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ON THE PARAMETRIZATION OF NONLINEAR BOUNDARY VALUE PROBLEMS WITH NONLINEAR BOUNDARY CONDITIONS

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Abstract. We consider a boundary value problem with nonlinear boundary conditions. By using a suitable parametrization, we reduce the original problem to the parametrized one containing linear boundary restrictions. We construct a numerical-analytic scheme that is suitable for the study of the solutions of the transformed boundary value problem.

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1. Introduction

Our aim is to show that, for some types of nonlinear boundary value problems with nonlinear boundary conditions, it is useful to introduce certain parametrization techniques.

2. Problem setting

We consider the following nonlinear two-point boundary value problem subjected to the nonlinear boundary conditions

$$\frac{dx(t)}{dt} = f(t, x(t)), \, t \in [0, T], \, x \in \mathbb{R}^n, \tag{2.1}$$

$$g(x(0), x(T)) = 0, \tag{2.2}$$

where $f : [0, T] \times D \to \mathbb{R}^n$ and $g : D \times D \to \mathbb{R}^n$ $(n \geq 2)$ are continuous, $D \subset \mathbb{R}^n$ is a closed and bounded domain.

We have to find a continuously differentiable solution of the system of differential equations (2.1) satisfying the nonlinear boundary restrictions (2.2).

3. Construction of an equivalent problem with linear boundary conditions

To pass to the linear boundary conditions in (2.2), we replace the values of the components of the solution (2.1), (2.2) at the points $t=0$, $t=T$ by parameters z_1, $z_2, \ldots, z_n$ and $\lambda_1, \lambda_2, \ldots, \lambda_n$:

$$\begin{aligned} x(0) = col\,(x_1(0), x_2(0), \ldots, x_n(0)) = col\,(z_1, z_2, \ldots, z_n), \\ x(T) = col\,(x_1(T), x_2(T), \ldots, x_n(T)) = col\,(\lambda_1, \lambda_2, \ldots, \lambda_n). \end{aligned} \tag{3.1}$$

Let us rewrite the boundary conditions (2.2) in the form:

$$x(T) + g\,(x(0), x(T)) = x(T). \tag{3.2}$$

Using the parametrization (3.1), the non–linear boundary restrictions (3.2) can be written as:

$$\lambda + g\,(z, \lambda) = x(T). \tag{3.3}$$

Let us put:

$$d(z,\lambda) := \lambda + g\,(z,\lambda), \tag{3.4}$$

where

$$\begin{aligned} z := x(0) = col\,(z_1, z_2, \ldots, z_n), \\ \lambda := x(T) = col\,(\lambda_1, \lambda_2, \ldots, \lambda_n). \end{aligned} \tag{3.5}$$

Taking into account (3.4), the parametrized boundary conditions (3.3) can be written in the form:

$$x(T) = d(z,\lambda), \tag{3.6}$$

where $A \equiv 0$, $C \equiv I_n$ and I_n is an initial $(n \times n)$ matrix.

So instead of the original boundary value problem of (2.1)and (2.2), we obtain an equivalent parametrized one (2.1), (3.6).

Remark 1. The set of the solutions of the non–linear boundary value problem (2.1), (2.2) coincides with the set of the solutions of the problem (2.1), (3.6) satisfying the additional conditions (3.1).

4. Construction of successive approximations

Let us introduce the vector

$$\delta_D(f) := \frac{1}{2}\left[\max_{(t,x)\in[0,T]\times D} f(t,x) - \min_{(t,x)\in[0,T]\times D} f(t,x)\right], \tag{4.1}$$

satisfying the inequality:

$$\delta_D(f) \le \max_{(t,x)\in[0,T]\times D} |f\,(t,x)|.$$

The given boundary value problem (2.1), (2.2) is such that the subset

$$D_\beta := \left\{ z \in D : B\left(z, \max_{t\in[0,T]} \left| z + \frac{t}{T}(d(z,\lambda) - z) \right| \subset D, \forall \lambda \in D \right) \right\}$$

is non–empty

$$D_\beta \neq \varnothing. \tag{4.2}$$

Assume that the function $f(t,x)$ is continuous in the domain $[0,T] \times D$ and satisfies a Lipschitz condition of the form

$$|f(t,u) - f(t,v)| \leq K\,|u - v|, \tag{4.3}$$

for all $t \in [0,T]$, $\{u,v\} \subset D$ with some non–negative constant matrix $K = \left(k_{ij}\right)_{i,j=1}^{n}$.

Moreover, we suppose that the spectral radius $r(K)$ of the matrix K satisfies the following inequality

$$r(K) < \frac{10}{3T}. \tag{4.4}$$

Let us connect with the parametrized boundary–value problem (2.1), (3.6) the sequence of functions:

$$x_m(t,z,\lambda) := z + \int_0^t f(s, x_{m-1}(s,z,\lambda))ds - \\ - \frac{t}{T}\int_0^T f(s, x_{m-1}(s,z,\lambda))ds + \frac{t}{T}\left[d(z,\lambda) - z\right], \tag{4.5}$$

where $m = 1,2,3,\ldots,$

$$x_0(t,z,\lambda) = z + \frac{t}{T}(d(z,\lambda) - z) \in D_\beta, \tag{4.6}$$

$\lambda = col(\lambda_1, \lambda_2, \ldots, \lambda_n) \in D$,
$x_m(t,z,\lambda) = col\left(x_{m,1}(t,z,\lambda), x_{m,2}(t,z,\lambda), \ldots, x_{m,n}(t,z,\lambda)\right)$ and z, λ are parameters.

It is easy to check that the functions $x_m(t,z,\lambda)$ satisfy linear boundary conditions (3.6) for all $m \geq 1$, $\lambda \in D$ and $z \in D_\beta$.

The following statement establishes the convergence of the sequence (4.5) and its relation to the original boundary–value problem (2.1), (2.2).

Theorem 1. *Assume that the function $f : [0,T] \times D \to \mathbb{R}^n$ in the right side of the system of differential equations (2.1) and the parametrized boundary restrictions (3.6) satisfy conditions (4.2)–(4.4).*

Then for all fixed $\lambda \in D$, $z \in D_\beta$:

(1) *The functions of the sequence (4.5) are continuously differentiable and satisfy the parametrized boundary conditions (3.6):*

$$x_m(T,z,\lambda) = d\,(z,\lambda)\,, \tag{4.7}$$

m=1,2,3,....

(2) *The sequence of functions (4.5) for $t \in [0,T]$ converges uniformly as $m \to \infty$ to the limit function*

$$x^*(t,z,\lambda) = \lim_{m\to\infty} x_m(t,z,\lambda). \tag{4.8}$$

(3) *The limit function $x^*(t,z,\lambda)$ satisfies the initial conditions*

$$x^*(0,z,\lambda) = z,$$

and the parametrized linear boundary conditions:

$$x^*(T,z,\lambda) = \lambda + g\,(z,\lambda)\,.$$

(4) *The limit function (4.8) for all $t \in [0,T]$ is a unique continuously differentiable solution of the integral equation*

$$x(t) = z + \int_0^t f(s,x(s))ds - \frac{t}{T}\int_0^T f(s,x(s))ds + \frac{t}{T}\left[d\,(z,\lambda) - z\right], \tag{4.9}$$

i. e. it is the solution of the Cauchy problem for the modified system of differential equations:

$$\frac{dx}{dt} = f(t,x) + \Delta(z,\lambda), \tag{4.10}$$

$$x(0) = z, \tag{4.11}$$

where

$$\Delta\,(z,\lambda) := \frac{1}{T}\left[d\,(z,\lambda) - z\right] - \frac{1}{T}\int_0^T f(s,x(s))ds. \tag{4.12}$$

(5) *The following error estimation holds:*

$$\left|x^*(t,z,\lambda) - x_m(t,z,\lambda)\right| \le \frac{20}{9}t\left(1 - \frac{1}{T}\right)Q^m(I_n - Q)^{-1}\delta_D(f), \tag{4.13}$$

where the matrix

$$Q := \frac{3T}{10}K. \tag{4.14}$$

Proof. We will prove that the sequence of functions (4.5) is a Cauchy sequence in the Banach space $C([0,T],\mathbb{R}^n)$.

First we show that $x_m(t,z,\lambda) \in D$, for all $(t,z,\lambda) \in [0,T] \times D_\beta \times D$, $m \in \mathbb{N}$.

Indeed, using the estimation of Lemma 2.3 from [3] (see also Lemma 3 [2] and Lemma 2 [1]):

$$\left|\int_0^t \left[f(\tau)-\frac{1}{T}\int_0^T f(s)ds\right]d\tau\right| \leq \frac{1}{2}\alpha_1(t)\left[\max_{t\in[0.T]} f(t)-\min_{t\in[0,T]} f(t)\right], \quad (4.15)$$

where

$$\alpha_1(t)=2t\left(1-\frac{t}{T}\right),\ |\alpha_1(t)|\leq\frac{T}{2},\ t\in[0,T], \quad (4.16)$$

relation (4.5) for $m=0$ implies that:

$$|x_1(t,z,\lambda)-x_0(t,z,\lambda)|\leq\left|\int_0^t\left[f(t,z)-\frac{1}{T}\int_0^T f(s,z)ds\right]dt\right|\leq$$

$$\leq\alpha_1(t)\delta_D(f)\leq\frac{T}{2}\delta_D(f). \quad (4.17)$$

Therefore, by virtue of (4.17), we conclude that $x_1(t,z,\lambda)\in D$ whenever $(t,z,\lambda)\in[0,T]\times D_\beta\times D$.

By induction we can easily establish that all functions (4.5) are also contained in the domain D for $\forall m=1,2,3,\ldots,\ t\in[0,T],\ z\in D_\beta,\ \lambda\in D$.

Now, consider the difference of functions:

$$x_{m+1}(t,z,\lambda)-x_m(t,z,\lambda)=\int_0^t[f(s,x_m(s,z,\lambda))-f(s,x_{m-1}(s,z,\lambda))]ds-$$

$$-\frac{t}{T}\int_0^T[f(s,x_m(s,z,\lambda))-f(s,x_{m-1}(s,z,\lambda))]ds, \quad (4.18)$$

for m=1,2,3, ...
and introduce the notation:

$$r_m(t,z,\lambda):=|x_m(t,z,\lambda)-x_{m-1}(t,z,\lambda)|,\ m=1,2,3,\ldots.$$

By virtue of the estimation (4.15) and of the Lipschitz condition (4.3), we have:

$$r_{m+1}(t,z,\lambda)\leq K\left[\left(1-\frac{t}{T}\right)\int_0^t r_m(s,z,\lambda)ds+\frac{t}{T}\int_t^T r_m(s,z,\lambda)ds\right], \quad (4.19)$$

for m=0,1,2, ...

According to (4.17)

$$r_1(t,z,\lambda)=|x_1(t,z,\lambda)-x_0(t,z,\lambda)|\leq\alpha_1(t)\delta_D(f). \quad (4.20)$$

By virtue of the statement of Lemma 3 [1] of the form

$$\alpha_{m+1}(t)\leq\frac{10}{9}\left(\frac{3}{10}T\right)^m\alpha_1(t),\ m=0,1,2,\ldots \quad (4.21)$$

we obtain for the sequence of functions

$$\alpha_{m+1}(t) = \left(1-\frac{t}{T}\right)\int_0^t \alpha_m(s)ds + \frac{t}{T}\int_t^T \alpha_m(s)ds,\ m=0,1,2,\ldots, \tag{4.22}$$

$$\alpha_0(t) = 1,\ \alpha_1(t) = 2t\left(1-\frac{t}{T}\right),$$

and the statement (4.22), from (4.19) in the case of $m=1$ follows:

$$r_2(t,z,\lambda) \le K\delta_D(f)\left[\left(1-\frac{t}{T}\right)\int_0^t \alpha_1(s)ds + \frac{t}{T}\int_t^T \alpha_1(s)ds\right] \le K\alpha_2(t)\delta_D(f)$$

By induction we can easily obtain

$$r_{m+1}(t,z,\lambda) \le K^m \alpha_{m+1}(t)\delta_D(f), \tag{4.23}$$

$m=0,1,2,\ldots,$
where $\alpha_{m+1}(t)$, $\alpha_m(t)$ are calculated according to (4.22), and $\delta_D(f)$ is given by (4.1).

By virtue of the estimate (4.21), from (4.23) we have

$$r_{m+1}(t,z,\lambda) \le \frac{10}{9}\alpha(t)\Big[Q^m\delta_D(f) + KQ^{m-1}\big|d(z,\lambda)-z\big|\Big], \tag{4.24}$$

$\forall m=1,2,3,\ldots,$ where the matrix Q is given by (4.14).

Therefore, in view of (4.24)

$$\begin{aligned}\big|x_{m+j}(t,z,\lambda)-x_m(t,z,\lambda)\big| &\le \big|x_{m+j}(t,z,\lambda)-x_{m+j-1}(t,z,\lambda)\big| + \\ +\big|x_{m+j-1}(t,z,\lambda)-x_{m+j-2}(t,z,\lambda)\big| &+ \ldots + |x_{m+1}(t,z,\lambda)-x_m(t,z,\lambda)| = \\ = \sum_{i=1}^{j} r_{m+i}(t,z,\lambda) &\le \frac{10}{9}\alpha_1(t)\sum_{i=1}^{j} Q^{m+i}\delta_D(f) = \\ &= \frac{10}{9}\alpha_1(t)Q^m\sum_{i=0}^{j-1} Q^i\delta_D(f).\end{aligned} \tag{4.25}$$

Since, due to condition (4.4), the maximum eigenvalue of the matrix Q of the form (4.14) does not exceed the unity, we have

$$\sum_{i=0}^{j-1} Q^i \le (I_n - Q)^{-1},\ \lim_{m\to\infty} Q^m = [0].$$

Therefore we can conclude from (4.25) that, according to the Cauchy criterium, the sequence $\{x_m(t,z,\lambda)\}$ of the form (4.5) uniformly converges in the domain $(t,z,\lambda) \in [0,T]\times D_\beta \times D$ to the limit function $x^*(t,z,\lambda)$. Since all functions $x_m(t,z,\lambda)$ of the sequence (4.5) satisfy the boundary conditions (3.6) for all values of the artificially introduced parameters, the limit function $x^*(t,z,\lambda)$ also satisfies these conditions.

Passing to the limit as $m \to \infty$ in equality (4.5) we show that the limit function satisfies both the integral equation (4.9) and the Cauchy problem (4.10), (4.11), where $\Delta(z,\lambda)$ is given by (4.12).

□

Consider the Cauchy problem

$$\frac{dx}{dt} = f(t,x) + \mu, t \in [0,T] \tag{4.26}$$

$$x(0) = z, \tag{4.27}$$

where $\mu = col(\mu_1, \dots, \mu_n)$ is control parameter.

Theorem 2. *Under the conditions of Theorem 1, the solution $x = x(t,z,\lambda,\mu)$ of the initial value problem (4.26), (4.27) satisfies the boundary conditions (3.6) if and only if $x = x(t,z,\lambda,\mu)$ coincides with the limit function $x^*(t,z,\lambda,\mu)$ of the sequence (4.5). Besides*

$$\mu = \mu_{z,\lambda} = \frac{1}{T}[d(z,\lambda) - z] - \frac{1}{T}\int_0^T f(s, x^*(s,z,\lambda))ds. \tag{4.28}$$

Proof. Sufficiency. Let us suppose that $\mu_{z,\lambda}$ in the right side of the system of differential equations (4.26) is given by (4.28). By virtue of Theorem 1, the limit function (4.8) of the sequence (4.5) is the unique solution of the boundary–value problem (4.26), (3.6) for fixed values of parameters z and λ when $\mu = \mu_{z,\lambda}$. Besides the limit function $x^*(t,z,\lambda,\mu)$ satisfies initial conditions (4.27), i.e. it is a solution of the Cauchy problem (4.26), (4.27) when $\mu = \mu_{z,\lambda}$.
Necessity. Let us fix an arbitrary $\bar{\mu} \in \mathbb{R}^n$ and assume that the initial value problem (4.29), (4.27):

$$\frac{dx}{dt} = f(t,x) + \bar{\mu}, t \in [0,T] \tag{4.29}$$

has a solution $\bar{x}(t)$, that satisfies the two–point boundary conditions (3.6). Then $\bar{x}(t)$ satisfies an integral equation:

$$\bar{x}(t) = z + \int_0^t f(s,\bar{x}(s))ds + \bar{\mu}t \tag{4.30}$$

for all $t \in [0,T]$.
When $t = T$ from (4.30) we get:

$$T\bar{\mu} = \bar{x}(T) - z - \int_0^T f(s,\bar{x}(s))ds. \tag{4.31}$$

Under the assumption $\bar{x}(t)$ satisfies boundary restrictions (3.6):

$$\bar{x}(T) = d(z,\lambda), \tag{4.32}$$

and the initial condition

$$\bar{x}(0) = z.$$

Substituting (4.32) in (4.31) we get:

$$\bar{\mu} = \frac{1}{T}d(z,\lambda) - \frac{1}{T}z - \frac{1}{T}\int_0^T f(s,\bar{x}(s))ds. \tag{4.33}$$

On the other hand, it is proved that the limit function $x^*(t,z,\lambda,\mu)$ is a solution of the initial value problem (4.26), (4.27) for $\mu = \mu_{z,\lambda}$ and is given by formula (4.28) and satisfies boundary conditions (3.6).

By analogy

$$x^*(t,z,\lambda,\mu) = z + \int_0^t f(s,x^*(s,z,\lambda,\mu))ds + \mu_{z,\lambda}t, \tag{4.34}$$

$$T\mu_{z,\lambda} = x^*(T,z,\lambda,\mu) - z - \int_0^T f(s,x^*(s,z,\lambda,\mu))ds, \tag{4.35}$$

$$x^*(T,z,\lambda,\mu) = d(z,\lambda), \tag{4.36}$$

$$x^*(0,z,\lambda,\mu) = z.$$

By virtue of (4.34)–(4.36) it is easy to get that

$$\mu_{z,\lambda} = \frac{1}{T}d(z,\lambda) - \frac{1}{T}z - \frac{1}{T}\int_0^T f(s,x^*(s,z,\lambda,\mu))ds. \tag{4.37}$$

Substituting (4.33) in (4.30) and (4.37) in (4.34), we get that for all $t \in [0,T]$

$$\bar{x}(t) = z + \int_0^t f(s,\bar{x}(s))ds + \frac{1}{T}[d(z,\lambda) - z] - \frac{1}{T}\int_0^T f(s,\bar{x}(s))ds, \tag{4.38}$$

$$x^*(t,z,\lambda,\mu) = z + \int_0^t f(s,x^*(s,z,\lambda,\mu))ds + \frac{1}{T}[d(z,\lambda) - z] - \\ - \frac{1}{T}\int_0^T f(s,x^*(s,z,\lambda,\mu))ds. \tag{4.39}$$

Using Theorem 1 $\bar{x}(t) \in D$ and $x^*(t,z,\lambda,\mu) \in D$. Obviously that form (4.38), (4.39) implies that

$$x^*(t,z,\lambda,\mu) - \bar{x}(t) = \int_0^t [f(s,x^*(s,z,\lambda,\mu)) - f(s,\bar{x}(s))]ds - \\ - \frac{1}{T}\int_0^T [f(s,x^*(s,z,\lambda,\mu)) - f(s,\bar{x}(s))]ds. \tag{4.40}$$

On the bases of the Lipschitz condition (4.3), from the relation (4.40) we get that the function

$$\omega(t) = |x^*(t,z,\lambda,\mu) - \bar{x}(t)|, t \in [0,T] \tag{4.41}$$

satisfies integral inequalities:

$$\omega(t) \leq K\left[\int_0^t \omega(s)ds + \frac{t}{T}\int_0^T \omega(s)ds\right] \leq K\alpha_1(t) \max_{s\in[0,T]} \omega(s), t \in [0,T], \tag{4.42}$$

where $\alpha_1(t)$ is given by (4.16).

Using (4.42) recursively, we come to an inequality:

$$\omega(t) \leq K^m \alpha_m(t) \max_{s\in[0,T]} \omega(s), t \in [0,T], \tag{4.43}$$

where $m \in \mathbb{N}$ and functions $\alpha_m(t)$ are given by the formula (4.22).

Considering the estimations (4.21), from (4.43) for each $m \in \mathbb{N}$ we get an estimation:

$$\omega(t) \leq K\alpha_1(t)\frac{10}{9}\left(\frac{3T}{10}K\right)^{m-1} \cdot \max_{s\in[0,T]} \omega(s), t \in [0,T]. \tag{4.44}$$

Letting $m \to \infty$ in the last inequality and by virtue of (4.4), we come to the conclusion that

$$\max_{s\in[0,T]} \omega(s) \leq Q^m \max_{s\in[0,T]} \omega(s) \to 0.$$

It means that the function $\bar{x}(t)$ coincides with $x^*(t,z,\lambda,\mu)$. Starting with (4.33) and (4.37), we come to the conclusion that $\bar{\mu} = \mu_{z,\lambda}$.

□

Let's find out the relation of the limit function $x = x^*(t,z,\lambda)$ of the sequence of functions (4.5) to the solution of the parametrized boundary value problem (2.1), (3.6) or the equivalent problem (2.1), (2.2).

Theorem 3. *Let the conditions (4.2)–(4.4) are hold for the boundary–value problem (2.1), (2.2).*

Then the pair $(x^*(\cdot,z^*,\lambda^*),\lambda^*)$ is the solution of the parametrized boundary value problem (2.1), (3.6) if and only if $z^* = (z_1^*, z_2^*, \ldots, z_n^*)$, $\lambda^* = (\lambda_1^*, \lambda_2^*, \ldots, \lambda_n^*)$ satisfy the following determining system of algebraic or transcendental equations

$$\Delta(z,\lambda) = \frac{1}{T}[d(z,\lambda) - z] - \frac{1}{T}\int_0^T f(s, x^*(s,z,\lambda))ds = 0, \tag{4.45}$$

$$x^*(T,z,\lambda) = \lambda. \tag{4.46}$$

Proof. It suffices to apply Theorem 2 and notice that the differential equation in (4.10) coincides with (2.1) if and only if pair (z^*,λ^*) satisfies an equation

$$\Delta(z^*,\lambda^*)=0.$$

Taking into account (3.1) and the equivalence (2.1), (2.2) and (2.1), (3.6), it is clear that $(x^*(\cdot,z^*,\lambda^*),\lambda^*)$ coincides with the solution of the parametrized boundary value problem (2.1), (3.1), (3.6) if and only if $(x^*(\cdot,z^*,\lambda^*),\lambda^*)$ satisfies an equation

$$x^*\left(T,z,\lambda^*\right)=\lambda^*.$$

It means that the pair $(x^*(\cdot,z^*,\lambda^*),\lambda^*)$ is the solution of the parametrized boundary value problem (2.1), (3.6) if and only if (4.45), (4.46) is hold.

□

The next statement proves that the system of determining equations (4.45), (4.46) defines all possible solutions of the original boundary value problem (2.1), (2.2).

Lemma 1. *Let all conditions of Theorem 1 be satisfied. Besides there exist some vectors $z\in D_\beta$ and $\lambda\in D$ that satisfy the system of determining equations (4.45), (4.46).*

Then the non-linear boundary–value problem (2.1), (2.2) has the solution $x(\cdot)$ such that:

$$\begin{aligned} x(0)&=z,\\ x(T)&=\lambda. \end{aligned} \tag{4.47}$$

Moreover this solution is given by formula

$$x(t)=x^*(t,z,\lambda),t=[0,T], \tag{4.48}$$

where $x^*(t,z,\lambda)$ is the limit function of the sequence (4.5). And if the boundary–value problem (2.1), (2.2) has a solution $x(\cdot)$, then this solution is given by (4.48), and the system of determining equations (4.45), (4.46) is satisfied when

$$z=x(0),$$

$$\lambda=x(T).$$

Proof. We will apply Theorems 2 and 3. If there exist such $z\in D_\beta$ and $\lambda\in D$ that satisfy determining system (4.45), (4.46), then, according to Theorem 3,the function (4.48) is a solution of the original boundary–value problem (2.1), (2.2). On the other hand, if $x(\cdot)$ is the solution of the original boundary–value problem (2.1), (2.2), then this function is the solution of the Cauchy problem (4.26), (4.27) for

$$\mu=0,$$

$$z=x(0).$$

As $x(\cdot)$ satisfies boundary restrictions (2.2) and equivalent conditions (3.6), by virtue of Theorem 2, the equality (4.48) holds. Besides,

$$\begin{aligned} \mu = \mu_{z,\lambda} = 0, \\ z = x(0), \end{aligned} \tag{4.49}$$

where vector λ is defined by (3.5). But, $\mu_{z,\lambda}$ is given by formula (4.28), that's why the first equation (4.45) of the determining system is satisfied, if

$$z = x(0), \lambda = col(x_1(T), \ldots, x_n(T)):$$

$$\Delta(x(0), \lambda) = 0. \tag{4.50}$$

From (3.6) follows that the second equation (4.46) of the determining system holds, too. So we specified such pairs $(z, \lambda) = (x(0), x(T))$ that satisfy the system of determining equations (4.45), (4.46), this proves the Lemma.

□

Remark 2. The main difficulty of the realization of this method is to find the limit function $x^*(\cdot, z, \lambda)$. But in most cases this problem can be solved using the properties of the approximate solution $x_m(\cdot, z, \lambda)$ that was built in an analytic form.

For $m \geq 1$ let us define the function $\Delta_m : D_\beta \times D \to \mathbb{R}^n$ by formula

$$\Delta_m(z, \lambda) := \frac{1}{T}[d(z, \lambda) - z] - \frac{1}{T}\int_0^T f(s, x_m(s, z, \lambda))ds, \tag{4.51}$$

where z and λ are given by the relation (3.5). To investigate the solubility of the parametrized boundary value problem (2.1), (3.6) we observe an approximate determining system of algebraic or transcendental equations of the form

$$\Delta_m(z, \lambda) = \frac{1}{T}[d(z, \lambda) - z] - \frac{1}{T}\int_0^T f(s, x_m(s, z, \lambda))ds = 0, \tag{4.52}$$

$$x_m(T, z, \lambda) = \lambda, \tag{4.53}$$

where $x_m(\cdot, z, \lambda)$ is a vector–function, that is defined by the recursive relation (4.5). By increasing m in the systems (4.45), (4.46) and (4.52), (4.53), one can achieve the needed precision of the approximate solution of the original boundary value problem (2.1), (2.2).

5. Example

Consider the system

$$\begin{cases} \frac{dx_1}{dt} = x_2 = f_1(t, x_1, x_2), \\ \frac{dx_2}{dt} = \frac{9}{32} + \frac{1}{16}t^2 + \frac{1}{8}tx_2 - \frac{1}{2}x_2^2 - \frac{1}{2}x_1 = f_2(t, x_1, x_2), \end{cases} \tag{5.1}$$

where $t \in [0, 1]$,

with non–linear two–point boundary conditions

$$\begin{cases} g_1(x(0),x(1)) := x_1(0)+x_1(1)-x_2(1)^2-\frac{3}{16}=0, \\ g_2(x(0),x(1)) := x_2(0)+x_1(1)-x_2(1)+\frac{1}{16}=0. \end{cases} \tag{5.2}$$

It is easy to check that an exact solution of the problem (5.1), (5.2) are the functions

$$\begin{cases} x_1^* = \frac{1}{8}t^2+\frac{1}{16}, \\ x_2^* = \frac{1}{4}t. \end{cases} \tag{5.3}$$

Suppose that the boundary–value problem (5.1), (5.2) is considered in the domain

$$D = \left\{(x_1,x_2) : |x_1| \le 1, |x_2| \le \frac{3}{4}\right\}. \tag{5.4}$$

Boundary conditions (5.2) can be rewritten in the form

$$x(1)+g(x(0),x(1)) = x(1), \tag{5.5}$$

where $g(x(0),x(1)) = col\,(g_1(x(0),x(1)), g_2(x(0),x(1)))$.

Let us replace the values of the components of the solution of the boundary–value problem (5.1), (5.2) at the points $t=0$ and $t=1$ by parameters z_1, z_2 and λ_1, λ_2 :

$$\begin{aligned} x(0) &= col\,(x_1(0),x_2(0)) = col\,(z_1,z_2), \\ x(1) &= col\,(x_1(1),x_2(1)) = col\,(\lambda_1,\lambda_2). \end{aligned} \tag{5.6}$$

Using (5.6), the boundary restrictions (5.5) can be rewritten as

$$\lambda + g(z,\lambda) = x(1), \tag{5.7}$$

where

$$\begin{aligned} z &= col\,(z_1,z_2), \\ \lambda &= col\,(\lambda_1,\lambda_2). \end{aligned} \tag{5.8}$$

Let us put

$$d(z,\lambda) := Az+\lambda+g(z,\lambda), \tag{5.9}$$

where z and λ are given by (5.8).

Using (5.9), the parametrized boundary conditions (5.7) can be written in the form:

$$Ax(0)+x(1) = d(z,\lambda). \tag{5.10}$$

It is easy to check that the matrix K from the Lipschitz condition (4.3) is

$$K = \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{7}{8} \end{pmatrix},$$

and

$$r(K) < 1.27 < \frac{10}{3T},$$

when $T=1$.

Vector $\delta_D(f)$ can be chosen as

$$\delta_D(f) \le \begin{pmatrix} \frac{3}{4} \\ \frac{355}{512} \end{pmatrix}.$$

One can verify that, for the parametrized boundary value problem in this example, all needed conditions are fulfilled. So, we can proceed with application of the numerical-analytic scheme described above, and thus construct the sequence of approximate solutions.

The result of the first iteration is:

$$
\begin{gathered}
x_{11} = z_1 + 0.5t^2\lambda_2 + 0.3125t^2 + 1.5t\lambda_1 + 0.21875t - t\lambda_2^2, \\
x_{12} = z_2 + 0.02278645833t^3 - 0.1666666666t^3\lambda_1^2 + \\
+0.02083333333t^3\lambda_1 - 0.5t^2 z_2\lambda_1 + 0.03125t^2 z_2 - 0.5t^2\lambda_1 + 0.25t^2\lambda_2^2 + \\
+0.046875t^2 - 0.007161458333t + 0.1666666667t\lambda_1^2 + 1.479166667t\lambda_1 + \\
+0.5t z_2\lambda_1 - 0.03125 z_2 t - 0.25t\lambda_2^2,
\end{gathered}
$$

for all $t \in [0,1]$.

The computation shows that the approximate solutions of the approximate determining equation are

$$
\begin{gathered}
z_1 := z_{11} = 0.06249675051, \\
z_2 := z_{12} = 0.00001071252252, \\
\lambda_1 := \lambda_{11} = 0.1875172131, \\
\lambda_2 := \lambda_{12} = 0.2500279256.
\end{gathered}
$$

The first and second components of the first approximation are

$$
\begin{gathered}
x_{11} = 0.06249675051 + 0.1250086066t^2 + 0.0000118560t, \\
x_{12} = 0.00001071252252 + 0.02083261607t^3 - \\
-0.03125578527t^2 + 0.2604403824t.
\end{gathered}
$$

The first approximation and the exact solution of the original boundary–value problem is shown on Figure 1.

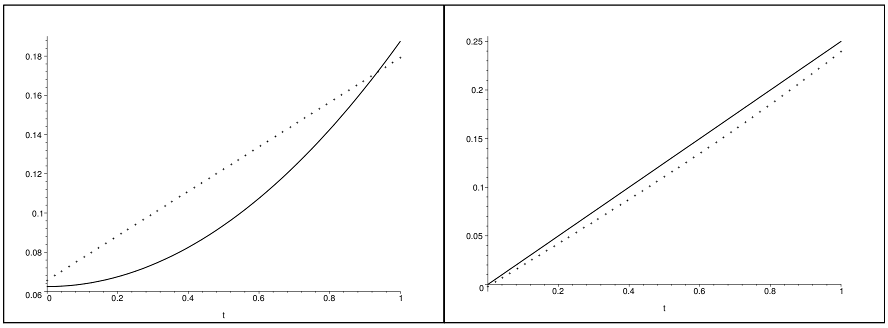

FIGURE 1. The first components of the exact solution (solid line) and its first approximation (drawn with dots)

The error of the first approximation is

$$
\max_{t\in[0,1]} \left| x_1^*(t) - x_{11}(t) \right| \le 0.33 \cdot 10^{-5},
$$

$$\max_{t\in[0,1]}\left|x_2^*(t)-x_{12}(t)\right|\le 0.98\cdot 10^{-3}.$$

The approximate solutions of the third approximate determining equation are:

$$z_1 := z_{31} = 0.06250000284,$$
$$z_2 := z_{32} = 0.0000001048559559,$$
$$\lambda_1 := \lambda_{31} = 0.1875000988,$$
$$\lambda_2 := \lambda_{32} = 0.2500002034.$$

So the first and second components of the third approximation have the form:

$$x_{31} = 0.00001089110198t + 0.1249893435t^2 - 0.000003875246484t^8 - \\ -0.0001130313603t^4 + 0.00001550100414t^7 - 0.0001103154996t^6 + \\ +0.0002115888487t^5 + 0.06250000284,$$

$$x_{32} = 0.2500862247t - 0.0008671786995t^3 - 0.000001635343948t^8 + \\ +0.001302085198t^4 + 0.00001180290239t^7 - 0.00002203673649t^6 - \\ -0.0005095320438t^5.$$

The third approximation and the exact solution of the original boundary value problem is shown on Figure 2.

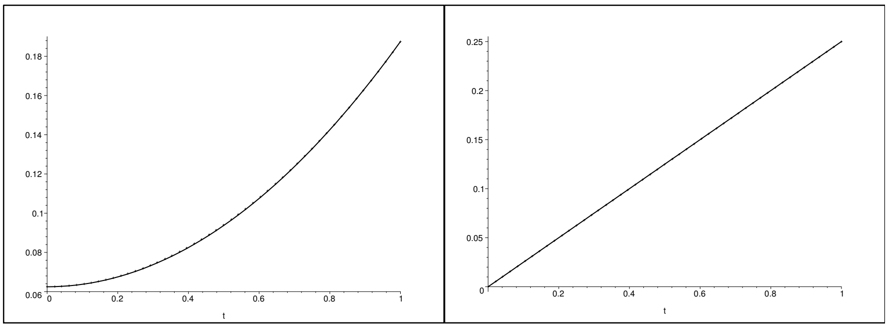

FIGURE 2. The first components of the exact solution (solid line) and its first approximation (drawn with dots)

The error of the third approximation is

$$\max_{t\in[0,1]}\left|x_1^*(t)-x_{31}(t)\right|\le 0.14\cdot 10^{-5},$$

$$\max_{t\in[0,1]}\left|x_2^*(t)-x_{32}(t)\right|\le 0.13\cdot 10^{-4}.$$

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