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OPTIMAL BOUNDARY CONTROL FOR DEGENERATE PARABOLIC FREE BOUNDARY PROBLEM

By Hardy-Poincare inequality we prove the solvability for optimal control problem for degenerate evolutional variational inequality in the case when the degenerate weight function is the function of potential type.

За допомогою нерівності Харді-Пуанкаре в роботі доведена розв'язність задачі оптимального керування для виродженої еволюційної варіаційної нерівності у випадку, коли вироджена вагова функція є функцією потенціального типу.

1. Introduction.

The main object of investigation of the given paper is the optimal boundary control problem for degenerate parabolic inequality with non-gomogeneous initial data. The corresponding problems "without degeneration" are the mathematical models, in particular, for one-phase Stefan problems and free boundary problems. The classical theorems on solvability for optimal control problems based on the direct variational method provide the existence of solution for such problems ([1, 2]). In the case when we consider the inequality with weight ρ that is locally integrated with its converse the situation with solvability becomes more complicate. Namely, for the differential operator that is related with inequality standard conditions for existence of solution of corresponding evolution object can be broken. This problem related to the fact that the weight function ρ can be unbounded on the domain Ω or reach zero on subsets of zero Lebesgue measure. It can leads, for example, to the non-uniqueness of setting of the main boundary problem (the so-called Lavrentiev phenomenon), and, as the result, non-uniqueness of setting of the optimal control problem.

The aim of the paper is to prove the solvability for the optimal control problem of degenerate parabolic inequality in the case when the weight function is the function of potential type. By Hardy-Poincare inequality, it is justified the existence of optimal solution for such optimal control problem in weight Sobolev space.

2. Setting of the problem.

Let Ω be a bounded open set in $\mathbb{R}^N$ with rather smooth boundary and let $0 \in \mathbb{R}^N$ be an internal point of the set Ω. Let the boundary $\partial\Omega$ consists of three smooth parts Γ_1, Γ_2 and Γ_3, which are pairs disjoint, Γ_1 and Γ_2 have not the common boundary and Lebesgue measure of Γ_1 is not equal to zero. Let $Q = (0, T) \times \Omega$ be a cylinder in $\mathbb{R}^1 \times \mathbb{R}^N$, where $T < +\infty$, and $\Sigma_i = (0, T) \times \Gamma_i$, $i = 1, 2, 3$ are corresponding parts of lateral surface. Let the function $\rho : \Omega \to \mathbb{R}$ satisfies the next conditions: $\rho > 0$ a.e. on Ω and

$$\rho,\, \rho^{-1} \in L^1(\Omega),\ \nabla \ln \rho \in L^2(\Omega, \mathbb{R}^N),\ \rho + \rho^{-1} \notin L^\infty(\Omega). \tag{1}$$

Thus, the function ρ can be identified with Radon measure on Ω, if we set $\rho(E) = \int_E \rho(x)dx$ for an arbitrary measurable set $E \subset \Omega$. Recall that as a non-negative Radon measure on Ω we can consider a non-negative Borel measure that is finite on every compact set. Further we will suggest that there is the closed subset Ω_* of Ω such that

$$\operatorname{dist}(\partial\Omega_*, \partial\Omega) = \delta,\ \rho > \sigma \text{ a.e. on } \Omega \setminus \Omega_*, \text{ and } \rho \in L^\infty(\Omega \setminus \Omega_*) \tag{2}$$

for some $\delta > 0$ and $\sigma > 0$. That is, it is suggested that conditions (1) are not typical for the boundary layer of the set Ω.

Further a non-negative function ρ with properties (1)–(2) we will call a degenerate weight function. Let us consider weight Hilbert spaces $L^2(\Omega, \rho dx)$ and $L^2(\Omega, \rho^{-1}dx)$, which are related to the weight ρ, where, in particular, $L^2(\Omega, \rho dx)$ is the Hilbert space of measurable functions $f : \Omega \longrightarrow \mathbb{R}$, for which

$$\|f\|_{L^2(\Omega,\rho dx)} = (f, f)_{L^2(\Omega,\rho dx)} = \int_\Omega f^2 \rho dx < +\infty.$$

Let us consider the next spaces: $C_0^\infty(\mathbb{R}^N, \Gamma_2) = \{\varphi \in C_0^\infty(\mathbb{R}^N) : \varphi = 0 \text{ on } \Gamma_2\}$, and let $W^{1,1}(\Omega; \Gamma_2)$ and $W^{1,2}(\Omega; \Gamma_2)$ be the closing of $C_0^\infty(\mathbb{R}^N; \Gamma_2)$ with regard to norms

$$\|y\|_{W^{1,1}(\Omega;\Gamma_2)} = \|y\|_{L^1(\Omega)} + \|\nabla y\|_{L^1(\Omega)^N}$$

and

$$\|y\|_{W^{1,2}(\Omega;\Gamma_2)} = \left(\int_\Omega y^2 dx + \int_\Omega |\nabla y|^2_{\mathbb{R}^N} dx \right)^{1/2}$$

correspondingly. Let us denote by $W^{1,2}(\Omega; \Gamma_2, \rho dx)$ the closing $C_0^\infty(\mathbb{R}^N; \Gamma_2)$ with regard to the norm

$$\|y\|^2_{W^{1,2}(\Omega;\Gamma_2,\rho dx)} = \int_\Omega y^2 \rho dx + \int_\Omega |\nabla y|^2_{\mathbb{R}^N} \rho dx.$$

Let $\lambda_* = (N-2)^2/4$. Then for an arbitrary open bounded domain $\Omega \subset \mathbb{R}^N$ with rather regular boundary $\partial\Omega$ there exists the constant $C(\Omega) > 0$ such that

$$\int_\Omega \left[|\nabla y|^2_{\mathbb{R}^N} - \lambda_* \frac{y^2}{|x|^2_{\mathbb{R}^N}} \right] dx \geq C(\Omega) \int_\Omega y^2 dx, \quad \forall y \in W^{1,2}(\Omega, \Gamma_2). \tag{3}$$

In the literature the relation (3) is called the Hardy-Poincare inequality (see [4]). Note, that in that case when $0 < \lambda < \lambda_*$ the relations: $\left(\int_\Omega \left[|\nabla y|^2_{\mathbb{R}^N} - \lambda \frac{y^2}{|x|^2_{\mathbb{R}^N}} \right] dx \right)^{1/2}$ i $\left(\int_\Omega y^2 dx + \int_\Omega |\nabla y|^2_{\mathbb{R}^N} dx \right)^{1/2}$ are equivalent norms in Sobolev space $W^{1,2}(\Omega, \Gamma_2)$ (see, for example, [5]).

Similarly to [6] let us consider a non-empty convex closed set $K = \{v | v \in W^{1,2}(\Omega; \Gamma_2, \rho dx),\ v \geq 0 \text{ a.e. in } \Omega\}$ in $W^{1,2}(\Omega; \Gamma_2, \rho dx)$, that is sequentially closed with regard to the norm:

$$\|y\|^2_\rho := \int_\Omega y^2 \rho\, dx + \int_\Omega \left| \nabla y + \frac{y}{2} \nabla \ln \rho \right|^2_{\mathbb{R}^N} \rho\, dx. \tag{4}$$

Also let us consider a convex closed subset $\mathcal{K}$ of the space $L^2(0,T;W^{1,2}(\Omega;\Gamma_2,\rho dx))$, that is determined by the next way:

$$\mathcal{K} = \{v| v \in L^2(0,T;W^{1,2}(\Omega;\Gamma_2,\rho dx)),\ v(t) \in K \text{ a.e.}\} =$$
$$= \{v|v \in L^2(0,T;W^{1,2}(\Omega;\Gamma_2,\rho dx)),\ v \geq 0 \text{ a.e. в } Q\},$$

that is sequentially closed with regard to the norm:

$$\|y\|^2_{\rho(0,T)} := \int_0^T \int_\Omega y^2 \rho\, dx dt + \int_0^T \int_\Omega \left|\nabla y + \frac{y}{2}\nabla \ln \rho\right|^2_{\mathbb{R}^N} \rho\, dx dt. \qquad (5)$$

Let $f_0 \in L^2(0,T;L^2(\Omega,\rho^{-1}dx))$, $y_0 \in W^{1,2}(\Omega)$, $y_0 \geq 0$ a.e. in Ω be the given function. Let U_∂ be a non-empty convex closed subset in $L^2(0,T;L^2(\Gamma_1,\rho^{-1}d\xi))$ such that $U_\partial = \{u \in L^2(0,T;L^2(\Gamma_1,\rho^{-1}d\xi)) : u \geq 0 \text{ a.e. in } \Sigma_1\}$, where the space $L^2(0,T;L^2(\Gamma_1,\rho^{-1}d\xi))$ is the space of measurable functions $u : [0,T] \to L^2(\Gamma_1,\rho d\xi)$, for which the norm $\|u\|^2_{L^2(0,T;L^2(\Gamma_1,\rho^{-1}d\xi))} = \int_0^T\int_\Omega \frac{u^2}{\rho} d\xi dt$, $\xi \in \Gamma_1$ is finite.

In the cylinder Q let us consider the the next optimal control problem for the degenerate parabolic problem with mixed boundary conditions:

$$I(u,y) = \|y\|^2_{L^2(0,T;L^2(\Omega,\rho dx))} + \|u\|^2_{L^2(0,T;L^2(\Gamma_1,\rho^{-1}d\xi))} + \|y(T)\|^2_{L^2(\Omega,\rho dx)} \to \inf, \qquad (6)$$

$$\rho\dot{y} - \operatorname{div}(\rho(x)\nabla y) \geq f_0,\ y \geq 0 \text{ a.e. in } Q, \qquad (7)$$

$$\rho\dot{y} - \operatorname{div}(\rho(x)\nabla y) = f_0 \text{ a.e. in } [(x,t) \in Q :\ y(x,t) > 0], \qquad (8)$$

$$\frac{\partial y}{\partial n_A} + \rho y = u \text{ a.e. in } \Sigma_1, \qquad (9)$$

$$y = 0 \text{ a.e. in } \Sigma_2, \qquad (10)$$

$$\frac{\partial y}{\partial n_A} = 0 \text{ a.e. in } \Sigma_3, \qquad (11)$$

$$\sqrt{\rho}y(x,0) = y_0(x),\ x \in \Omega, \qquad (12)$$

$$u \in U_\partial, \qquad (13)$$

where $\frac{\partial y}{\partial n_A} = \sum_{i,j=1}^{n} \rho(x)\frac{\partial y}{\partial x_j}\cos(n,x_i)$.

It is known (see [1–3]), that the optimal control problem (6)-(13) can be reduced to the optimal control problem for the degenerate evolution variation inequality:

$$I(u,y) = \|y\|^2_{L^2(0,T;L^2(\Omega,\rho dx))} + \|u\|^2_{L^2(0,T;L^2(\Gamma_1,\rho^{-1}d\xi))} + \|y(T)\|^2_{L^2(\Omega,\rho dx)} \to \inf, \qquad (14)$$

$$\int_0^T\int_\Omega \rho\dot{y}(v-y)dxdt + \int_0^T\int_\Omega \sum_{i,j=1}^{N} \rho\frac{\partial y}{\partial x_i}\frac{\partial(v-y)}{\partial x_j}dxdt +$$
$$+\int_0^T\int_{\Gamma_1} \rho y(v-y)d\xi dt \geq \int_0^T\int_\Omega f_0(v-y)dxdt + \int_0^T\int_{\Gamma_1} u(v-y)d\xi dt, \qquad (15)$$
$$\forall v \in \mathcal{K},\ \dot{y} \in L^2(0,T;(W^{1,2}(\Omega;\Gamma_2,\rho dx))^*),$$

$$u \in U_\partial, \quad y \in \mathcal{K}, \qquad (16)$$

$$\sqrt{\rho}y(x,0) = y_0(x),\ x \in \Omega. \tag{17}$$

Thus, we have to find the pair of functions

$$(u^0, y^0) \in L^2(0,T; L^2(\Gamma_1, \rho^{-1}d\xi)) \times L^2(0,T; W^{1,2}(\Omega;\Gamma_2,\rho dx)),$$

for which conditions (15)-(17) are fulfilled and the functional (14) reach its least value.

3. The previous analysis of the optimal control problem (14)-(17).

In order to show the correctness of the setting of the problem (14)-(17), let us pass (14)-(17) to new variables in relations, setting $y(x,t) = \frac{z(x,t)}{\sqrt{\rho}}$. From the formal transformation we have: $-\mathrm{div}(\rho\nabla y) = -\sqrt{\rho}\Delta z - \frac{1}{2}\sqrt{\rho}V(x)z$, where $V(x) = -\Delta \ln \rho(x) - \frac{1}{2}|\nabla \ln \rho(x)|^2_{\mathbb{R}^N}$ (see for details [5]). Moreover, applying formal transformations to $\frac{\partial y}{\partial n_A}$, we obtain: $\frac{\partial y}{\partial n_A} = \sqrt{\rho}\frac{\partial z}{\partial n} - \frac{1}{2}\sqrt{\rho}z\frac{\partial \ln\rho}{\partial n}$. Taking into account the boundary conditions (9) and (11), we will have: $\frac{\partial z}{\partial n} - \frac{1}{2}z\frac{\partial \ln \rho}{\partial n} + z = \frac{u}{\sqrt{\rho}}$ в Σ_1, $\frac{\partial z}{\partial n} - \frac{1}{2}z\frac{\partial \ln\rho}{\partial n} = 0$ в Σ_3.

Moreover, for such change of variables the next result is fulfilled (see [6, Proposition 1]).

Statement 1. *For an arbitrary $y \in L^2(0,T; W^{1,2}(\Omega;\Gamma_2,\rho dx))$ the representation $y = \frac{z}{\sqrt{\rho}}$ takes place, at that $z = \sqrt{\rho}y \in L^2(0,T; W^{1,1}(\Omega;\Gamma_2) \cap L^2(\Omega))$.*

Let us note that the map

$$\varphi : L^2(0,T; W^{1,2}(\Omega;\Gamma_2,\rho dx)) \to L^2(0,T; W^{1,1}(\Omega;\Gamma_2) \cap L^2(\Omega)),$$

that is identified as $\varphi(y) = y\sqrt{\rho}$, is not surjective. But in the space

$$L^2(0,T; W^{1,1}(\Omega;\Gamma_2) \cap L^2(\Omega))$$

the set of its images $\varphi(L^2(0,T; W^{1,2}(\Omega;\Gamma_2,\rho dx)))$ is dense. Similarly to [6] we can see that for an arbitrary

$$z \in L^2(0,T; C_0^\infty(\mathbb{R}^N,\Gamma_2)) \subset L^2(0,T; W^{1,1}(\Omega;\Gamma_2) \cap L^2(\Omega)),$$

we have $\frac{z}{\sqrt{\rho}} \in L^2(0,T; W^{1,2}(\Omega;\Gamma_2,\rho dx))$.

Thus, as a consequence of this result and continuity of the embedding

$$L^2(0,T; C_0^\infty(\mathbb{R}^N,\Gamma_2)) \subset L^2(0,T; W^{1,2}(\Omega;\Gamma_2)) \subset L^2(0,T; W^{1,1}(\Omega;\Gamma_2) \cap L^2(\Omega)),$$

we can say that there exists such dense set $\mathcal{D}_\rho \subset L^2(0,T; W^{1,2}(\Omega;\Gamma_2))$, that $\frac{z}{\sqrt{\rho}} \in L^2(0,T; W^{1,2}(\Omega;\Gamma_2,\rho dx))\ \forall z \in \mathcal{D}_\rho$.

Let us consider the linear map

$$\mathcal{F} : \mathcal{D}_\rho \subset L^2(0,T; W^{1,2}(\Omega;\Gamma_2)) \to L^2(0,T; W^{1,2}(\Omega;\Gamma_2,\rho dx)),$$

where $\mathcal{F}z = \frac{z}{\sqrt{\rho}}$. Since the domain $\mathcal{D}_\rho$ of the given map is dense set of Banach space $L^2(0,T; W^{1,2}(\Omega;\Gamma_2))$, then for $\mathcal{F}$, as for densely defined operator, there exists the conjugated operator, that in the general case is not densely defined (see for details [6, 7]).

In view of obtained results let us note the next property of the set $\mathcal{K}$. Since $\mathcal{K}$ is closed subset of the space $L^2(0,T;\mathcal{W}_\rho)$, where $\mathcal{W}_\rho$ is the closing of the space of finite functions $C_0^\infty(\mathbb{R}^N;\Gamma_2)$ with regard to the norm (4), then $\forall y \in \mathcal{K}$ we obtain: $y = \mathcal{F}z = \frac{z}{\sqrt{\rho}}$, where $z \in L^2(0,T;W^{1,1}(\Omega;\Gamma_2)\cap L^2(\Omega))$ and $\|y\|_{\rho(0,T)} = \left\|\frac{z}{\sqrt{\rho}}\right\|_{\rho(0,T)} < +\infty$ (by original suggestions). Hence, we have that $z \in L^2(0,T;W^{1,2}(\Omega;\Gamma_2))$. Thus, $\mathcal{K} \subset \mathcal{F}(\mathcal{D}_\rho)$.

Let us consider the conception of the weight function of potential type.

Definition 1. *We will say that $\rho : \Omega \to \mathbb{R}$ is the weight function of potential type if $\rho > 0$ a.e. on Ω, $\rho \in L^1(\Omega)$, $\rho^{-1} \in L^1(\Omega)$, $\nabla \ln \rho \in L^2(\Omega;\mathbb{R}^N)$ and there are exist constants $\hat{C}(\Omega) > 0$, $\tilde{C} > 0$ and subregion $\Omega_* \subset \Omega$ such that $\rho \in C^1(\overline{\Omega \setminus \Omega_*})$, where $\operatorname{dist}(\partial\Omega, \partial\Omega_*) > \delta$ for some $\delta > 0$, and the next inequalities are fulfilled:*

$$\rho(x) \geq \sigma \text{ on } \Omega \setminus \Omega_* \text{ for some } \sigma > 0, \tag{18}$$

$$\frac{1}{2}\frac{\partial \ln \rho}{\partial n} < 1 \text{ on } \Gamma_1; \tag{19}$$

$$\frac{\partial \ln \rho}{\partial n} = 0 \text{ on } \Gamma_3; \tag{20}$$

$$-\hat{C}(\Omega) \leq -\Delta \ln \rho(x) - \frac{1}{2}|\nabla \ln \rho|^2_{\mathbb{R}^N} < \frac{2\lambda_*}{|x|^2_{\mathbb{R}^N}} = \frac{(N-2)^2}{2|x|^2_{\mathbb{R}^N}} \text{ in } \Omega. \tag{21}$$

In the given case the function $V(x) = -\Delta \ln \rho(x) - \frac{1}{2}|\nabla \ln \rho|^2_{\mathbb{R}^N}$ is called Hardy potential for the weight function ρ (see for details [4,5]).

Let us construct sets

$$K_1 = \{\eta \in W^{1,2}(\Omega;\Gamma_2) | \eta = \sqrt{\rho} y, \forall y \in K \subset W^{1,2}(\Omega;\Gamma_2,\rho dx)\}$$

and

$$\mathcal{K}_1 = \{\eta \in L^2(0,T;W^{1,2}(\Omega;\Gamma_2)) | \eta(t) \in K_1 \text{ a.e. on } [0,T]\} =$$
$$= \left\{ \begin{array}{c} \eta \in L^2(0,T;W^{1,2}(\Omega;\Gamma_2)) : \\ \eta = \sqrt{\rho} y, \forall y \in \mathcal{K} \subset L^2(0,T;W^{1,2}(\Omega;\Gamma_2,\rho dx)) \end{array} \right\},$$

which by construction and original suggestions are convex closed subsets of spaces $W^{1,2}(\Omega;\Gamma_2)$ and $L^2(0,T;W^{1,2}(\Omega;\Gamma_2))$ correspondingly.

Together with the original optimal control problem (14)-(17), in the case when the function $\rho : \Omega \to \mathbb{R}_+$ satisfies the conditions of Definition 1, let us consider the next problem:

$$J(p,z) = \|z\|^2_{L^2(0,T;L^2(\Omega))} + \|p\|^2_{L^2(0,T;L^2(\Gamma_1))} + \|z(T)\|^2_{L^2(\Omega)} \to \inf, \tag{22}$$

$$\begin{gathered} \int_0^T\int_\Omega \dot{z}(w-z)dxdt + \int_0^T\int_\Omega (\nabla z, \nabla w - \nabla z)_{\mathbb{R}^N} dxdt - \\ -\frac{1}{2}\int_0^T\int_\Omega V(x) z(w-z)dxdt + \int_0^T\int_{\Gamma_1} \left(1 - \frac{1}{2}\frac{\partial \ln \rho}{\partial n}\right) z(w-z)d\xi dt \geq \\ \geq \int_0^T\int_\Omega \frac{f_0}{\sqrt{\rho}}(w-z)dxdt + \int_0^T\int_{\Gamma_1} p(w-z)d\xi dt, \\ \forall z \in \mathcal{K}_1, \ \dot{z} \in L^2(0,T;(W^{1,2}(\Omega;\Gamma_2))^*), \end{gathered} \tag{23}$$

$$p \in P_\partial, \ z \in \mathfrak{K}_1, \tag{24}$$

$$z(0,x) = y_0(x), \ x \in \Omega, \tag{25}$$

where $p = \frac{u}{\sqrt{\rho}}$, $V(x) = -\Delta \ln \rho(x) - \frac{1}{2}|\nabla \ln \rho|^2_{\mathbb{R}^N}$, and the set P_∂ is defined by the next way:

$$P_\partial = \{p \in L^2(0,T;L^2(\Gamma_1)) : \ p \geq 0 \text{ a.e. in } \Sigma_1\}.$$

Further let us consider the next sets:

$$\Xi_1 = \{(u,y) \in L^2(0,T;L^2(\Gamma_1,\rho^{-1}d\xi)) \times L^2(0,T;W^{1,2}(\Omega;\Gamma_2,\rho dx))| \\ u \text{ and } y \text{ are related by conditions (15)} - (17)\}$$

and

$$\Xi_2 = \{(p,z) \in L^2(0,T;L^2(\Gamma_1)) \times L^2(0,T;W^{1,2}(\Omega;\Gamma_2))| \\ p \text{ and } z \text{ are related by conditions (23)} - (25)\}$$

which we will call the sets of admissible solutions for the optimal control problems (14)–(17) and (22)–(25), correspondingly.

Definition 2. *We will say that pairs of functions $(u^0,y^0) \in \Xi_1$ and $(p^0,z^0) \in \Xi_2$ are the optimal solutions for problems (14)–(17) and (22)–(25) correspondingly, if*

$$\inf_{(u,y)\in\Xi_1} I(u,y) = I(u^0,y^0), \quad \inf_{(p,z)\in\Xi_2} J(p,z) = J(p^0,z^0).$$

The solvability of the problem (22)-(25). Let us show that in the case when as the weight function $\rho : \Omega \to \mathbb{R}$ we consider the function of potential type, the optimal control problem (22)-(25) will have at least one solution.

Theorem 1. *Let $\rho : \Omega \to \mathbb{R}_+$ be a weight function of potential type. Let $f_0 \in L^2(0,T;L^2(\Omega,\rho^{-1}dx))$, $y_0 \in W^{1,2}(\Omega)$, $y_0 \geq 0$ a.e. in Ω be the given functions, and for every $\varphi \in C_0^\infty(\mathbb{R}^N;\Gamma_2)$ the next equality takes place*

$$\lim_{t\to 0+} \langle z(t),\varphi\rangle_{L^2(\Omega)} = \langle y_0,\varphi\rangle_{L^2(\Omega)}.$$

Then the optimal control problem (22)-(25) is solvable in the space

$$L^2(0,T;L^2(\Gamma_1)) \times L^2(0,T;W^{1,2}(\Omega;\Gamma_2)).$$

Proof. First let us show that the optimal control problem (22)-(25) is regular, that is let us show that the problem (23)-(25) has at least one solution for some $p \in L^2(0,T;L^2(\Gamma_1))$.

Let us set

$$V = \left\{ w \in W^{1,2}(\Omega;\Gamma_2) : \ \int_\Omega \left[|\nabla w|^2_{\mathbb{R}^N} - \frac{1}{2}V(x)w^2 \right] dx < +\infty \right\}, \ H = L^2(\Omega).$$

Using the suggestions from the proof of Theorem 3.2 in [5], we can see that the space V coincides with the space $W^{1,2}(\Omega;\Gamma_2)$. Thus, in view of Rellich-Kondrashov theorem, we obtain the next compact embeddings:

$$L^2(0,T;W^{1,2}(\Omega;\Gamma_2)) \subset L^2(0,T;L^2(\Omega)) \subset L^2(0,T;(W^{1,2}(\Omega;\Gamma_2))^*). \tag{26}$$

Let us consider related to the inequality (23) linear symmetric operator

$$B : L^2(0,T;W^{1,2}(\Omega;\Gamma_2)) \to L^2(0,T;(W^{1,2}(\Omega;\Gamma_2))^*),$$

that is defined by the rule:

$$\begin{gathered}\langle Bz, w\rangle_{L^2(0,T;W^{1,2}(\Omega;\Gamma_2))} = \int_0^T\int_\Omega (\nabla z, \nabla w)_{\mathbb{R}^N}\,dxdt-\\ -\frac{1}{2}\int_0^T\int_\Omega V(x)zw\,dxdt + \int_0^T\int_{\Gamma_1}\left(1-\frac{1}{2}\frac{\partial \ln\rho}{\partial n}\right) zw\,d\xi dt,\\ \forall z(t)\in W^{1,2}(\Omega;\Gamma_2),\ t\in[0,T].\end{gathered}\tag{27}$$

Taking into account the definition of the weight function of potential type, Hardy-Poincare inequality (3) and Sobolev trace theorem, we obtain:

$$\begin{gathered}|\langle Bz, w\rangle_{L^2(0,T;W^{1,2}(\Omega;\Gamma_2))}| \le\\ \le (1+C_1)\|z\|_{L^2(0,T;W^{1,2}(\Omega;\Gamma_2))}\|w\|_{L^2(0,T;W^{1,2}(\Omega;\Gamma_2))},\\ \langle Bz, z\rangle_{L^2(0,T;W^{1,2}(\Omega;\Gamma_2))} \ge C_2\|z\|^2_{L^2(0,T;W^{1,2}(\Omega;\Gamma_2))},\end{gathered}\tag{28}$$

where C is the constant from the theorem on traces, $C_1 = \max\{C, \hat{C}(\Omega)\}$, C_2 is the constant from the equivalence of norms in the space $L^2(0,T;W^{1,2}(\Omega;\Gamma_2))$.

Since the condition $f_0 \in L^2(0,T;L^2(\Omega,\rho^{-1}dx))$ provides $\frac{f_0}{\sqrt{\rho}} \in L^2(0,T;L^2(\Omega))$, and the condition $u \in L^2(0,T;L^2(\Gamma_1,\rho^{-1}d\xi))$ provides that $\frac{u}{\sqrt{\rho}} \in L^2(0,T;L^2(\Gamma_1)$, and the space $L^2(0,T;L^2(\Omega))$ is compactly embedded into $L^2(0,T;(W^{1,2}(\Omega;\Gamma_2))^*)$, then f, that is defined by

$$\langle f, v\rangle_{L^2(0,T;W^{1,2}(\Omega;\Gamma_2))} = \int_0^T\int_\Omega f_0 v\,dxdt + \int_0^T\int_{\Gamma_1} uv\,d\xi dt,\quad v\in\mathcal{K}_1,\tag{29}$$

is defined in $L^2(0,T;(W^{1,2}(\Omega;\Gamma_2))^*)$.

Under conditions (26), (28), (29), boundedness of the linear operator B, for $y_0 \in W^{1,2}(\Omega;\Gamma_2)$ such that $y_0 \ge 0$ a.e. in Ω, by [1, Теорема 4.6] the problem (23)-(25) has the unique solution. Moreover, the next estimate takes place:

$$\begin{gathered}\|z\|_{L^2(0,T;W^{1,2}(\Omega;\Gamma_2))} + \|z\|_{L^\infty(0,T;L^2(\Omega))}+\\ +\|\dot{z}\|_{L^2(0,T;(W^{1,2}(\Omega;\Gamma_2))^*)} \le L(1+\|p\|_{L^2(0,T;L^2(\Gamma_1))}),\end{gathered}\tag{30}$$

where $L > 0$. Moreover, if $p_n \to p$ weakly in $L^2(\Sigma_1)$ as $n\to\infty$, then $z_n \to z$, $n\to\infty$, in $L^2(0,T;L^2(\Omega))$ and weakly in $L^2(0,T;W^{1,2}(\Omega;\Gamma_2))$, where z_n is the solution of (23)-(25) with $p = p_n$.

Indeed, let us consider the weakly convergent in $L^2(0,T;L^2(\Gamma_1))$ sequence of controls $\{p_n\} \subset P_\vartheta$. Then from (30) we obtain, that there exists a subsequence of $\{z_n\}$ (which we identify as $\{z_n\}$ again) such that (see for details [3, Глава 1, Теорема 5.1]): $z_n \to z$ weakly in $L^2(0,T;W^{1,2}(\Omega;\Gamma_2))$ and strongly in $L^2(0,T;L^2(\Omega))$ and $\dot{z}_n \to \dot{z}$ в $L^2(0,T;(W^{1,2}(\Omega;\Gamma_2))^*)$ as $n\to\infty$.

Let us pass to the limit in the relation:

$$\int_0^T\int_\Omega \dot z_n(z_n-w)dxdt+\langle Bz_n,z_n-w\rangle_{L^2(0,T;W^{1,2}(\Omega;\Gamma_2))}\le$$

$$\le\int_0^T\int_\Omega \frac{f_0}{\sqrt{\rho}}(z_n-w)dxdt+\int_0^T\int_{\Gamma_1} p_n(z_n-w)d\xi dt\ \forall w\in\mathcal{K}_1$$

and use the compactness of the embedding $L^2(0,T;W^{1,2}(\Omega;\Gamma_2))\subset L^2(0,T;L^2(\Omega))$ and the semi-continuity from below for the norm in $L^2(0,T;W^{1,2}(\Omega;\Gamma_2))$ with regard to the weak convergence. As a result we obtain

$$\int_0^T\int_\Omega \dot z(z-w)dxdt+\langle Bz,z-w\rangle_{L^2(0,T;W^{1,2}(\Omega;\Gamma_2))}=$$

$$=\int_0^T\int_\Omega \dot z z dxdt-\int_0^T\int_\Omega \dot z w dxdt+\langle Bz,z\rangle_{L^2(0,T;W^{1,2}(\Omega;\Gamma_2))}-\langle Bz,w\rangle_{L^2(0,T;W^{1,2}(\Omega;\Gamma_2))}\le$$

$$\le\lim_{n\to\infty}\left(\int_0^T\int_\Omega \dot z_n(z_n-w)dxdt\right)+(1+C_1)\varliminf_{n\to\infty}\|z_n\|^2_{L^2(0,T;W^{1,2}(\Omega;\Gamma_2))}-$$

$$-\lim_{n\to\infty}\langle Bz_n,w\rangle_{L^2(0,T;W^{1,2}(\Omega;\Gamma_2))}\le$$

$$\le\varliminf_{n\to\infty}\left(\int_0^T\int_\Omega \dot z_n(z_n-w)dxdt+\langle Bz_n,z_n-w\rangle_{L^2(0,T;W^{1,2}(\Omega;\Gamma_2))}\right)\le$$

$$\le\lim_{n\to\infty}\left(\int_0^T\int_\Omega\frac{f_0}{\sqrt{\rho}}(z_n-w)dxdt+\int_0^T\int_{\Gamma_1}p_n(z_n-w)dxdt\right)=$$

$$=\left(\int_0^T\int_\Omega\frac{f_0}{\sqrt{\rho}}(z-w)dxdt+\int_0^T\int_{\Gamma_1}p(z-w)dxdt\right).$$

Since the set $\mathcal{K}_1$ is closed and convex then by Mazur theorem it is weakly convex. Thus, $z\in\mathcal{K}_1$. From upper suggestions we obtain that $(p,z)\in\Xi_2$, that is $z=z(p)$, and, hence, the set Ξ_2 is closed with regard to topology of weak convergence in $L^2(0,T;L^2(\Gamma_1))\times\times L^2(0,T;W^{1,2}(\Omega;\Gamma_2))$. Thus, the considered optimal control problem is regular. Let us show that it is solvable.

Let us rewrite (22) in the next view: $J(p,z)=\int_0^T(g(z)+h(p))dt+\Phi_0(z(T))$, where $g:L^2(\Omega)\to\mathbb{R}$ and it is defined by the rule $g(z)=\|z\|^2_{L^2(\Omega)}$, $h:L^2(\Gamma_1)\to\mathbb{R}$ and it is defined by the rule $h(p)=\|p\|^2_{L^2(\Gamma_1)}$, $\Phi_0:L^2(\Omega)\to\mathbb{R}$ and it is defined by the rule $\Phi_0(z(T))=\|z(T)\|^2_{L^2(\Omega)}$.

Note that the functional $h : L^2(\Gamma_1) \to \bar{\mathbb{R}}$ is convex and semi-continuous from below on $L^2(\Omega)$.

Let us show that for every $r > 0$ there exists $L_r > 0$ such that the next inequality takes place:

$$|g(z_1) - g(z_2)| + |\Phi_0(z_1) - \Phi_0(z_2)| \le L_r \|z_1 - z_2\|_{L^2(\Omega)} \quad \text{for } \|z_1\|_{L^2(\Omega)} + \|z_2\|_{L^2(\Omega)} \le r. \tag{31}$$

We obtain:

$$|g(z_1) - g(z_2)| \le \left(\int_\Omega (z_1 - z_2)^2 dx \right)^{1/2} \cdot \sqrt{2r^2 r^2} =$$

$$= \sqrt{2} r^3 \left(\int_\Omega (z_1 - z_2)^2 dx \right)^{1/2} = C_r \|z_1 - z_2\|_{L_2(\Omega)}.$$

By similar suggestions we can obtain that $\exists \hat{C}_r > 0$ such that

$$|\Phi_0(z_1) - \Phi_0(z_2)| \le \hat{C}_r \|z_1 - z_2\|_{L^2(\Omega)}.$$

Thus, the inequality (31) is fulfilled for $L_r = \max\{C_r, \hat{C}_r\}$.

Under conditions for the operator B (linearity, continuity, symmetry and the condition (28)), considered conditions for maps g, h and Φ_0, by [1, Твердження 5.1], the problem (22)-(25) has at least one solution $(p^0, z^0) \in L^2(0, T; L^2(\Gamma_1)) \times L^2(0, T; W^{1,2}(\Omega; \Gamma_2))$.

4. The solvability of the optimal control problem (14)–(17).

Theorem 2. *Let $\rho : \Omega \to \mathbb{R}_+$ be a weight function of potential type. Let $f_0 \in L^2(0, T; L^2(\Omega, \rho^{-1} dx))$, $y_0 \in W^{1,2}(\Omega)$, $y_0 \ge 0$ a.e.in Ω be the given functions, and for every $\varphi \in C_0^\infty(\mathbb{R}^N; \Gamma_2)$ the next equality takes place*

$$\lim_{t \to 0+} \langle z(t), \varphi \rangle_{L^2(\Omega)} = \langle y_0, \varphi \rangle_{L^2(\Omega)}.$$

Then the optimal control problem (14)-(17) has at least one solution (u^0, y^0) in the space $L^2(0, T; L^2(\Gamma_1, \rho^{-1} d\xi)) \times L^2(0, T; (W^{1,2}(\Omega; \Gamma_2, \rho\, dx))$.

Proof. First let us show that optimal control problems (14)–(17) and (22)-(25) are equivalent in the next sense: the admissible pair $(p^0, z^0) \in \Xi_2$ is the optimal one in the problem (22)-(25) if and only if

$$(u^0, y^0) := \left(\sqrt{\rho} p^0, \frac{z^0}{\sqrt{\rho}} \right) \tag{32}$$

is the solution of the original optimal control problem (14)–(17) on the set Ξ_1. And the next equality takes place

$$\inf_{(p,z) \in \Xi_2} J(p, z) = J(p^0, w^0) = I(u^0, y^0) = \inf_{(u,y) \in \Xi_1} I(u, y). \tag{33}$$

Let us consider the operator

$$A : L^2(0, T; W^{1,2}(\Omega, \Gamma_2, \rho dx)) \to L^2(0, T; (W^{1,2}(\Omega; \Gamma_2, \rho dx))^*),$$

that is defined by the rule:

$$\langle Ay, v\rangle_{L^2(0,T;W^{1,2}(\Omega;\Gamma_2\rho dx))} = \int_0^T\int_\Omega (\nabla y, \nabla v)\rho dxdt + \int_0^T\int_{\Gamma_1} \rho yvd\xi dt$$

and related to the inequality (15). Similarly to [7] we have, in particular, that for z from $\mathcal{D}_\rho$, where $\mathcal{D}_\rho$ is some dense subset in the space $L^2(0,T;W^{1,2}(\Omega;\Gamma_2))$, the element $\frac{z}{\sqrt{\rho}} \in L^2(0,T;W^{1,2}(\Omega;\Gamma_2,\rho dx))$ and $\nabla y = \nabla\left(\frac{z}{\sqrt{\rho}}\right) = \frac{1}{\sqrt{\rho}}\left(\nabla z - \frac{z}{2}\nabla \ln\rho\right)$. Taking into account these relations, that fact that for v and z from

$$\mathcal{D}_\rho \subset L^2(0,T;W^{1,2}(\Omega;\Gamma_2))$$

we have

$$\mathcal{F}z, \mathcal{F}v \in L^2(0,T;W^{1,2}(\Omega;\Gamma_2,\rho dx)),$$

and the view of the function $V(x)$, we obtain:

$$\langle A(\mathcal{F}z), \mathcal{F}w\rangle_{L^2(0,T;W^{1,2}(\Omega,\Gamma_2,\rho dx))} = \langle Bz, w\rangle_{L^2(0,T;W^{1,2}(\Omega,\Gamma_2))},$$

where $z, w \in L^2(0,T;W^{1,2}(\Omega,\Gamma_2))$. Since for an arbitrary $v \in \mathcal{K}$ there exists an element $w \in \mathcal{K}_1$ such that $v = \mathcal{F}w := \frac{w}{\sqrt{\rho}}$, then we will have:

$$\int_0^T\int_\Omega \dot{y}(v-y)\rho dxdt = \int_0^T\int_\Omega \dot{z}(w-z)dxdt;$$

$$\int_0^T\int_\Omega f_0(v-y)dxdt = \int_0^T\int_\Omega \frac{f_0}{\sqrt{\rho}}(w-z)dxdt, \int_0^T\int_{\Gamma_1} u(v-y)d\xi dt = \int_0^T\int_{\Gamma_1} \frac{u}{\sqrt{\rho}}(w-z)d\xi dt.$$

Thus, we can conclude that $y \in \mathcal{K}$ is the solution of the variation inequality (15) if and only if $z = \sqrt{\rho}y \in \mathcal{K}_1$ is the solution of the variation inequality (23).

Hence, solutions of variation inequalities (15) and (23) are related by $y = \frac{z}{\sqrt{\rho}}$. Let us consider in the space $L^2(0,T;L^2(\Gamma_1,\rho^{-1}\,d\xi))$ the map G that is defined by the rule: $G(u) = \left(\frac{u}{\sqrt{\rho}}\right)$, and show that it is isometrically maps the space $L^2(0,T;L^2(\Gamma_1,\rho^{-1}\,d\xi))$ on the space $L^2(0,T;L^2(\Gamma_1))$. For an arbitrary control $u \in L^2(0,T;L^2(\Gamma_1,\rho^{-1}\,d\xi))$ we have $p = G(u)$, where

$$\|u\|^2_{L^2(0,T;L^2(\Gamma_1,\rho^{-1}\,d\xi))} = \int_0^T\int_{\Gamma_1} \frac{u^2}{\rho}d\xi dt = \int_0^T\int_{\Gamma_1} p^2 d\xi dt = \|p\|^2_{L^2(0,T;L^2(\Gamma_1))};$$

Moreover, the condition (24) provides the equivalence of statements:

$$u \in U_\partial \Leftrightarrow p = \frac{u}{\sqrt{\rho}} \in P_\partial.$$

Thus, $(u,y) \in \Xi_1 \Leftrightarrow (p,z) \in \Xi_2$.

Note, that for every $\varphi \in C_0^\infty(\mathbb{R}^N;\Gamma_2)$ the equality

$$\lim_{t\to 0+}\langle y(t),\varphi\rangle_{L^2(\Omega,\sqrt{\rho}dx)} = \lim_{t\to 0+}\langle z(t),\varphi\rangle_{L^2(\Omega)} = (y_0,\varphi)_{L^2(\Omega)}$$

takes place.

Let us prove the equality (14) and (22) on corresponding admissible pairs:

$$I(u,y) = \|y\|^2_{L^2(0,T;L^2(\Omega,\rho dx))} + \|u\|^2_{L^2(0,T;L^2(\Gamma_1;\rho^{-1}d\xi)} + \|y(T)\|^2_{L^2(\Omega,\rho dx))} =$$

$$= \|\sqrt{\rho}y\|^2_{L^2(0,T;L^2(\Omega))} + \left\|\frac{u}{\sqrt{\rho}}\right\|^2_{L^2(0,T;L^2(\Gamma_1))} + \|\sqrt{\rho}y(T)\|^2_{L^2(\Omega)} =$$

$$= \|z\|^2_{L^2(0,T;L^2(\Omega))} + \|p\|^2_{L^2(0,T;L^2(\Gamma_1))} + \|z(T)\|^2_{L^2(\Omega)} = J(p,z),$$

that provides the equality (33). Thus, problems (14)-(17) and (22)-(25) are equivalent. In view of Theorem 1 the optimal control problem (22)-(25) has at least one solution in the space $L^2(0,T;L^2(\Gamma_1)) \times L^2(0,T;W^{1,2}(\Omega,\Gamma_2))$. Hence, the original optimal control problem (14)-(17) has at least one solution in the space

$$L^2(0,T;L^2(\Gamma_1,\rho^{-1}d\xi)) \times L^2(0,T;(W^{1,2}(\Omega;\Gamma_2,\rho\, dx)).$$

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