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## On numerical-analytic method for boundary-value problems with four-point nonlinear boundary conditions

Here we give a possible approach to transform a four-point boundary-value problem with nonlinear boundary restrictions into a two-point one with linear boundary conditions using the main ideas from [1-4].

According to the basic idea of the mentioned method, at first the original boundary-value problem is transformed to the two-point one which is replaced by the problem for the "perturbed" differential equation containing some new artificially introduced parameters, whose numerical values should be determined. The solutions of the parametrized problem are built in the analytic form by successive iterations with all iterations depending upon the artificially introduced parameters.

**Problem setting.** We consider the four-point boundary-value problem subjected to the nonlinear boundary conditions

$$\frac{dx(t)}{dt} = f(t, x(t)), t \in [0, T], x \in R^n, \tag{1}$$

$$g(x(0), x(t_1), x(t_2), x(T)) = 0, \quad 0 < t_1 < t_2 < T. \tag{2}$$

Here, we suppose that the functions

$$f : [0, T] \times D \to R^n, (n \geq 2)$$

and

$$g : D \times D \times D \times D \to R^n, (n \geq 2)$$

are continuous, where $D \subset R^n$ is a closed and bounded domain.

The problem is to find a continuously differentiable solution of the system of differential equations (1) satisfying nonlinear four-point boundary restrictions (2).

**Construction of an equivalent problem with linear two-point boundary conditions.** To pass to the linear two-point boundary conditions from (2), we replace the values of the components of the solution of (1), (2) at the points $t = 0, t = t_1, t = t_2, t = T$ by parameters:

$$\begin{aligned} z &:= x(0) = col(x_1(0), x_2(0), \ldots, x_n(0)) = col(z_1, z_2, \ldots, z_n), \\ \eta_1 &:= x(t_1) = col(x_1(t_1), x_2(t_1), \ldots, x_n(t_1)) = col(\eta_{11}, \eta_{12}, \ldots, \eta_{1n}), \\ \eta_2 &:= x(t_2) = col(x_1(t_2), x_2(t_2), \ldots, x_n(t_2)) = col(\eta_{21}, \eta_{22}, \ldots, \eta_{2n}), \\ \lambda &:= x(T) = col(x_1(T), x_2(T), \ldots, x_n(T)) = col(\lambda_1, \lambda_2, \ldots, \lambda_n). \end{aligned} \tag{3}$$

Let us rewrite the boundary conditions (2) in the form:

$$Ax(0) + Cx(T) + g(x(0), x(t_1), x(t_2), x(T)) = Ax(0) + Cx(T). \tag{4}$$

where $A$ is some given $n \times n$ matrix and $C \equiv I_n$, $I_n$ is a unit $n \times n$ matrix.

Using parametrization (3), the nonlinear four-point boundary restrictions (4)can be written as the two-point ones:

$$Ax(0) + x(T) = Az + \lambda - g(z, \eta_1, \eta_2, \lambda). \tag{5}$$

Let us put:

$$d(z, \eta_1, \eta_2, \lambda) := Az + \lambda - g(z, \eta_1, \eta_2, \lambda). \tag{6}$$

Taking into account (6) the parametrized boundary conditions (5) can be rewritten in the form:

$$Ax(0) + x(T) = d(z, \eta_1, \eta_2, \lambda). \tag{7}$$

Let us consider the special case of (7) when we take instead of $A$ the zero matrix:

$$x(0)=z,\ x(T)=d(z,\eta_1,\eta_2,\lambda). \tag{8}$$

So, instead of the original four-point boundary-value problem (1), (2) with nonlinear boundary conditions we study an equivalent parametrized two-point one (1), (8), containing already linear separated boundary restrictions.

*Remark 1.* The set of the solutions of the non-linear four-point boundary-value problem (1), (2) coincides with the set of the solutions of the two-point problem (1), (8) satisfying additional conditions (3).

**Construction of the successive approximations.** Let us introduce the vector

$$\delta_D(f):=\frac{1}{2}\left[\max_{(t,x)\in[0,T]\times D} f(t,x)-\min_{(t,x)\in[0,T]\times D} f(t,x)\right].$$

The original boundary-value problem (1), (2) is such that the subset

$$D_\beta:=\left\{z\in D: B\left(z,\ \max_{t\in[0,T]}\left|z+\frac{t}{T}[d(z,\eta_1,\eta_2,\lambda)]-z\right|\right)\subset D, \forall \eta_1,\ \eta_2,\lambda\in D\right\}$$

is non-empty

$$D_\beta\neq \text{O}. \tag{9}$$

Assume that the function $f(t,x)$ satisfies Lipschitz condition of the form

$$|f(t,u)-f(t,v)|\le K|u-v|, \tag{10}$$

for all $t\in[0,T]$, $\{u,v\}\subset D$ with some non-negative constant matrix $K=(k_{ij})_{i,j=1}^n$.

Moreover, we suppose that the spectral radius $r(K)$ of the matrix $K$ satisfies the following inequality:

$$r(K)<\frac{10}{3T}. \tag{11}$$

Let us connect with the parametrized boundary-value problem (1), (8) the sequence of functions:

$$x_m(t,z,\eta_1,\eta_2,\lambda):=z+\int_0^t f(s,x_{m-1}(t,z,\eta_1,\eta_2,\lambda))ds-$$
$$-\frac{t}{T}\int_0^T f(s,x_{m-1}(t,z,\eta_1,\eta_2,\lambda))ds+\frac{t}{T}[d(z,\eta_1,\eta_2,\lambda)-z], \tag{12}$$

where $m=1,2,3,\ldots,$

$$x_0(t,z,\eta_1,\eta_2,\lambda):=z+\frac{t}{T}[d(z,\eta_1,\eta_2,\lambda)-z]\in D_\beta,$$
$$x_m(t,z,\eta_1,\eta_2,\lambda)=col(x_{m,1}(t,z,\eta_1,\eta_2,\lambda),x_{m,2}(t,z,\eta_1,\eta_2,\lambda),\ldots,x_{m,n}(t,z,\eta_1,\eta_2,\lambda))$$

and $z,\eta_1,\eta_2,\lambda$ are considered as parameters.

It is easy to check that the functions $x_m(t,z,\eta_1,\eta_2,\lambda)$ satisfy linear parametrized boundary conditions (8) for all $m\ge 1$, $z\in D_\beta,\eta_1,\eta_2,\lambda\in D$.

The following statement establishes the convergence of the sequence (12).

***Theorem 1.*** *Assume that the function $f$ in the right side of the system of differential equations (1) and the parametrized boundary restrictions (8) satisfy conditions (9)-(11).*

*Then for all fixed $z\in D_\beta,\eta_1,\eta_2,\lambda\in D$:*

*(1) The functions of the sequence (12) are continuously differentiable and satisfy the parametrized boundary conditions (8):*

$$x_m(0,z,\eta_1,\eta_2,\lambda)=z,$$
$$x_m(T,z,\eta_1,\eta_2,\lambda)=d(z,\eta_1,\eta_2,\lambda),$$

$m=1,2,3,\ldots.$

(2) *The sequence of functions (12) for* $t \in [0,T]$ *converges uniformly as* $m \to \infty$ *to the limit function*

$$x^*(t,z,\eta_1,\eta_2,\lambda) = \lim_{m\to\infty} x_m(t,z,\eta_1,\eta_2,\lambda). \tag{13}$$

(3) *The limit function* $x^*(t,z,\eta_1,\eta_2,\lambda)$ *satisfies the parametrized linear two-point boundary conditions:*

$$x^*(0,z,\eta_1,\eta_2,\lambda) = z,$$
$$x^*(T,z,\eta_1,\eta_2,\lambda) = d(z,\eta_1,\eta_2,\lambda).$$

(4) *The limit function (13) for all* $t \in [0,T]$ *is a unique continuously differentiable solution of the integral equation*

$$x(t) = z + \int_0^t f(s,x(s))ds - \frac{t}{T}\int_0^T f(s,x(s))ds + \frac{t}{T}[d(z,\eta_1,\eta_2,\lambda) - z].$$

(5) *The following error estimation holds:*

$$\left|x^*(t,z,\eta_1,\eta_2,\lambda) - x_m(t,z,\eta_1,\eta_2,\lambda)\right| \le \frac{20}{9} t\left(1 - \frac{t}{T}\right) Q^m (I_n - Q^{-1})\delta_D(f),$$

*where*

$$Q := \frac{3T}{10} K.$$

Consider the Cauchy problem

$$\frac{dx(t)}{dt} = f(t,x(t)) + \mu, \quad t \in [0,T], \tag{14}$$

$$x(0) = z, \tag{15}$$

where $\mu = col(\mu_1,\mu_2,\ldots,\mu_n)$ is a control parameter.

***Theorem 2.*** *Under the conditions of Theorem 1 the solution* $x = x(\cdot,z,\eta_1,\eta_2,\lambda,\mu)$ *of the initial value problem (14), (15) satisfies the boundary conditions (8) if and only if* $x = x(\cdot,z,\eta_1,\eta_2,\lambda,\mu)$ *coincides with the limit function* $x^*(\cdot,z,\eta_1,\eta_2,\lambda,\mu)$ *of the sequence (12.) Moreover*

$$\mu = \mu_{z,\eta_1,\eta_2,\lambda} = \frac{1}{T}[d(z,\eta_1,\eta_2,\lambda) - z] - \frac{1}{T}\int_0^T f(s,x^*(s,z,\eta_1,\eta_2,\lambda))ds.$$

Let's find out the relation of the limit function $x = x^*(t,z,\eta_1,\eta_2,\lambda)$ of the sequence (12) to the solution of the parametrized two-point linear boundary-value problem (1), (8) or the equivalent four-point nonlinear problem (1), (2).

***Theorem 3.*** *Let the conditions (9)-(11) are hold for the original boundary-value problem (1),(2). Then the* $x^*(\cdot,z^*,\eta_1^*,\eta_2^*,\lambda^*)$ *is the solution of the parametrized boundary-value problem (1), (8) if and only if the parameters* $z^*$, $\eta_1^*$, $\eta_2^*$, $\lambda^*$ *satisfy the determining system of algebraic or transcendental equations*

$$\Delta(z,\eta_1,\eta_2,\lambda) := \frac{1}{T}[d(z,\eta_1,\eta_2,\lambda)] - \frac{1}{T}\int_0^T f(s,x^*(s,z,\eta_1,\eta_2,\lambda))ds = 0, \tag{16}$$

$$x^*(t_1,z,\eta_1,\eta_2,\lambda) = \eta_1, \tag{17}$$

$$x^*(t_2,z,\eta_1,\eta_2,\lambda) = \eta_2, \tag{18}$$

$$x^*(T,z,\eta_1,\eta_2,\lambda) = \lambda. \tag{19}$$

The next statement shows that the system of determining equations (16)-(19) defines all possible solutions of the original four-point boundary-value problem (1), (2).

***Lemma 1.*** *Let all conditions of Theorem 1 be satisfied. Furthemore there exist some vectors* $z \in D_\beta, \eta_1, \eta_2, \lambda \in D$ *that satisfy the system of determining equations (16)-(19).*

Then the non-linear four-point boundary-value problem (1), (2) has the solution $x(\cdot)$ that:

$$x(0) = z,$$
$$x(t_1) = \eta_1,$$
$$x(t_2) = \eta_2,$$
$$x(T) = \lambda.$$

*Moreover this solution is given by formula*

$$x = x^*(t, z, \eta_1, \eta_2, \lambda),\ t \in [0, T], \tag{20}$$

*where* $x^*(t, z, \eta_1, \eta_2, \lambda)$ *is the limit function of the sequence (12).*

*And if the boundary-value problem (1), (2) has a solution* $x(\cdot)$, *then this solution is given by (20), and the system of determining equations (16)-(19) is satisfied when*

$$z = x(0),$$
$$\eta_1 = x(t_1),$$
$$\eta_2 = x(t_2),$$
$$\lambda = x(T).$$

*Remark 2.* The main difficulty of realization of this method is to find the limit function $x^*(t, z, \eta_1, \eta_2, \lambda)$. But in most cases this problem can be solved using the properties of the approximate solution $x_m(t, z, \eta_1, \eta_2, \lambda)$ built in an analytic form.

For $m \geq 1$ let us define the function $\Delta_m : D_\beta \times D \times D \times D \to R^n$ by formula

$$\Delta_m(z, \eta_1, \eta_2, \lambda) := \frac{1}{T}[d(z, \eta_1, \eta_2, \lambda) - z] - \frac{1}{T}\int_0^T f(s, x_m(s, z, \eta_1, \eta_2, \lambda))ds,$$

where $z, \eta_1, \eta_2, \lambda$ are given by the relation (3).

To investigate the solubility of the parametrized boundary-value problem (1), (8) we observe an approximate determining system of algebraic or transcendental equations of the form

$$\Delta_m(z, \eta_1, \eta_2, \lambda) = \frac{1}{T}[d(z, \eta_1, \eta_2, \lambda) - z] - \frac{1}{T}\int_0^T f(s, x_m(s, z, \eta_1, \eta_2, \lambda))ds = 0, \tag{21}$$

$$x_m(t_1, z, \eta_1, \eta_2, \lambda) = \eta_1, \tag{22}$$

$$x_m(t_2, z, \eta_1, \eta_2, \lambda) = \eta_2, \tag{23}$$

$$x_m(T, z, \eta_1, \eta_2, \lambda) = \lambda. \tag{24}$$

where $x_m(t, z, \eta_1, \eta_2, \lambda)$ is a vector-function, that defines with the recursive relation (12).

Increasing $m$ systems (16)-(19) and (21)-(24) are close enough to provide needed precision of finding an approximate solution of the original boundary-value problem (1), (2).

**Example.** Consider the system

$$\begin{cases} \dfrac{dx_1(t)}{dt} = 0.05x_2 + x_1x_2 - 0.01t^3 - 0.005t^2 + 0.1\ (:= f_1(t, x_1, x_2)), \\ \dfrac{dx_2(t)}{dt} = 0.5x_1 - x_2^2 + 0.01t^4 + 0.15t\ (:= f_2(t, x_1, x_2)), \end{cases} \tag{25}$$

where $t \in \left[0, \frac{1}{2}\right]$,

with non-linear four-point boundary conditions

$$
\begin{cases}
g_1\left(x(0), x\left(\frac{1}{8}\right), x\left(\frac{1}{4}\right), x\left(\frac{1}{2}\right)\right) := x_1\left(\frac{1}{2}\right) - x_2^2(0) - x_1\left(\frac{1}{8}\right) - 0.0375 = 0, \\
g_2\left(x(0), x\left(\frac{1}{8}\right), x\left(\frac{1}{4}\right), x\left(\frac{1}{2}\right)\right) := x_1(0) + x_2\left(\frac{1}{2}\right) - x_1\left(\frac{1}{2}\right) + 2x_2\left(\frac{1}{4}\right) + 0.0125 = 0
\end{cases}
\tag{26}
$$

in the domain

$$D = \{(x_1, x_2) : |x_1| \le 0.42, |x_2| \le 0.4\}.$$

It is easy to check that an exact solution of the problem (25), (26) are the functions

$$x_1^*(t) = 0.1t,$$

$$x_2^*(t) = 0.1t^2.$$

According to the main idea of parametrization the boundary conditions (26) can be written in the form:

$$x(0) = z, \quad x\left(\frac{1}{2}\right) = d(z, \eta_1, \eta_2, \lambda), \tag{27}$$

where $z := x(0), \eta_1 := x\left(\frac{1}{8}\right), \eta_2 := x\left(\frac{1}{4}\right), \lambda := x\left(\frac{1}{2}\right)$, $d(z, \eta_1, \eta_2, \lambda) := \lambda - g(z, \eta_1, \eta_2, \lambda)$.

One can varify that, for the parametrized boundary-value problem (25), (27), all needed conditions are fulfilled. So, we can proceed with application of the numerical-analytic scheme described above and thus construct the sequence of approximate solutions.

The components of the iteration sequence (12) for the boundary-value problem (25) under the linear separated parametrized two-point boundary conditions (27) have the form

$$x_{m,1}(t, z, \eta_1, \eta_2, \lambda) := z_1 + \int_0^t f_1(s, x_{m-1,1}(t, z, \eta_1, \eta_2, \lambda))ds -$$

$$- 2t \int_0^{1/2} f_1(s, x_{m-1,1}(t, z, \eta_1, \eta_2, \lambda))ds + 2t\left[z_2^2 + \eta_{11} - z_1 + 0.0375\right],$$

$$x_{m,2}(t, z, \eta_1, \eta_2, \lambda) := z_2 + \int_0^t f_2(s, x_{m-1,2}(t, z, \eta_1, \eta_2, \lambda))ds -$$

$$- 2t \int_0^{1/2} f_2(s, x_{m-1,2}(t, z, \eta_1, \eta_2, \lambda))ds + 2t\left[2\eta_{21} - z_1 - 0.0125\right],$$

$m = 1, 2, 3, \ldots,$

$$x_{0,1}(t, z, \eta_1, \eta_2, \lambda) := z_1 + 2t\left[z_2^2 + \eta_{11} - z_1 + 0.0375\right],$$

$$x_{0,2}(t, z, \eta_1, \eta_2, \lambda) := z_2 + 2t\left[2\eta_{21} - z_1 - 0.0125\right].$$

The system of $m$- approximate determining equations (21)-(24) depending on the number of iterations for the given example is

$$\Delta_{m,1}(z, \eta_1, \eta_2, \lambda) = 2\left[z_2^2 + \eta_{11} - z_1 + 0.0375\right] - 2\int_0^{1/2} f_1(s, x_{m,1}(s, z, \eta_1, \eta_2, \lambda))ds = 0,$$

$$\Delta_{m,2}(z, \eta_1, \eta_2, \lambda) = 2\left[2\eta_{21} - z_1 - 0.0125\right] - 2\int_0^{1/2} f_2(s, x_{m,2}(s, z, \eta_1, \eta_2, \lambda))ds = 0,$$

$$x_{m,1}\left(\frac{1}{8}, z, \eta_1, \eta_2, \lambda\right) = \eta_{11}, \quad x_{m,2}\left(\frac{1}{8}, z, \eta_1, \eta_2, \lambda\right) = \eta_{12},$$

$$x_{m,1}\left(\frac{1}{4}, z, \eta_1, \eta_2, \lambda\right) = \eta_{21}, \quad x_{m,2}\left(\frac{1}{4}, z, \eta_1, \eta_2, \lambda\right) = \eta_{22},$$

$$x_{m,1}\left(\frac{1}{2}, z, \eta_1, \eta_2, \lambda\right) = \lambda_1, \quad x_{m,2}\left(\frac{1}{2}, z, \eta_1, \eta_2, \lambda\right) = \lambda_2.$$

The error of the first approximation by Maple 13 is

$$\max_{t \in \left[0, \frac{1}{2}\right]} \left| x_{11}(t) - x_1^*(t) \right| \le 3 \cdot 10^{-5},$$

$$\max_{t \in \left[0, \frac{1}{2}\right]} \left| x_{12}(t) - x_2^*(t) \right| \le 7 \cdot 10^{-4}$$

and on the third iteration step it is given by the following inequalities:

$$\max_{t \in \left[0, \frac{1}{2}\right]} \left| x_{31}(t) - x_1^*(t) \right| \le 1.72 \cdot 10^{-8},$$

$$\max_{t \in \left[0, \frac{1}{2}\right]} \left| x_{32}(t) - x_2^*(t) \right| \le 1.73 \cdot 10^{-7}.$$

The graphics of the exact and approximate solutions on the third iteration are given on the Figure 1.

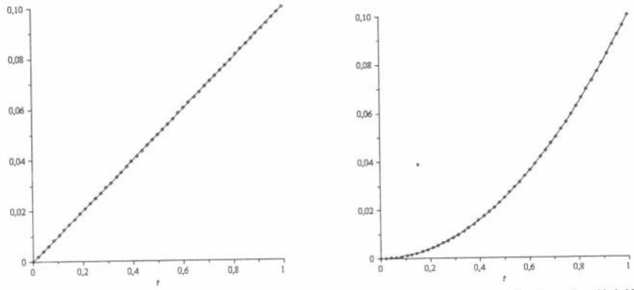


Figure 1. The graphics of the first and the second components of the exact solution (solid line ) and its third approximation (drawn with dots)

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