall t \geq s \geq$ $\geq \tau \geq 0$, $\forall x \in X$, $\forall h \geq 0$.

$$S_d(t, \tau, x) = x,$$

$$S_d(t, \tau, x) \subset S_d(t, s, S_d(s, \tau, x)),$$

$$S_d(t + h, \tau + h, x) \subset S_{d(\cdot+h)}(t, \tau, x).$$

It is easy to verify that $\{S_d\}_{d\in U}$ satisfies cocycle property:

$$S_d(t+h,0,x) \subset S_d(t+h,h,S_d(h,0,x)) \subset S_{d(\cdot+h)}(t,0,S_d(h,0,x)),$$

and $\forall \varphi \in K_d^\tau$ $\varphi(t) \in S_d(t,s,\varphi(s))$.

In particular, $\forall \varphi \in K_d^0$, $\forall t,h \geq 0$

$$\varphi(t+h) \in S_d(t+h,h,\varphi(h)) \subset S_{d(\cdot+h)}(t,0,\varphi(h)). \tag{8}$$

(S4) Moreover, if $\forall s \geq \tau$, $\forall \psi \in K_d^\tau$, $\forall \varphi \in K_d^s$ with $\psi(s) = \varphi(s)$ the function

$$\Theta(p) = \begin{cases} \psi(p), & p \in [\tau,s], \\ \varphi(p), & p \geq s, \end{cases}$$

belongs to K_d^τ, then inclusion $S_d(t,\tau,x) \subset S_d(t,s,S_d(s,\tau,x))$ takes place.

(S5) If $\forall h \geq 0$, $\forall \varphi \in K_{d(\cdot+h)}^\tau$ we have that $\varphi(\cdot - h) \in K_d^{\tau+h}$, then inclusion $S_d(t+h,\tau+h,x) \subset S_{d(\cdot+h)}(t,\tau,x)$ takes place.

Lemma 3. *Under conditions* (U), (S1)–(S5) *for the semiprocess family* $\{S_d\}_{d\in U}$ *we have that* $\{S_d\}_{d\in U}$ *is strict, i.e.*

$$S_d(t,\tau,x) = S_d(t,s,S_d(s,\tau,x)),$$

$$S_d(t+h,\tau+h,x) = S_{d(\cdot+h)}(t,\tau,x),$$

$$S_d(t+h,0,x) = S_{d(\cdot+h)}(t,0,S_d(h,0,x)).$$

In particular, in the undisturbed case ($d \equiv 0$)

$$S_0(t+h,0,x) = S_0(t,0,S_0(h,0,x)),$$

so S_0 is a strict m-semiflow.

The next theorem is the main abstract result of the paper.

Theorem 1. *Assume that m-semiflow S_0 is generated by family of maps K satisfying* (K1), (K2), *S_0 is strict, has compact values, and possesses stable global attractor Θ.*

Additionally, exists locally bounded function $c : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\forall r > 0,\ \forall t \geq 0,$$

$$\|x_1\| \leq r,\ \|x_2\| \leq r \ \Rightarrow\ dist(S_0(t,0,x_1), S_0(t,0,x_2)) \leq e^{c(r)t}\|x_1 - x_2\|. \tag{9}$$

Assume that $\{S_d\}_{d\in U}$ is the family of m-semiprocesses satisfying (U), (S1)–(S5), *where $d \in U$ is disturbances of the initial system S_0.*

Assume that $\exists \sigma \in \mathcal{K}$, exists continuous function $D : \mathbb{R}_+^2 \to \mathbb{R}_+$ such that $\forall r > 0$ $\overline{\lim\limits_{t\to 0+}} \frac{d(r,t)}{t} < \infty$, and $\forall t \geq 0$

$$\|d\|_\infty \leq r,\ \|x\| \leq r \ \Rightarrow$$

$$dist(S_d(t,0,x), S_0(t,0,x)) \leq D(r,t)\sigma(\|d\|_\infty). \tag{10}$$

Assume, that

$$\forall r > 0 \text{ the set } \bigcup_{t\geq 0} \bigcup_{\|d\|_\infty \leq r} \bigcup_{\|x\|\leq r} S_d(t,0,x) \text{ is bounded.} \tag{11}$$

Then $\{S_d\}_{d\in U}$ is local ISS w.r.t. Θ, i.e., inequality (5) holds.

Proof. First let us prove that $\forall r>0\ \exists \underline{\psi}, \overline{\psi}, \alpha \in \mathcal{K}$, exists Lipschitz continuous function V with Lipschitz constant equals 1, such that

$$\underline{\psi}(\|x\|_\Theta) \le V(x) \le \overline{\psi}(\|x\|_\Theta),\ \forall \|x\|_\Theta \le r, \tag{12}$$

$$\dot{V}_0(x) := \overline{\lim_{t\to 0^+}} \frac{1}{t} dist(V(S_0(t,0,x)), V(x)) \le -\alpha(\|x\|_\Theta)\ \forall \|x\|_\Theta \le r, \tag{13}$$

where here and after for $A \subset X,\ V(A) = \bigcup\limits_{a\in A} V(a)$.

For this purpose, we choose function β from (7), fix $r_0 > 0$ and $\forall \varepsilon > 0$ let $T = T(r_0, \varepsilon)$ be such that

$$\beta(r_0, t) \le \varepsilon\ \ \forall t \ge T. \tag{14}$$

We put

$$V^\varepsilon(x) := e^{-(c_0+c)T} \sup_{t\ge 0}(e^{ct}\eta_\varepsilon(\|S_0(t,0,x)\|_\Theta)), \|x\|_\Theta < r_0,$$

where $c_0 = c(r_0)$ is taken from (9), $c > 0$ will be fixed throughout the proof, $\eta_\varepsilon(r) := \max\{0, r-\varepsilon\}$. Due to (14)

$$V^\varepsilon(x) = e^{-(c_0+c)T} \sup_{t\in[0,T]} (e^{ct}\eta_\varepsilon(\|S_0(t,0,x)\|_\Theta)).$$

Using elementary properties of η_ε:

$$\eta_\varepsilon(r) \le r, \quad |\eta_\varepsilon(r_1) - \eta_\varepsilon(r_2)| \le |r_1 - r_2|,$$

we get the following properties of V^ε:

$$V^\varepsilon(x) \le e^{-c_0 T} \sup_{t\in[0,T]} \eta_\varepsilon(\|S_0(t,0,x)\|_\Theta) \le \beta(\|x\|_\Theta, 0), \quad \forall \|x\|_\Theta \le r_0,$$

and

$$\begin{aligned} |V^\varepsilon(x) - V^\varepsilon(y)| &\le e^{-(c_0+c)T} \times \\ &\quad \times \sup_{t\in[0,T]} |e^{ct}\eta_\varepsilon(\|S_0(t,0,x)\|_\Theta) - e^{ct}\eta_\varepsilon(\|S_0(t,0,y)\|_\Theta)| \le \\ &\le e^{-c_0T} \sup_{t\in[0,T]} |\|S_0(t,0,x)\|_\Theta) - \|S_0(t,0,y)\|_\Theta)| \le \\ &\le e^{-c_0T} \sup_{t\in[0,T]} \text{dist}\,(S_0(t,0,x), S_0(t,0,y)) \le \\ &\le e^{-c_0T} e^{c_0T} \|x-y\| = \\ &= \|x-y\|, \quad \forall\, \|x\|_\Theta \le r_0, \quad \forall\, \|y\|_\Theta \le r_0. \end{aligned}$$

Here, we utilized the inequality

$$\text{dist}\,(A,B) \le \text{dist}\,(A,C) + \text{dist}\,(C,B),$$

with $A = S_0(t,0,x),\ B = \Theta,\ C = S_0(t,0,y)$.

Due to compactness of Θ we have that $\forall \|x\|_\Theta < r_0$

$$\|x\|_\Theta = \inf_{\xi\in\Theta} \|x-\xi\| = \|x-\xi_0\|,\ \xi_0 \in \Theta.$$

Then due to (9)

$$\mathrm{dist}\,(S_0(t,0,x),S_0(t,0,\xi_0)) \le e^{c_0 t}\|x-\xi_0\|.$$

Invariance of Θ implies the inclusion

$$S_0(t,0,\xi_0)\subset\Theta.$$

Therefore,

$$\mathrm{dist}\,(S_0(t,0,x),S_0(t,0,\xi_0)) \ge \|S_0(t,0,x)\|_\Theta.$$

So, from the strict inequality $\|x\|_\Theta < r_0$ we derive that for sufficiently small $\tau>0$

$$\|S_0(\tau,0,x)\| < r_0.$$

Then $\forall\varphi\in K:\ \varphi(0)=x$, we get from the strictness of S_0

$$\begin{aligned}V^\varepsilon(\varphi(\tau)) &= e^{-(c_0+c)T}\sup_{t\ge 0}(e^{ct}\eta_\varepsilon(\|S_0(t,0,\varphi(\tau))\|_\Theta)) \le\\ &\le e^{-(c_0+c)T}\sup_{t\ge 0}(e^{ct}\eta_\varepsilon(\|S_0(t+\tau,0,x)\|_\Theta)) \le\\ &\le e^{-c\tau}V^\varepsilon(x) \text{ for sufficiently small } \tau>0.\end{aligned}$$

Due to compactness of $S_0(t,0,x)$ we deduce: for every small $\tau>0$ $\exists\varphi\in$ $\in K,\ \varphi(0)=x$ such that

$$\mathrm{dist}\,(V^\varepsilon(S_0(\tau,0,x)),V^\varepsilon(x)) = V^\varepsilon(\varphi(\tau))-V^\varepsilon(x) \le (e^{-c\tau}-1)V^\varepsilon(x).$$

Therefore,

$$\dot{V}_0^\varepsilon(x) := \overline{\lim_{t\to 0+}}\frac{1}{t}dist(V^\varepsilon(S_0(t,0,x)),V^\varepsilon(x)) \le -cV^\varepsilon(x),\ \|x\|_\Theta<r_0.$$

Now, for every $\|x\|_\Theta\le r_0$, we put

$$V(x) := \sum_{k=1}^{\infty}2^{-k}V^{\frac{1}{k}}(x).$$

Then from the previous arguments, we get

$$\begin{gathered}V(x)\le\beta(\|x\|_\Theta,0),\ \|x\|_\Theta\le r_0,\\ |V(x)-V(y)|\le\|x-y\|,\ \|x\|_\Theta\le r_0,\ \|y\|_\Theta\le r_0,\end{gathered}$$

$\forall\varphi\in K,\ \varphi(0)=x$ for sufficiently small $\tau>0$

$$V(\varphi(\tau))\le e^{-c\tau}V(x),$$

and therefore,

$$\mathrm{dist}\,(V(S_0(\tau,0,x)),V(x)) \le (e^{-c\tau}-1)V(x).$$

So,

$$\dot{V}_0(x)\le -cV(x),\ \|x\|_\Theta<r_0.$$

Moreover, inequality

$$\sup_{t\geq 0}\left(e^{ct}\eta_{\frac{1}{k}}(\|S_0(t,0,x)\|_\Theta)\right)\geq \eta_{\frac{1}{k}}(\|x\|_\Theta),$$

implies

$$V(x)\geq \sum_{k=1}^{\infty}2^{-k}e^{-(c_0+c)T(\frac{1}{k})}\eta_{\frac{1}{k}}(\|x\|_\Theta),\ \|x\|_\Theta\leq r_0.$$

Finally, denoting

$$\overline{\psi}(r)=\beta(r,0)+r,$$
$$\underline{\psi}(r)=\sum_{k=1}^{\infty}2^{-k}e^{-(c_0+c)T(\frac{1}{k})}\eta_{\frac{1}{k}}(r),$$
$$\alpha(r)=c\underline{\psi}(r),$$

we obtain (12),(13).

Then for $\forall\|x\|_\Theta<1,\ \forall u\in U:\ \|u\|_\infty\leq 1,\ \forall\varphi\in K_u^0:\varphi(0)=x$, let us consider for $t>0$ the upper right-hand Dini derivative [23]

$$\overline{D}^+V(\varphi(t))=\overline{\lim_{\tau\to 0+}}\frac{1}{\tau}(V(\varphi(t+\tau))-V(\varphi(t))).$$

According to property (8)

$$\varphi(t+\tau)\in S_u(t+\tau,0,x)\subset S_{u(\cdot+t)}(\tau,0,\varphi(t)).$$

From (11), for some $r_0>0$, $\|\varphi(t)\|<r_0\ \forall t\geq 0$. We fix such r_0 in all previous arguments. So, in view of (10), we can write

$$\begin{aligned}V(\varphi(t+\tau))-V(\varphi(t))&\leq \operatorname{dist}(V(S_{u(\cdot+t)}(\tau,0,\varphi(t))),V(\varphi(t)))\leq\\ &\leq \operatorname{dist}\left(V(S_{u(\cdot+t)}(\tau,0,\varphi(t))),V(S_0(\tau,0,V(S_{u(\cdot+t)}(\tau,0,\varphi(t)))))\right)+\\ &\quad+\operatorname{dist}(V(S_0(\tau,0,V(S_{u(\cdot+t)}(\tau,0,\varphi(t))))),V(\varphi(t)))\leq\\ &\leq d(r_0,\tau)\sigma(\|u\|_\infty)+(e^{-c\tau}-1)V(\varphi(t)).\end{aligned}$$

It means that

$$\overline{D}^+V(\varphi(t))\leq -cV(\varphi(t))+\overline{d}\sigma(\|u\|_\infty),\quad \forall t>0,\tag{15}$$

where $\overline{d}=\overline{\lim_{\tau\to 0+}}\frac{d(r_0,\tau)}{\tau}$.

Due to the properties of upper limit, we get from (15):

$$\overline{D}^+\left(V(\varphi(t))e^{ct}\right)\leq -\overline{D}^+\left(-\frac{\overline{d}\sigma(\|u\|_\infty)}{c}e^{ct}\right),$$
$$\overline{D}^+\left(V(\varphi(t))e^{ct}-\frac{\overline{d}\sigma(\|u\|_\infty)}{c}e^{ct}\right)\leq 0.\tag{16}$$

Then inequality (16) implies that (see [23])

$$V(\varphi(t))e^{ct}-\frac{\overline{d}\sigma(\|u\|_\infty)}{c}e^{ct}\leq V(x)-\frac{\overline{d}\sigma(\|u\|_\infty)}{c},\quad \forall t\geq 0.$$

So,

$$V(\varphi(t)) \le V(x)e^{-ct} + \frac{\overline{d}}{c}\sigma(\|u\|_\infty), \quad \forall t \ge 0.$$

Finally,

$$\begin{aligned} \underline{\psi}(\|\varphi(t)\|_\Theta) &\le \overline{\psi}(\|x\|_\Theta)e^{-ct} + \frac{\overline{d}}{c}\sigma(\|u\|_\infty), \\ \|\varphi(t)\|_\Theta &\le \underline{\psi}^{-1}(\overline{\psi}(\|x\|_\Theta)e^{-ct} + \frac{\overline{d}}{c}\sigma(\|u\|_\infty)) \le \\ &\le \frac{1}{2}\underline{\psi}^{-1}\left(2\overline{\psi}(\|x\|_\Theta)e^{-ct}\right) + \frac{1}{2}\underline{\psi}^{-1}\left(\frac{2\overline{d}}{c}\sigma(\|u\|_\infty)\right). \end{aligned} \tag{17}$$

If we denote

$$\begin{aligned} \beta(r,s) &:= \frac{1}{2}\underline{\psi}^{-1}\left(2\overline{\psi}(\|x\|_\Theta)e^{-cs}\right), \\ \gamma(r) &:= \frac{1}{2}\underline{\psi}^{-1}\left(\frac{2\overline{d}}{c}\sigma(r)\right), \end{aligned}$$

then inequality (17) implies the required local ISS property.

Theorem is proved.

4. Application to reaction-diffusion systems. We consider the problem

$$\begin{cases} u_t = a\Delta u - f(u) + h(x) + g(u)d(t), \ x \in \Omega, t > 0, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{18}$$

Suppose that all components of functions f, g belong to the class $C(\mathbb{R})$, $h \in (L^2(\Omega))^N$, $\exists C_1, C_2, C_3 > 0$, $\gamma_i > 0, p_i \ge 2$, $i = \overline{1,N}$ such that $\forall v \in \mathbb{R}^N$

$$\sum_{i=1}^N |f^i(v)|^{\frac{p_i}{p_i-1}} \le C_1(1 + \sum_{i=1}^N |v^i|^{p_i}), \tag{19}$$

$$\sum_{i=1}^N f^i(v)v^i \ge \sum_{i=1}^N \gamma_i |v^i|^{p_i} - C_2, \tag{20}$$

$$\|g(v)\|^2 := \sum_{i,j=1}^N |g^{ij}(v)|^2 \le C_3. \tag{21}$$

We will use the following standard functional spaces:

$$H = (L^2(\Omega))^N \text{ and } V = (H_0^1(\Omega))^N.$$

Let us denote

$$p = (p_1, \ldots, p_N), \ L^p(\Omega) = L^{p_1}(\Omega) \times \ldots \times L^{p_N}(\Omega).$$

It is known [24] that under assumptions (19)–(21) for any disturbances $d \in L^\infty(\mathbb{R}_+; H)$ (even for $d \in L^2_{loc}(\mathbb{R}_+; H)$) the problem (18) is globally resolvable in weak sense in the phase space H, i.e., for every $u_0 \in H$ there exists (maybe not unique) a function $u = u(t,x) \in L^2_{loc}(0,+\infty;V) \bigcap L^p_{loc}(0,+\infty;L^p(\Omega))$ such that for any $T > 0,\ v \in V \cap L^p(\Omega)$

$$\frac{d}{dt}\int\limits_{\Omega} u(t,x)v(x)dx + \int\limits_{\Omega}\Big(a\nabla u(t,x)\nabla v(x) +$$
$$+ f(u(t,x))v(x) - g(x)v(x) - d(t,x)v(x)\Big)dx = 0.$$

in the sense of scalar distributions on $(0,T)$, and $u(0,x) = u_0(x)$. Due to inclusion $u \in C([0,+\infty);H)$ the last equality makes sense. Moreover, every weak solutions of (18) belongs to the class of absolutely continuous functions from $[\tau,T]$ to H for every $T > \tau$, and for positive constants v, c_1, c_2 for a.a. $t > \tau$

$$\frac{d}{dt}\|u(t)\|^2 + v\|u(t)\|^2 \le c_1 + c_2\|d\|^2_\infty.$$

So,

$$\|u(t)\|^2 \le \|u(\tau)\|^2 e^{-v(t-\tau)} + \frac{1}{v}(c_1 + c_2\|d\|^2_\infty),\ \forall t \ge \tau. \tag{22}$$

Moreover, if $u_0^n \to u_0$ weakly in $H,\ d_n \to d$ weakly in $L^2(0,T)\ \forall T > 0$ then up to subsequence

$$\forall t > 0\ u_n(t) \to u(t) \text{ in } H, \tag{23}$$

where u is a solution of (1) with initial data u_0 and disturbances d.

These statements allow us to claim that for $d \equiv 0$ all weak solutions K of undisturbed problem (2) generate strict m-semiflow S according to the formula (8), and properties (22), (23) imply (G1) – (G4). So, due to Lemma 1, m-semiflow S has stable global attractor $\Theta \subset H$.

Moreover, from estimate (22) we get that property (6) takes place. So, Lemma 2 implies the robust stability estimate (6) for our problem (2).

Theorem 2. *Suppose that (19) – (21) takes place and, additionally, components of f belong to the class $C^1(\mathbb{R}^n)$, and the corresponding Jacobian matrix Df satisfies the following inequality:*

$$\exists C_4 > 0\ \forall v \in \mathbb{R}^n\ Df(v) \ge -C_4. \tag{24}$$

Then the formula

$$S_d(t,\tau,u_\tau) := \{u(t)|\ u(\cdot) \text{ is a solution of (2) on } [\tau,+\infty),\ u(\tau) = u_\tau\}, \tag{25}$$

generates the family of semiprocesses $\{S_d\}_{d\in U}$ with $U = L^\infty(0,+\infty)$, which is locally ISS w.r.t. Θ, i.e., property (5) takes place.

Proof. It can be proved that the family of mappings $\{S_d\}_{d\in U}$ defined by (25), satisfies (S1) – (S5). So, Lemma 3 implies that $\{S_d\}_{d\in U}$ is the strict family of semiprocesses. Moreover, inequality (24) allows us to prove that S_0 and S_d satisfy properties (9), (10). Estimate (22) gives property (11). Thus, we can apply Theorem 1 and obtain required result.

5. Conclusions. In this work, we considered the stability of the limit modes of an infinite-dimensional system of the reaction-diffusion type in relation to external disturbing signals. The main result is the estimation of the deviation of the trajectories of the disturbed system from the uniform attractor set (global attractor) of the undisturbed system in terms of the amplitude of the external signal. At the same time, the obtained results can be applied to wide classes of reaction-diffusion systems under rather general assumptions on coefficients, including systems with non-smooth interaction functions, multi-dimensional Lotka-Volterra systems with diffusion, FitzHugh–Nagumo systems and others for which the uniqueness of the solution of the Cauchy problem is not guaranteed. Therefore, we can conclude that this robust stability with respect to disturbances is the interior property of evolutionary processes which are modeled by reaction-diffusion systems.

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Капустян О. В., Юсипів Т. В. Стійкість граничних режимів для загального випадку систем типу реакція-дифузія.

У цій статті ми розглядаємо стійкість граничних режимів для загального класу нелінійних розподілених математичних моделей, які називаються моделями реакції-дифузії. Системи реакції-дифузії природно виникають у багатьох застосуваннях. Наприклад, при математичному моделюванні в біології та у теорії передачі сигналів широко використовується модель ФітцХью–Нагумо (FitzHugh–Nagumo model), розподілений варіант якої є окремим випадком загальної системи реакції-дифузії. Досліджено проблему стійкості притягуючих множин для нескінченновимірної системи реакції-дифузії відносно обмежених зовнішніх сигналів (збурень). Функції взаємодії, а також нелінійні збурення не вважаються неперервними за Ліпшицем. Отже, ми не можемо очікувати єдиності розв'язку для відповідної початкової задачі, і ми повинні використовувати багатозначний напівгруповий підхід. Вважається, що незбурена система має глобальний атрактор, тобто мінімальну компактну рівномірно притягаючу множину. Основною метою дослідження є оцінка відхилення траєкторії збуреної системи від глобального атрактора незбуреної як функції величини зовнішніх сигналів. Таку оцінку можна отримати в рамках теорії стійкості від входу до стану (ISS). У статті запропоновано новий підхід до отримання оцінок робастної стійкості атрактора у випадку багатозначного еволюційного оператора. Зокрема, доведено, що багатозначна напівгрупа, породжена слабкими розв'язками нелінійної системи типу реакції-дифузії, має властивість локальної ISS відносно атрактора незбуреної системи.

Ключові слова: система реакція-дифузія, система без єдиності розв'язку, стійкість від входу до стану, робастна стійкість, глобальний атрактор.

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