On Non-Linear Boundary Value Problems for Iterative Differential Equations

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We study the general form boundary value problem

$$\frac{dx(t)}{dt} = f(t, x(t), x(x(t))), \quad t \in [a,b], \tag{1}$$

for the system of so called iterative differential equations (see, e.g., [1,5] and the references therein) under the non-linear boundary conditions

$$\Phi(x(t), x(x(t))) = d, \tag{2}$$

where $f \in C([a,b] \times D \times D; \mathbb{R}^n)$, $d \in \mathbb{R}^n$ is a given vector, Φ is a continuous n-dimensional vector functional and there exist some $n \times n$ matrices K_1, K_2 with non-negative entries such that for all $t \in [a,b]$, $u_i, v_i \in D$, $i = 1,2$ the inequality

$$\big|f(t, u_1, u_2) - f(t, v_1, v_2)\big| \le K_1|u_1 - v_1| + K_2|u_2 - v_2| \tag{3}$$

holds.

The domain $D \sqsubseteq [a,b]^n$ will be defined in Eqs. (10) and (11).

We deal only with such solutions

$$x : [a,b] \to D \sqsubseteq [a,b]^n, \tag{4}$$

of problem (1), (2), which belong to the set

$$S := \Big\{x \in C([a,b]; D) : \ |x(t_1) - x(t_2)| \le L|t_1 - t_2|, \ \forall\, t_1, t_2 \in [a,b]\Big\}, \tag{5}$$

where L is a given diagonal matrix with non-negative entries $L = diag(L_1, \dots, L_n)$. On the base of conditions (3) and (5), we obtain

$$\big|f(t, u_1, u_2) - f(t, v_1, v_2)\big| \le K_1|u_1 - v_1| + K_2L|u_1 - v_1| = [K_1 + K_2L]\,|u_1 - v_1|, \tag{6}$$

$t \in [a,b]$. Thus, we prescribed some restrictions for the values of the derivative of the possible solutions similarly to that of [5] and [1].

To study the BVP (1), (2) we will use an approach similar to [2]. Note that this technique can be applied also in the case when, instead of (5), the condition

$$S := \Big\{ x \in C([a,b];[a_1,b_1]^n) : \ |x(t_1) - x(t_2)| \le L|t_1 - t_2|, \ \forall\, t_1, t_2 \in [a_1,b_1] \Big\}$$

is fulfilled and in addition there are given some initial functions

$$\beta \in C([a_1,a],D), \quad \gamma \in C([b,b_1],D).$$

For vectors $x = col(x_1,\dots,x_n) \in \mathbf{R}^n$ the obvious notation $|x| = col(|x_1|,\dots,|x_n|)$ is used and the inequalities between vectors are understood componentwise. The same convention is adopted for operations like "max" and "min".

$\mathbf{I}_n$ and $\mathbf{0}_n$ are the unit and zero matrices of dimension n, respectively. $r(K)$ is the maximal (in modulus) eigenvalue of the matrix K.

For any non-negative vector $\rho \in \mathbf{R}^n$ under the componentwise ρ-neighbourhood of a point $z \in \mathbf{R}^n$, we understand the set

$$O_\rho(z) := \big\{ \xi \in \mathbf{R}^n : \ |\xi - z| \le \rho \big\}. \tag{7}$$

Similarly, the ρ-neighbourhood of a domain $\Omega \subset \mathbf{R}^n$ is defined as

$$O_\rho(\Omega) := \bigcup_{z \in \Omega} O_\rho(z). \tag{8}$$

A particular kind of vector ρ will be specified below in relation (11).

Let us choose certain compact convex sets $D_a \subset \mathbb{R}^n$, $D_b \subset \mathbb{R}^n$ and define the set

$$D_{a,b} := (1-\theta)z + \theta\eta, \ \ z \in D_a, \ \ \eta \in D_b, \ \ \theta \in [0,1], \tag{9}$$

moreover, according to (8) its ρ- neighbourhood

$$D = O_\rho(D_{a,b}) \tag{10}$$

with a non-negative vector $\rho = col(\rho_1,\dots,\rho_n) \in \mathbb{R}^n$, such that

$$\rho \ge \frac{b-a}{2}\, \delta_{[a,b],D\times D}(f), \tag{11}$$

where $\delta_{[a,b],D\times D}(f)$ denotes the half of the oscillation of the function f over $[a,b] \times D \times D$, i.e.,

$$\delta_{[a,b],D\times D}(f) := \frac{\max\limits_{(t,x,y)\in[a,b]\times D\times D} f(t,x,y) - \min\limits_{(t,x,y)\in[a,b]\times D\times D} f(t,x,y)}{2}. \tag{12}$$

Instead of the original boundary value problem (1), (2), we will consider the following auxiliary two-point parametrized boundary value problem

$$\frac{dx(t)}{dt} = f(t, x(t), x(x(t))), \ \ t \in [a,b], \tag{13}$$

$$x(a) = z, \quad x(b) = \eta, \tag{14}$$

where z and η are treated as free parameters.

Let us connect with problem (13), (14) the sequence of functions

$$x_{m+1}(t,z,\eta)=z+\int\limits_a^t f\big(s,x_m(s,z,\eta),x_m(x_m(s,z,\eta),z,\eta)\big)\,ds \\ -\frac{t-a}{b-a}\int\limits_a^b f\big(s,x_m(s,z,\eta),x_m(x_m(s,z,\eta),z,\eta)\big)\,ds \\ +\frac{t-a}{b-a}\,[\eta-z],\ \ t\in[a,b],\ \ m=0,1,2,\dots, \tag{15}$$

satisfying (14) for arbitrary $z,\eta\in\mathbb{R}^n$, where

$$x_0(t,z,\eta)=z+\frac{t-a}{b-a}\,[\eta-z]=\Big(1-\frac{t-a}{b-a}\Big)z+\frac{t-a}{b-a}\,\eta,\ \ t\in[a,b]. \tag{16}$$

It is easy to see from (16) that $x_0(t,z,\eta)$ is a linear combination of vectors z and η, when $z\in D_a$ and $\eta\in D_b$.

The following statement establishes the uniform convergence of sequence (15) to some parameterized limit function.

Theorem 1. *Let conditions* (6), (11) *be fulfilled, moreover, for the matrix*

$$Q=\frac{3(b-a)}{10}\,K,\ \ K=K_1+K_2L \tag{17}$$

the inequality

$$r(Q)<1 \tag{18}$$

hold.

Then, for all fixed $(z,\eta)\in D_a\times D_b$:

1. *The functions of sequence* (15) *belonging to the domain D of form* (10) *are continuously differentiable on the interval $[a,b]$ and satisfy conditions* (14).
2. *The sequence of functions* (15) *for $t\in[a,b]$ uniformly converges as $m\to\infty$ with respect to the domain $(t,z,\eta)\in[a,b]\times D_a\times D_b$ to the limit function*

$$x_\infty(t,z,\eta)=\lim_{m\to\infty}x_m(t,z,\eta), \tag{19}$$

satisfying conditions (14).

3. *The function $x_\infty(t,z,\eta)$ for all $t\in[a,b]$ is a unique continuously differentiable solution of the integral equation*

$$x(t)=z+\int\limits_a^t f(s,x(s),x(x(s)))\,ds-\frac{t-a}{b-a}\int\limits_a^b f(s,x(s),x(x(s)))\,ds+\frac{t-a}{b-a}\,[\eta-z], \tag{20}$$

i.e., it is the solution to the Cauchy problem for the modified system of integro-differential equations:

$$\frac{dx}{dt}=f(t,x(t),x(x(t)))+\frac{1}{b-a}\,\Delta(z,\eta), \\ x(a)=z \tag{21}$$

where $\Delta(z,\eta): D_a\times D_b\to\mathbb{R}^n$ *is a mapping given by the formula*

$$\Delta(z,\eta)=[\eta-z]-\int\limits_a^b f\big(s,x_\infty(s,z,\eta),x_\infty(x_\infty(s,z,\eta),z,\eta)\big)\,ds. \tag{22}$$

4. *The error estimation*

$$\big|x_\infty(t,z,\eta)-x_m(t,z,\eta)\big|\leqslant\frac{10}{9}\,\alpha_1(t)Q^m(1_n-Q)^{-1}\delta_{[a,b],D\times D}(f),\ \ t\in[a,b],\ \ m\ge 0 \tag{23}$$

holds, where

$$\alpha_1(t)=2(t-a)\Big(1-\frac{t-a}{b-a}\Big)\le\frac{b-a}{2}\,,\ \ t\in[a,b].$$

The following statement gives a relation of the parameterized limit function $x_\infty(t,z,\eta)$ to the solution of the original boundary value problem $(1),(2)$.

Theorem 2. *Under the assumptions of Theorem 1, the limit function*

$$x_\infty(t,z,\eta)=\lim_{m\to\infty}x_m(t,z,\eta)$$

of sequence (15) *is a solution of the boundary value problem* $(1),(2)$ *with property* (5) *if and only if the pair of parameters* (z,η) *satisfies the system of* $2n$ *algebraic equations*

$$\begin{gathered}\Delta(z,\eta):=[\eta-z]-\int\limits_a^b f\big(s,x_\infty(s,z,\eta),x_\infty(x_\infty(s,z,\eta),z,\eta)\big)\,ds=0,\\ \Phi(z,\eta):=\Phi(x_\infty(t,z,\eta)),\quad x_\infty(x_\infty(t,z,\eta))-d=0.\end{gathered} \tag{24}$$

We apply the above techniques to the following model BVP in $\mathbf{R}^2$

$$\begin{aligned}\frac{dx_1(t)}{dt}&=\big[x_1(x_1(t))\big]^2-\frac18\,x_2(t)+\frac12=f_1\big(x_1,x_2,x_1(x_1(t)),x_2(x_2(t))\big),\ \ t\in[a,b]=\Big[0,\frac12\Big],\\ \frac{dx_2(t)}{dt}&=x_2(x_2(t))-\frac t2\,x_1(t)\cdot x_2(t)+t=f_2\big(x_1,x_2,x_1(x_1(t)),x_2(x_2(t))\big),\end{aligned} \tag{25}$$

with the iterative integral boundary conditions

$$\begin{aligned}\Phi_1\big(x(t),x(x(t))\big)&=\int\limits_0^{1/2}\big[x_1(s)+x_2(s)\big]\,ds=\frac1{12}\,,\\ \Phi_2\big(x(t),x(x(t))\big)&=\int\limits_0^{1/2}\big[x_1(x_1(s))\big]^2\,ds=\frac1{384}\,.\end{aligned} \tag{26}$$

Clearly, problem $(25),(26)$ is a particular case of $(1),(2)$ with $a=0$, $b=\frac12$, $d=col(\frac18,\frac1{384})$. It is easy to check that $x_1(t)=\frac t2$, $x_2(t)=\frac{t^2}2$ is a continuously differentiable solution to problem $(25),(26)$.

One can check that all the conditions of Theorem 1 for this example are fulfilled for the following choosing and computation of corresponding sets, vectors, matrices:

$$D_a=D_b=\big\{(x_1,x_2):\ -0.05\le x_1\le 0.3,\ -0.05\le x_2\le 0.2\big\},\ \ D_{a,b}=D_a=D_b, \tag{27}$$

$$\rho := col(0.15, 0.15), \;\; O\rho(D_{a,b}) = D = \big\{(x_1, x_2): \; -0.2 \le x_1 \le 0.45, \; -0.2 \le x_2 \le 0.35\big\},$$

$$K_1 = \begin{bmatrix} 0 & \frac{1}{8} \\ 0.25 & 0.25 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$K = K_1 + K_2 L = \begin{bmatrix} 1 & \frac{1}{8} \\ 0.25 & 1.25 \end{bmatrix}, \quad Q = \frac{3(b-a)}{10}\, K = \begin{bmatrix} 0.15 & 0.01875 \\ 0.0375 & 0.1875 \end{bmatrix}, \quad r(Q) \approx 0.2 < 1,$$

$$\delta_{[a,b], D\times D}(f) ::= \begin{bmatrix} 0.176875 \\ 0.415 \end{bmatrix}, \quad \rho = \begin{bmatrix} 0.15 \\ 0.15 \end{bmatrix} \ge \frac{b-a}{2}\, \delta_{[a,b], D\times D}(f) = \begin{bmatrix} 0.03546875 \\ 0.08125 \end{bmatrix}.$$

In the case of Maple computations for iterative systems it is more appropriate to use instead of (15) a scheme with polynomial interpolation, [3,4], when instead of (15), we introduce the sequence $\{x^{q+1}_{m+1}(t,z,\eta)\}_{m=0}^{\infty}$ of vector polynomials $x^{q+1}_{m+1}(t,z,\eta) = col(x^{q+1}_{m+1,1}(t,z,\eta), x^{q+1}_{m+1,2}(t,z,\eta))$ of degree $(q+1)$

$$\begin{aligned} x^{q+1}_{m+1,j}(t,z,\eta) &:= a_{m+1,j,0}(z,\eta) + a_{m+1,j,1}(z,\eta)t + a_{m+1,j,2}(z,\eta)t^2 + \dots + a_{m+1,j,q+1}(z,\eta)t^{q+1} \\ &= z + \int\limits_a^t \Big[A_{m,j,0}(z,\eta) + A_{m,j,1}(z,\eta)t + A_{m,j,2}(z,\eta)t^2 + \dots + A_{m,j,q}(z,\eta)t^q\Big]\, dt \\ &\quad - \frac{t-a}{b-a} \int\limits_a^b \Big[A_{m,j,0}(z,\eta) + A_{m,j,1}(z,\eta)t + A_{m,j,2}(z,\eta)t^2 + \dots + A_{m,j,q}(z,\eta)t^q\Big]\, dt \\ &\quad + \frac{t-a}{b-a}\, [\eta_j - zj], \;\; t \in [a,b], \;\; m = 0,1,2,\dots, \;\; j = 1,2, \end{aligned} \tag{28}$$

where

$$A_{m,j,0}(z,\eta) + A_{m,j,1}(z,\eta)t + A_{m,j,2}(z,\eta)t^2 + \dots + A_{m,j,q}(z,\eta)t^q, \;\; j = 1,2$$

are the Lagrange interpolation polynomials of degree q on the Chebyshev nodes, translated from $(-1,1)$ to the interval (a,b), corresponding to the functions

$$f_j\Big(t, x^{q+1}_{m,1}(t,z,\eta), x^{q+1}_{m,2}(t,z,\eta), x^{q+1}_{m,1}\big(x^{q+1}_{m,1}(t,z,\eta)\big), x^{q+1}_{m,2}\big(x^{q+1}_{m,2}(t,z,\eta)\big)\Big), \;\; j = 1,2$$

respectively in (25). Note that the coefficients of the interpolation polynomials depend on the parameters z and η. On the basis of (28), instead of (24) let us define the mth approximate polynomial determining system, which consists of four algebraic equations when $j = 1,2$,

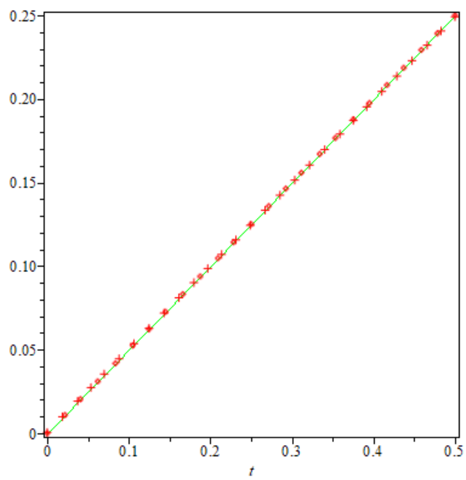
$$\begin{aligned} \Delta^q_{m,j}(z,\eta) &:= [\eta_j - z_j] \\ &\quad - \int\limits_a^b \Big[A_{m,j,0}(z,\eta) + A_{m,j,1}(z,\eta)t + A_{m,j,2}(z,\eta)t^2 + \dots + A_{m,j,q}(z,\eta)t^q\Big]\, dt = 0, \end{aligned} \tag{29}$$

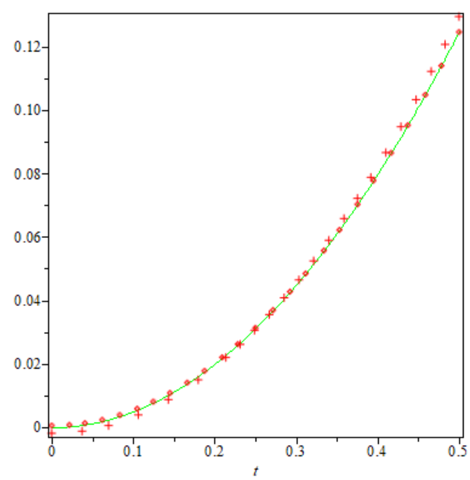
$$\Phi^q_{m,j}(z,\eta) := \Phi_j\Big(x^{q+1}_{m,1}(t,z,\eta), x^{q+1}_{m,2}(t,z,\eta), x^{q+1}_{m,1}\big(x^{q+1}_{m,1}(t,z,\eta)\big), x^{q+1}_{m,2}\big(x^{q+1}_{m,2}(t,z,\eta)\big)\Big) - d_j = 0.$$

By choosing $q = 3$, using (28) and solving (29) (applying Maple 14) we obtain the approximate numerical values for the introduced parameters given in table.

The graphs of the zeroth ($\times$), sixth ($\diamond$) approximation and the exact solution (solid line) to problem (25), (26) are shown in figure.

	z_1	z_2	η_1	η_2
$m = 0$	$0.5332693 \cdot 10^{-3}$	-0.194303210^{-2}	0.2491448305	0.1294598825
$m = 3$	$1.4024463 \cdot 10^{-7}$	$0.4841504 \cdot 10^{-3}$	0.2500002840	0.1245609255
$m = 6$	$1.3907241 \cdot 10^{-7}$	$0.4841505 \cdot 10^{-3}$	0.2500002846	0.1245609251
Exact	0	0	0.25	0.125





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