УДК 512.53, 512.64
DOI https://doi.org/10.24144/2616-7700.2023.43(2).62-66

M. V. Styopochkina

Polissia National University,
Associate professor of the department of higher and applied mathematics,
Candidate of physical and mathematical sciences
stmar@ukr.net
ORCID: https://orcid.org/0000-0002-7270-9874

THE COEFFICIENTS OF TRANSITIVITY OF THE POSETS MINIMAX ISOMORPHIC TO THE SYPERCRITICAL NON-PRIMITIVE POSET

The representations of partially ordered sets (abbreviated as posets), introduced by L. A. Nazarova and A. V. Roiter (in matrix form) in 1972, play an important role in the modern representation theory and its applications. After Yu. A. Drozd proved in 1974 that a poset S has finite representation type if and only if its Tits quadratic form

$$q_S(z) =: z_0^2 + \sum_{i \in S} z_i^2 + \sum_{i<j, i,j \in S} z_i z_j - z_0 \sum_{i \in S} z_i$$

is weakly positive (i.e., positive on the set of non-negative vectors), but not enough to be positive as for quivers, problems related to the positive and also non-negative Tits quadratic form began to be of great interest from various points of view.

In this paper we continue to study combinatorial properties of posets that are minimal with non-negative Tits quadratic form.

Key words: supercritical poset, positive and weakly positive quadratic forms, Tits quadratic form, finite representation type, minimax equivalence and isomorphism, coefficient of transitivity, nodal and neighboring elements.

1. Introduction. This paper continues the study of combinatorial properties of posets, minimax isomorphic to the supercritical posets. The cases of primitive supercritical posets were considered in [1–3]. In this paper it is considered the case of non-primitive poset, i.e. $(N, 5)$.

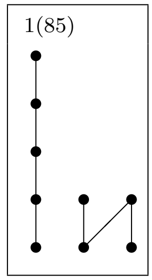
The importance of studying minimax isomorphic posets (infroduced by V. M. Bondarenko) is determined by the fact that their Tits quadratic forms are Z-equivalent, and minimax isomorphism itself is a fairly general constructively defined Z-equivalence of the Tits quadratic forms for posets.

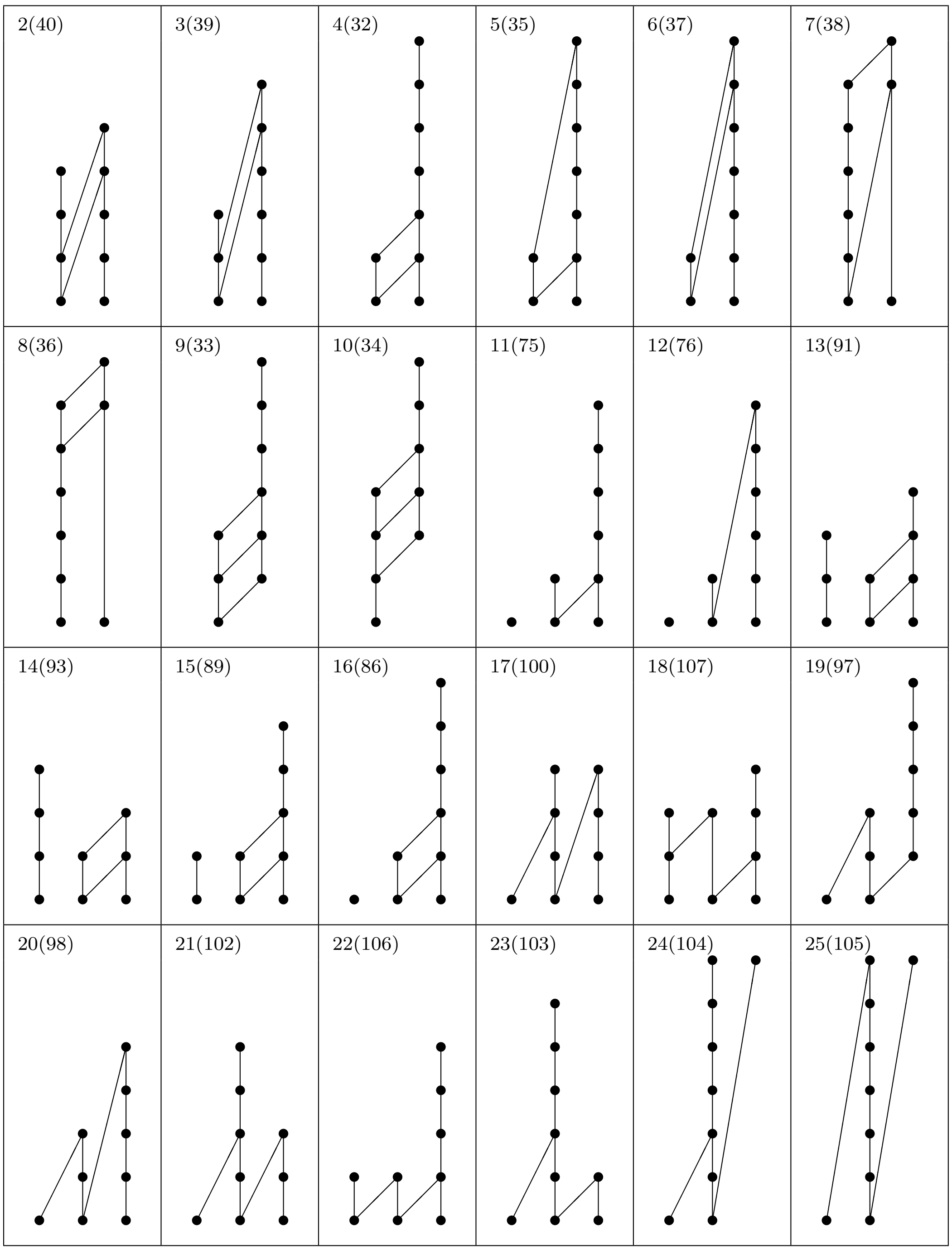
2. The list of posets minimax isomorphic to $(N, 5)$.

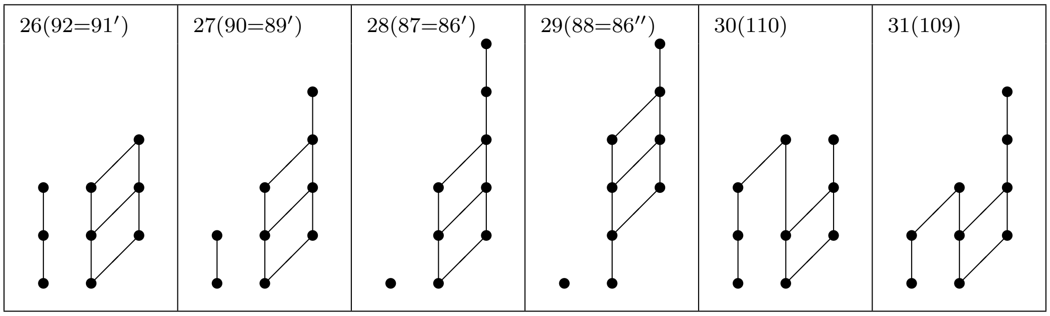
Let P be a poset. For a minimal (resp. maximal) element a of S, denote by $T = S_a^{\uparrow}$ (respect. $T = S_a^{\downarrow}$) the following poset: $T = S$ as usual sets, $T \backslash a = S \backslash a$ as posets, the element a is maximal (resp. minimal) in T, and a is comparable with x in T if and only if they are incomparable in S. Two posets S and T are called (min, max)-*equivalent* [4] if there are posets $S_1, ..., S_p (p \geq 0)$ such that, if we put $S = S_0$ and $T = S_{p+1}$, then, for every $i = 0, 1, ..., p$, either $S_{i+1} = (S_i)_{x_i}^{\uparrow}$ or $S_{i+1} = (S_i)_{y_i}^{\downarrow}$. Since some time the term *minimax equivalence* are also used.

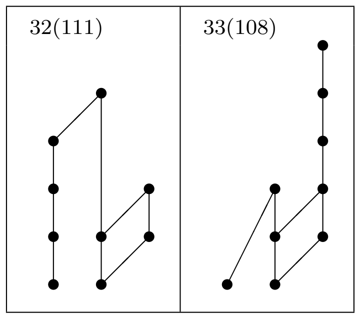
The notion of minimax equivalence can be naturally continued to the notion of *minimax isomorphism*: posets S and S' are minimax isomorphic if there exists a poset T, which is minimax equivalent to S and isomorphic to S'.

From the results of [5] it follows that the following table contains all (up to isomorphism and duality) posets which are minimax isomorphic to the the only non-primitive supercritical poset $(N, 5)$:









In the parentheses it is placed the nimbers from the table of [5].

3. Coefficients of transitivity. Main results. Let S be a poset and $S^2_< := \{(x, y) \,|\, x, y \in S, x < y\}$. If $(x, y) \in S^2_<$ and there is no z satisfying $x < z < y$, then x and y are called *neighboring*. Put $n_w = n_w(S) := |S^2_<|$ and denote by $n_e = n_e(S)$ the number of pairs of neighboring elements. The ratio $k_t = k_t(S)$ of the numbers $n_w - n_e$ and n_w we call *the coefficient of transitivity of* S. If $n_w = 0$ (then $n_e = 0$), we assume $k_t = 0$ (see [6]). Obvioisly, dual poset have the same coefficient of transitivity.

The aim of this paper is to calculate k_t for all posets minimax isomorphic to $N_6 = (N, 5) = \{1, 2, \cdots, 9 \,|\, 1 \prec 2 \prec 3 \prec 4 \prec 5, 6 \prec 7, 8 \prec 9, 6 \prec 9\}$, which is the only non-primitive supercritical poset.

Theorem 1. *The following holds for posets* 1 – 33 *minimax isomorphic to* N_6*:*

N	n_e	n_w	k_t	N	n_e	n_w	k_t	N	n_e	n_w	k_t
1	7	13	0,46154	12	7	17	0,58824	23	8	21	0,61905
2	9	19	0,52632	13	8	15	0,46667	24	8	27	0,70370
3	9	21	0,57143	14	8	13	0,38462	25	8	23	0,65217
4	9	33	0,72727	15	8	19	0,57895	26	9	15	0,4
5	9	29	0,68966	16	8	25	0,68	27	9	19	0,52632
6	9	25	0,64	17	8	15	0,46667	28	9	25	0,64
7	9	25	0,64	18	8	15	0,46667	29	9	25	0,64
8	9	29	0,68966	19	8	19	0,57895	30	9	17	0,47059
9	10	33	0,69697	20	8	15	0,46667	31	9	19	0,52632
10	10	33	0,69697	21	8	17	0,52941	32	9	17	0,47059
11	7	21	0,66667	22	8	14	0,42857	33	9	23	0,60870

The transitivity coefficients are written out with an accuracy of five decimal places. The value is exact if and only if the number of decimal places is less than five, and two values equal to exactly five digits are equal at all.

The proof is carried out by direct calculations.

Recall that the greatest length among the lengths of all linear ordered subsets of a poset S is called its *height*. An element of a poset is called *nodal*, if it is comparable with all the others elements. A subposet X of T is said to be *dense* if there is not $x_1, x_2 \in X$, $y \in T \setminus X$ such that $x_1 < y < x_2$.

Corollary 1. *The coefficient $k_t(S)$ of a poset S is the largest among all the posets minimax isomorphic to N_6 if and only if S contains a dense subposet that consists of five nodal elements.*

4. Conclusions. In this paper we investigate combinatorial spects of the only supercritical non-primitive poset, namely, describe the coefficients of transitivity for all posets that are minimax isomorphic to it. The importance of studying minimax isomorphic posets is determined by the fact that their Tits quadratic forms are Z-equivalent. The obtained results can be used in the study of combinatorial aspects of other classes of posets.

References

1. Bondarenko, V. M., & Styopochkina, M. V. (2018). On properties of posets of MM-type (1,3,5). *Scien. Bull. of Uzhhorod Univ. Series of Math. and Inform.*, 32(1), 50–53.
2. Bondarenko, V. M., & Styopochkina, M. V. (2019). On properties of posets of MM-type (1, 2, 7). *Applied problems of mech. and math.*, 17, 7–10.
3. Bondarenko, V. M., Stepochkina, M. V. , & Stoika, M. V. (2020). The coefficients of transitiveness of the posets of MM-type being the smallest sypercritical poset of width 3. *Applied problems of mech. and math.*, 18, 11–13.
4. Bondarenko, V. M. (2005). On (min, max)-equivalence of posets and applications to the Tits forms. *Bull. of Taras Shevchenko University of Kyiv. (series: Physics & Mathematics)*, 1, 24–25.
5. Bondarenko, V. M., & Stepochkina, M. V. (2009). Description of posets critical with respect to the nonnegativity of the quadratic Tits form. *Ukrainian Math. J.*, 61(5), 734–746.
6. Bondarenko, V. M., & Styopochkina, M. V. (2017). Coefficients of transitiveness of P-critical posets. *Proc. Inst. Math. of NAS of Ukraine*, 14(1), 46–51.

Стьопочкіна М. В. Коефіцієнти транзитивності частково впорядкованих множин мінімаксно ізоморфних суперкритичній непримітивній множині.

Зображення частково впорядкованих (скорочено ч. в.) множин, які введені Л. А. Назаровою і А. В. Ройтером (в матричній формі) в 1972 р., відіграють важливу роль в сучасній теорії зображень. Після того, як Ю. А. Дрозд у 1974 р. довів, що ч. в. множина S має скінченний зображувальний тип тоді і лише тоді, коли її квадратична форма Тітса

$$q_S(z) =: z_0^2 + \sum_{i \in S} z_i^2 + \sum_{i<j, i,j \in S} z_i z_j - z_0 \sum_{i \in S} z_i$$

є слабко додатною (тобто додатною на множині невід'ємних векторів), але не досить, щоб додатною як для сагайдаків, задачі, пов'язані з додатними, а також невід'ємними, квадратичними формами Тітса стали цікавими з різних точок зору.

У цій статті ми продовжуємо вивчати комбінаторні властивості ч. в. множин, що є мінімальними, для яких квадратична форма Тітса не є невід'ємною.

Ключові слова: суперкритична ч. в. множина, додатні і слабко додатні квадратичні форми, квадратична форма Тітса, скінченний зображувальний тип, мінімаксна

еквівалентність і мінімаксний ізоморфізм, коефіцієнт транзитивності, вузлові і сусідні елементи.

Список використаної літератури

1. Bondarenko V. M., Styopochkina M. V. On properties of posets of MM-type (1,3,5). *Науковий вісник Ужгородського університету: Серія «Математика і інформатика».* 2018. Вип. 32, № 1. С. 50–53.
2. Bondarenko V. M., Styopochkina M. V. On properties of posets of MM-type (1; 2; 7). *Прикладні проблеми механіки і математики.* 2019. Вип. 17. С. 7–10.
3. Bondarenko V. M., Stepochkina M. V. , Stoika M. V. The coefficients of transitiveness of the posets of MM-type being the smallest sypercritical poset of width 3. *Прикладні проблеми механіки і математики.* 2020. Вип. 18. С. 11–13.
4. Bondarenko V. M. On (min, max)-equivalence of posets and applications to the Tits forms. *Вісник Київського національного університету імені Тараса Шевченка. Математика. Механіка.* 2005. № 1. С. 24–25.
5. Бондаренко В. М., Степочкина М. В. Описание частично упорядоченных множеств, критических относительно неотрицательности квадратичной формы Титса. *Укр. мат. журнал.* 2009. Т. 61, № 5. С. 611–624.
6. Bondarenko V. M., Styopochkina M. V. Coefficients of transitiveness of P-critical posets. *Зб. праць Ін-ту математики НАН України.* 2017. Т. 14, № 1. С. 46–51.

Одержано 03.10.2023