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QUASI MULTIPLICATIVE BASES FOR BIMODULE PROBLEMS FROM SOME CLASS

We consider the class of separated bimodule problems (K, V) such that $\operatorname{Ob}\mathsf{K} = \operatorname{Ob}\mathsf{K}^+ \cup \operatorname{Ob}\mathsf{K}^-$, which contains the linear matrix problems. For a bimodule problem satisfying some conditions with $|\operatorname{Ob}\mathsf{K}^+| = 1$, we construct explicitly the analogue of multiplicative basis which we call quasi multiplicative. This basis makes it possible to use the covering technique.

Розглянуто клас розділених бімодульних задач (K, V), який містить лінійні матричні задачі і таких, що $\operatorname{Ob}\mathsf{K} = \operatorname{Ob}\mathsf{K}^+ \cup \operatorname{Ob}\mathsf{K}^-$. У випадку $|\operatorname{Ob}\mathsf{K}^+| = 1$ та виконання деяких умов скінченності побудовано аналог мультиплікативного базису бімодульної задачі, який ми називаємо квазі мультиплікативним. Цей базис дозволяє використовувати техніку накриттів.

A classification of problems of finite and tame representation type and their indecomposable representations together with a description of their representation categories belongs to the most important problems of representation theory [1–3]. A very important and effective tool for a solution of the finiteness problem is so called "covering method". This method is especially effective when the basis is multiplicative, see [4, 5]. We give a generalization of the notion of a multiplicative basis for a class of bimodule problems containing linear matrix problems. We formulate a number of conditions for "small" subproblems providing the finiteness of their representation type. For a bimodule problem satisfying these conditions with $|\operatorname{Ob}\mathsf{K}^+| = 1$, we construct explicitly the quasi multiplicative basis. This basis makes it possible to use the covering technique for bimodule problems from considered class.

Let $\Bbbk$ be algebraically closed field. Unless otherwise stated, all the categories we consider are the categories over $\Bbbk$, all morphism spaces are finite dimensional, and all functors are $\Bbbk$-linear. A category K is called *local*, provided for every $X \in \operatorname{Ob}\mathsf{K}$ the endomorphism algebra $\mathsf{K}(X, X)$ is local, and *regular*, if, in addition, every invertible morphism is automorphism. A category K is called *fully additive* or *Krull-Schmidt category* if K is a category with finite direct sums and every idempotent from K splits, i. e. it has kernel and cokernel.

A full subcategory $\mathsf{K}_0 \subset \mathsf{K}$ will be called *an additive skeleton* of K, provided K_0 is regular and every $X \in \operatorname{Ob}\mathsf{K}$ is isomorphic to a finite direct sum of objects from K_0. For a local category K and for every $X \in \operatorname{Ob}\mathsf{K}$ there exists the decomposition $\mathsf{K}(X, X) = \Bbbk 1_X \oplus \operatorname{Rad} X$, where $\operatorname{Rad} X$ is the Jacobson radical of the algebra $\mathsf{K}(X, X)$. If K is regular, then we denote by $\operatorname{Rad}\mathsf{K}$ the *radical* of K, i. e. an ideal in K such that $\operatorname{Rad}\mathsf{K}(X, Y) = \mathsf{K}(X, Y)$ for $X \neq Y$, and $\operatorname{Rad}\mathsf{K}(X, X) = \operatorname{Rad} X$, $X, Y \in \operatorname{Ob}\mathsf{K}$.

Let V be a K-bimodule ([6]). A category K (a bimodule V) is called *locally finite dimensional*, if for any $X \in \operatorname{Ob}\mathsf{K}$ the spaces $\bigoplus_{Y \in \operatorname{Ob}\mathsf{K}} \mathsf{K}(X, Y)$ and $\bigoplus_{Y \in \operatorname{Ob}\mathsf{K}} \mathsf{K}(Y, X)$ ($\bigoplus_{Y \in \operatorname{Ob}\mathsf{K}} \mathsf{V}(X, Y)$ and $\bigoplus_{Y \in \operatorname{Ob}\mathsf{K}} \mathsf{V}(Y, X)$) are finite dimensional, and *finite dimensional*, provided all the spaces $\bigoplus_{X, Y \in \operatorname{Ob}\mathsf{K}} \mathsf{K}(X, Y)$ ($\bigoplus_{X, Y \in \operatorname{Ob}\mathsf{K}} \mathsf{V}(X, Y)$) are finite dimensional.

Given a category K, we denote by $\operatorname{add}\mathsf{K}$ an *additive hull* of K, i. e. a minimal fully additive category which contains K. For a K-bimodule V, we denote by $\operatorname{add}\mathsf{V}$ the corresponding $\operatorname{add}\mathsf{K}$-bimodule.

A pair $\mathcal{A} = (\mathsf{K}, \mathsf{V})$ consisting of a category K and a K-bimodule V is called a *bimodule problem* over K or shortly bimodule problem. A bimodule problem $\mathcal{A}$ will be called *normal*, provided the category K is regular, and both K and V are locally finite dimensional All the bimodule problems we will consider are assumed to be normal. Given some $S \subset \operatorname{Ob}\mathsf{K}$ denote by K_S the full subcategory of K with $\operatorname{Ob}\mathsf{K}_S = S$, and by V_S the subbimodule $\mathsf{V}|_S = \mathsf{K}_S\mathsf{V}\mathsf{K}_S$. A bimodule problem $\mathcal{A}_S = (\mathsf{K}_S, \mathsf{V}_S)$ is called the *restriction* of $\mathcal{A}$ to S.

For a bimodule problem $\mathcal{A} = (\mathsf{K}, \mathsf{V})$, *a representation* M of $\mathcal{A}$ is a pair $M = (M_{\mathsf{K}}, M_{\mathsf{V}})$, where $M_{\mathsf{K}} \in \operatorname{Ob}\operatorname{add}\mathsf{V} = \operatorname{Ob}\operatorname{add}\mathsf{K}$ and $M_{\mathsf{V}} \in \operatorname{add}\mathsf{V}(M_{\mathsf{K}}, M_{\mathsf{K}})$. If M, N are two representations of $\mathcal{A}$, then a *morphism* f from M to N is a morphism $f \in \operatorname{add}\mathsf{K}(M_{\mathsf{K}}, N_{\mathsf{K}})$ such that $N_{\mathsf{V}} \cdot f - f \cdot M_{\mathsf{V}} = 0$. The composition of morphisms and the unit morphisms in the representation category $\operatorname{rep}\mathcal{A}$ and in the category $\operatorname{add}\mathsf{K}$ coincide.

Given two bimodule problems $\mathcal{A} = (\mathsf{K}, \mathsf{V})$ and $\mathcal{A}' = (\mathsf{K}', \mathsf{V}')$, a *morphism* of bimodule problems $\theta : \mathcal{A} \to \mathcal{A}'$ is a pair $\theta = (\theta_0, \theta_1)$, where $\theta_0 : \mathsf{K} \to \mathsf{K}'$ is a $\Bbbk$-functor, $\theta_1 : \mathsf{V} \to \mathsf{V}'$ is a K-bimodule morphism with the K-bimodule structure on V' induced by θ_0 ([6]).

Let $\mathcal{A} = (\mathsf{K}, \mathsf{V})$ be a normal bimodule problem. Bigraph $\Sigma (= \Sigma_{\mathcal{A}}) = (\Sigma_0, \Sigma_1)$ is called a *basis* of the bimodule problem $\mathcal{A}$, if $\Sigma_0 = \operatorname{Ob}\mathsf{K}$, $\Sigma_1^0(X, Y)$ is a basis of $\mathsf{V}(X, Y)$, and $\Sigma_1^1(X, Y)$ is a basis of $\operatorname{Rad}\mathsf{K}(X, Y)$, $X, Y \in \operatorname{Ob}\mathsf{K}$. For all $x, y \in \Sigma_1$ such that the product xy is not specified, we assume $xy = 0$.

Let V be a K-bimodule. We say that $x \in \operatorname{Rad}\mathsf{K}(X, Y)$ *annihilates the bimodule* V, if $xa = 0$, $bx = 0$ for any $Z \in \operatorname{Ob}\mathsf{K}$, $a \in \mathsf{V}(Z, X)$, $b \in \mathsf{V}(Y, Z)$. The ideal of the category K consisting of all annihilate elements is called the *annihilator* of V and is denoted by $\operatorname{Ann}_{\mathsf{K}}(\mathsf{V})$. A bimodule V is called *faithful* provided $\operatorname{Ann}_{\mathsf{K}}(\mathsf{V}) = 0$. We call a bimodule problem $\mathcal{A} = (\mathsf{K}, \mathsf{V})$ *faithful*, if the bimodule V is faithful. For a bimodule problem $\mathcal{A}$, a *faithful part* of $\mathcal{A}$ is defined as the faithful bimodule problem $\mathcal{A}_{\mathrm{red}}$, $\mathcal{A}_{\mathrm{red}} = (\mathsf{K}_{\mathrm{red}}, \mathsf{V})$, where $\mathsf{K}_{\mathrm{red}} = \mathsf{K}/\operatorname{Ann}_{\mathsf{K}}(\mathsf{V})$.

Let $\mathcal{A} = (\mathsf{K}, \mathsf{V})$ be a normal bimodule problem, $\mathsf{R} = \operatorname{Rad}\mathsf{K}$, Σ be a basis of $\mathcal{A}$. The integer N is called the global triangled height of $\mathcal{A}$ if $\mathsf{R}^{N+1} = 0$, $\mathsf{R}^N\mathsf{V} = 0$, and either or $\mathsf{R}^N \neq 0$ or $\mathsf{R}^{N-1}\mathsf{V} \neq 0$. Denote by $\mathsf{V}_i = \mathsf{R}^{i-1}\mathsf{V}$, $i = 1, \ldots, N$. We have two filtrations:

$$\mathsf{R} \supset \mathsf{R}^2 \supset \ldots \supset \mathsf{R}^N \supset 0, \qquad \mathsf{V}_1 \supset \mathsf{V}_2 \supset \ldots \supset \mathsf{V}_N \supset 0. \tag{1}$$

The map $h : \mathsf{R} \cup \mathsf{V} \to \mathbb{N}$ such that $h(x) = i$ if $x \in (\mathsf{R}^i \backslash \mathsf{R}^{i+1}) \cup (\mathsf{V}_i \backslash \mathsf{V}_{i+1})$, is called the *triangled height* of an element. The element $x \in \Sigma_1$ is called *minimal*, if $h(x) = 1$. Let $h(0) = \infty$. Then $h(xy) \geqslant h(x) + h(y)$ and $h(x + y) \geqslant \max\{h(x), h(y)\}$ for all $x, y \in \Sigma_1$. Let $\Sigma_1^{k(i)} = \{x \in \Sigma_1^k \mid h(x) = i\}$, $i = 1, \ldots, N$, $k = 0, 1$. Clearly, the set $\{\Sigma_1^{k(i)}, i = 1, \ldots, N\}$ is a partition of Σ_1^k, $k = 0, 1$.

Definition 1. *The basis Σ of bimodule problem $\mathcal{A}$ is called triangled (with respect to the filtration (1)), if $\bigcup_{l=i}^{N} \Sigma_1^{1(l)}$ is a basis of R^i, $\bigcup_{l=i}^{N} \Sigma_1^{0(l)}$ is a basis of V_i, $i = 1, \ldots, N$.*

Lemma 1 ([2]). *Every normal bimodule problem $\mathcal{A}$ with the nilpotent radical has a triangled basis.*

Definition 2. *A bimodule problem* $\mathcal{A} = (\mathsf{K}, \mathsf{V})$ *is called admitted if the set* $\operatorname{Ob}\mathsf{K}$ *can be decomposed in a disjoint union* $\operatorname{Ob}\mathsf{K} = \operatorname{Ob}\mathsf{K}^+ \cup \operatorname{Ob}\mathsf{K}^-$ *such that* $\mathsf{V}(X,Y) \neq 0$ *implies* $X \in \operatorname{Ob}\mathsf{K}^-$, $Y \in \operatorname{Ob}\mathsf{K}^+$ *and* $\operatorname{Rad}\mathsf{K}(X,Y) \neq 0$ *implies* $X, Y \in \operatorname{Ob}\mathsf{K}^+$.

The property of a bimodule problem $\mathcal{A}$ to be admitted depends only on the bigraph $\Sigma_{\mathcal{A}}$, therefore we will use the notation $\Sigma_0^+ = \operatorname{Ob}\mathsf{K}^+$ and $\Sigma_0^- = \operatorname{Ob}\mathsf{K}^-$.

Remark 1. *For a triangled basis* Σ *of an admitted bimodule problem, the following properties hold:*

1) $\Sigma_1^{1(i)}$ *is a basis of* $\mathsf{R}^i/\mathsf{R}^{i+1}$ *modulo* R^{i+1}, $\Sigma_1^{0(i)}$ *is a basis of* $\mathsf{V}_i/\mathsf{V}_{i+1}$ *modulo* V_{i+1}, $i = 1, \dots, N$;

2) for any $x \in \Sigma_1$, $h(x) = i$ *if and only if* $x \in {\Sigma_1}^{(i)}$;

3) for $x \in \mathsf{R}$ *the equality* $x = \sum\limits_{\varphi \in \Sigma_1^1} \lambda_\varphi \varphi$, $\lambda_\varphi \in \Bbbk$, *implies* $h(\varphi) \geqslant h(x)$ *for any* $\varphi \in \Sigma_1^1$ *with* $\lambda_\varphi \neq 0$;

4) for $x \in \mathsf{V}$ *the equality* $x = \sum\limits_{a \in \Sigma_1^0} \lambda_a a$, $\lambda_a \in \Bbbk$, *implies* $h(a) \geqslant h(x)$ *for any* $a \in \Sigma_1^0$ *with* $\lambda_a \neq 0$.

Remark 2. *Let* $\mathcal{A}$ *be an admitted bimodule problem with the global triangled height* N. *There are the decompositions of* $\Bbbk$*-vector spaces:*

$$\mathsf{V}_i = \bigoplus_{E \in \Sigma_0^-, A \in \Sigma_0^+} \mathsf{V}_i(E, A), \quad \mathsf{R}^i = \bigoplus_{A, B \in \Sigma_0^+} \mathsf{R}^i(A, B), \quad i = 1, \dots, N$$

with the multiplications:

$$\begin{aligned} \mathsf{R}^i(A,B) \times \mathsf{V}_j(E,A) &\to \mathsf{V}_{i+j}(E,B), \qquad A, B \in \Sigma_0^+, E \in \Sigma_0^-, \\ \mathsf{R}^i(B,C) \times \mathsf{R}^j(A,B) &\to \mathsf{R}^{i+j}(A,C), \qquad A, B, C \in \Sigma_0^+. \end{aligned}$$

Definition 3. *Let* $E \in \Sigma_0^-$, $A, B \in \Sigma_0^+$ *(in particular,* $A = B$*),* $a \in \mathsf{V}(E,A)$, $b \in \mathsf{V}(E,B)$. *We say that* $a \underset{\mathsf{R}}{<} b$ *if there exists* $r \in \mathsf{R}(A,B)$ *such that* $ra = b$, *we shall write or* $a <_r b$. *The partial order* $\underset{\mathsf{R}}{<}$ *on* V *is transitive due to the associativity: if* $a_1 <_{r_1} a_2$ *and* $a_2 <_{r_2} a_3$ *then* $a_1 <_r a_3$ *with* $r = r_2 r_1$. *The order* $\underset{\mathsf{R}}{<}$ *on* V *is non-reflexive due to the associativity and triangularity conditions. Two elements* $a, b \in \mathsf{V}$ *are called comparable is either* $a \underset{\mathsf{R}}{<} b$ *or* $b \underset{\mathsf{R}}{<} a$. *For* $A \in \Sigma_0^+$ *let* $\operatorname{ord} A = \dim_\Bbbk \sum\limits_{E \in \Sigma_0^-} \mathsf{V}(E,A)$.

Let us define the class $\mathcal{C}$ of bimodule problems we will consider. Let $\mathcal{A} \in \mathcal{C}$ if and only if $\mathcal{A}$ is normal admitted faithful bimodule problem with the nilpotent radical R and the triangled basis Σ with respect to the filtration (1) such that for any $E \in \Sigma_0^-$, $A, B \in \Sigma_0^+$, $A \neq B$:

1) $\operatorname{ord} A \leqslant 3$;

2) if $a_1, a_2 \in \mathsf{V}(E,A)$ are linearly independent, then a_1, a_2 are comparable;

3) if $\operatorname{ord} A = \operatorname{ord} B = 3$, then any $a \in \mathsf{V}(E,A)$, $b \in \mathsf{V}(E,B)$ are comparable;

4) if $\mathsf{R}(A,B) = \{\varphi\}$, then $\dim_\Bbbk (\varphi \sum\limits_{E \in \Sigma_0^-} \mathsf{V}(E,A)) < 3$.

Remark that if one of the conditions 1)–4) is not true, then bimodule problem $\mathcal{A}$ is of strictly unbounded type ([2]).

For any $x \in \mathsf{R} \cup \mathsf{V}$ there is a basis decomposition $x = \sum_{y \in \Sigma_1} \lambda_y y \in \Sigma_1$, $\lambda_y \in \Bbbk$. Denote by $\operatorname{con}_y x = \lambda_y$ the content of y in x. Two nonzero elements $x, y \in \mathsf{K} \cup \mathsf{V}$ are called proportional if $\Bbbk^* x = \Bbbk^* y$, in this case we write $x \approx y$.

Let $A, B \in \Sigma_0^+$ (in particular, $A = B$). For $\varphi, \psi \in \mathsf{R}(A, B)$ we say that $\varphi < \psi$ if $\psi \in \mathsf{R}(B, B)\varphi \cup \varphi\mathsf{R}(A, A) \cup \mathsf{R}(B, B)\varphi\mathsf{R}(A, A)$. If $\varphi < \psi$, then $h(\varphi) < h(\psi)$.

Definition 4. *Given $A, B \in \operatorname{Ob} \mathsf{K}^+$, define by $\mathcal{A}^{(A,B)} = (\mathsf{K}^{(A,B)}, \mathsf{V}^{(A,B)})$ the full restriction of the bimodule problem $\mathcal{A}$ to the set $S_{A,B} = \{A, B\} \cup \{s(a) \mid a \in \Sigma_1^0, e(a) \in \{A, B\}\}$, where $s, e : \Sigma_1 \to \Sigma_0$ are maps taking starting $s(a)$ and ending $e(x)$ vertices of an arrow a. Denote by $\Sigma^{(A,B)} \subset \Sigma$ the restriction of the basis Σ of $\mathcal{A}$ to $\mathcal{A}^{(A,B)}$. We will write $\mathcal{A}^{(A)}$ instead of $\mathcal{A}^{(A,A)}$ in the case $A = B$. The bimodule problems $\mathcal{A}^{(A,B)}$ and $\mathcal{A}^{(A)}$ inherit the triangled structure from $\mathcal{A}$ (and may have the proper one). Denote by $h' : \Sigma_1^{(A,B)} \to \mathbb{N}$ the induced triangled height.*

Let $S \subset \Sigma_0$, and $\mathcal{A}_S = (\mathsf{K}_S, \mathsf{V}_S)$ be a restriction of bimodule problem $\mathcal{A} = (\mathsf{K}, \mathsf{V})$ to the set S. Given a faithful bimodule problem $\mathcal{A}$, the bimodule problem $\mathcal{A}_S$ can be non-faithful.

Remark 3. *Let $\mathcal{A}$ be an admitted bimodule problem, and $A, B \in \Sigma_0^+$. If $\mathcal{A}$ is faithful, then bimodule problems $\mathcal{A}^{(A,B)}$ and $\mathcal{A}^{(A)}$ are faithful as well. Moreover, $\mathsf{R}(A, B) = \operatorname{Rad}(\mathcal{A}^{(A,B)})(A, B)$. This fact follows from the equality*

$$\operatorname{Ann}_{\mathsf{K}}(\mathsf{V}) = \bigcup_{A,B \in \operatorname{Ob} \mathsf{K}^+} \operatorname{Ann}_{\mathsf{K}^{(A,B)}}(\mathsf{V}^{(A,B)}).$$

Given a bimodule problem $\mathcal{A}$, a change of Σ consists of a family of changes of bases in all $\mathsf{V}(E, A)$ (the change of Σ_1^0) and all $\mathsf{R}(A, B)$ (the change of Σ_1^1), $A, B \in \Sigma_0^+$, $E \in \Sigma_0^-$. These new bases gives the new basis $\Sigma'_{\mathcal{A}}$ of $\mathcal{A}$. The change is called triangled, provided both $\Sigma_{\mathcal{A}}$ and $\Sigma'_{\mathcal{A}}$ allow a triangled filtration.

Definition 5. *Let $x, y \in \Sigma_1^i$, $i = 0, 1$, and $s(x) = s(y)$, $e(x) = e(y)$. The change of basis from Σ to Σ' is called an elementary change provided $x' = x + \lambda y$, $\lambda \in \Bbbk$, and $z' = z$ for all $z \in \Sigma_1^i \backslash \{x\}$. An elementary change is called correct, if $h(x) \leqslant h(y)$. The change of basis from Σ to Σ' is called standard if it is the superposition of correct elementary changes. We will use only standard changes of basis. Usually we do not modify the notations of basic elements after the elementary change of basis and write x instead x'.*

For $a, b \in \Sigma_1^0(E, A)$ let

$$\mathsf{S}(a, b) = \{\xi \in \Sigma_1^1(A, B) \mid \operatorname{con}_b(\xi a) \neq 0\},$$
$$\Sigma_1^1(a, b) = \{\xi \in \Sigma_1^1(A, B) \mid \xi a \approx b\} \subset \mathsf{S}(a, b).$$

For $\varphi \in \Sigma_1^1$, denote $\mathrm{P}_\varphi = \{(a, b) \in \Sigma_1^0 \times \Sigma_1^0 \mid \varphi \in \mathsf{S}(a, b)\}$.

Remark 4. *Let $a, b \in \Sigma_1^0$. If $\varphi, \psi \in \mathsf{S}(a, b)$ and $h(\varphi) \leqslant h(\psi)$, then there is a correct elementary change of basis $\varphi' = \varphi - \lambda\psi$, $\lambda \in \Bbbk^*$, such that $\mathsf{S}'(a, b) = \mathsf{S}(a, b) \backslash \{\varphi\}$. Indeed, since $h(\varphi) \leqslant h(\psi)$, then the elementary change of basis $\varphi' = \varphi - \dfrac{\operatorname{con}_b(\varphi a)}{\operatorname{con}_b(\psi a)}\psi$ is correct and leads to the condition $\mathsf{S}'(a, b) = \mathsf{S}(a, b) \backslash \{\varphi\}$.*

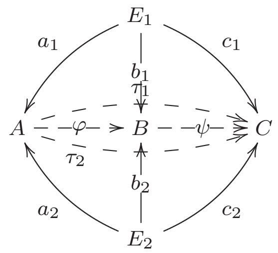
Definition 6. *The triangled basis Σ of a bimodule problem $\mathcal{A} \in \mathcal{C}$ is called quasi multiplicative if the following properties hold.*

1) For any $a \in \Sigma_1^0$ and $\varphi \in \Sigma_1^1$ there exists $b \in \Sigma_1^0$ such that $\varphi a \approx b$ whenever $\varphi a \neq 0$. In particular, $\mathsf{S}(a,b) = \Sigma_1^1(a,b)$.

2) Given $\varphi, \psi \in \Sigma_1^1$ with $\psi\varphi \neq 0$, one of the following conditions holds:

(a) there is $\tau \in \Sigma_1^1$ such that $\psi\varphi \approx \tau$;

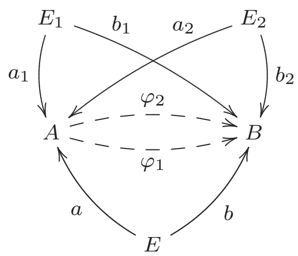
(b) there are $\tau_1, \tau_2 \in \Sigma_1^1$ such that $\psi\varphi = \lambda_1\tau_1 + \lambda_2\tau_2$, $\lambda_1, \lambda_2 \in \Bbbk^*$, and there exist $a_i, b_i, c_i \in \Sigma_1^0$, $i = 1, 2$ such that $\varphi a_i \approx b_i$, $\psi b_i \approx c_i$, $i = 1, 2$, and $\tau_j a_i = \delta_{ij} c_i$, $i, j = 1, 2$ (here δ_{ij} is a Kronecker symbol). It is possible here that two of the vertices A, B, C are equal.



3) For any $a, b \in \Sigma_1^0$ inequality $|\Sigma_1^1(a,b)| \leqslant 2$ holds. If $\Sigma_1^1(a,b) = \{\varphi_1, \varphi_2\}$, $a \in \Sigma_1^0(E, A)$, $b \in \Sigma_1^0(E, B)$, then $A \neq B$, $\operatorname{ord} A = \operatorname{ord} B = 3$, and the following conditions hold:

(a) $|\mathrm{P}_{\varphi_1}| = |\mathrm{P}_{\varphi_2}| = 2$ and $|\mathrm{P}_{\varphi_1} \cap \mathrm{P}_{\varphi_2}| = 1$;

(b) if $\mathrm{P}_{\varphi_i} = \{(a,b), (a_i, b_i)\}$, $i = 1, 2$, then $\Sigma_1^1(a_i, b_i) = \{\varphi_i\}$ and $a_1 \neq a_2$, $b_1 \neq b_2$.



Lemma 2. *Let $\mathcal{A} \in \mathcal{C}$ be a bimodule problem with triangled basis Σ, $A \in \Sigma_0^+$, $E \in \Sigma_0^-$, let $\mathcal{A}_{\{A,E\}}$ be the restriction of $\mathcal{A}$, let $\mathcal{A}' = \mathcal{A}'_{\{A,E\}} = (\mathsf{K}', \mathsf{V}')$ be the faithful part of $\mathcal{A}_{\{A,E\}}$, let Σ' be the restriction of Σ to $\mathcal{A}_{\{A,E\}}$ and $\mathsf{R}' = \mathsf{R}(A,A)/\operatorname{Ann}_{\mathsf{R}} \mathsf{V}'(E,A)$. Then $\Sigma'^0_1 = \Sigma_1^0(E,A)$, $\Sigma'^1_1 \subset \Sigma_1^1(A,A)$, $\mathsf{R}'^3 = 0$ and after the suitable correct in $\mathcal{A}$ triangled change of basis we obtain one of the following possible cases:*

1) $\dim_\Bbbk \mathsf{V}' = 1$, $\Sigma_1^0(E,A) = \{a_1\}$, $\mathsf{R}' = 0$.

2) $\dim_\Bbbk \mathsf{V}' = 2$, $\dim_\Bbbk \mathsf{V}'_2 = 1$ (see (1)). Then $\Sigma_1^0(E,A) = \{a_1, a_2\}$ with $a_2 \in \mathsf{V}'_2$, $a_1 \in \mathsf{V}'_1 \backslash \mathsf{V}_2$, and $\mathsf{R}' = \{\alpha_{12}\}$, $\alpha_{12} a_1 = a_2$.

3) $\dim_\Bbbk \mathsf{V}' = 3$, $\dim_\Bbbk \mathsf{V}'_1/\mathsf{V}'_2 = \dim_\Bbbk \mathsf{V}'_2/\mathsf{V}'_3 = \dim_\Bbbk \mathsf{V}'_3 = 1$. Let $\Sigma_1^0(E,A) = \{a_1, a_2, a_3\}$ with $a_3 \in \mathsf{V}'_3$, $a_2 \in \mathsf{V}'_2 \backslash \mathsf{V}'_3$, $a_1 \in \mathsf{V}'_1 \backslash \mathsf{V}'_2$, then for the radical R' the following holds: $\Sigma'^1_1(A,A) = \{\alpha_{12}, \alpha_{23}, \alpha_{13}\}$, where $\alpha_{12}, \alpha_{23} \in \mathsf{R}'/\mathsf{R}'^2$, $\alpha_{13} \approx \alpha_{23}\alpha_{12} \in \mathsf{R}'^2$, $\alpha_{ij} a_i \approx a_j$, and, probably, $\alpha_{12} = \alpha_{23}$.

Proof. By the definition of class $\mathcal{C}$, $\dim_\Bbbk \mathsf{V}' \leqslant 3$ and $\dim_\Bbbk \mathsf{V}'_i/\mathsf{V}'_{i+1} \leqslant 1$, $i = 1, 2$. Then $\mathsf{R}'^3 = 0$ since $\mathcal{A}$ is faithful.

Assume $\mathsf{R}' \neq 0$. Then $\mathsf{V}'_2 \neq 0$. Denote $\mathsf{V}'_1 = \{a_1\}$, $\mathsf{V}'_2 = \{a_2\}$. There is $\varphi \in \mathsf{R}'_1$ such that $\varphi a_1 \approx a_2$ (since $\mathsf{R}'_1 \cdot \mathsf{V}'_1 = \mathsf{V}'_2$). Choose the element φ from $X_{12} = \{\xi \in \mathsf{R}(A,A) \mid \mathrm{con}_{a_2}\,\xi a_1 \neq 0\}$ with the minimal height $h'(\varphi)$. If $|X_{12}| > 1$ then we do the correct elementary change of basis (as in remark 4) in order to obtain $X'_{12} = X_{12}\backslash\{\varphi\}$. Otherwise, one of the following cases occur:

1) $\varphi a_1 \approx a_2$;

2) $\dim_{\Bbbk}\mathsf{V}' = 3$, $\mathsf{V}'_3 = \{a_3\}$, and $\varphi a_1 = \lambda_2 a_2 + \lambda_3 a_3$; in this case we do the correct elementary change of basis $a'_2 = a_2 + \frac{\lambda_3}{\lambda_2} a_3$ to obtain $\varphi a_1 \approx a'_2$.

So it remains to consider the case $\dim_{\Bbbk}\mathsf{V}' = 3$, $\mathsf{V}'_3 = \{a_3\}$, and $X_{12} = \{\varphi_{12}\}$, $\varphi_{12}a_1 = \lambda_{12}a_2$, $\lambda_{12} \in \Bbbk$. For any $\psi \in X_{23} = \{\xi \in \mathsf{R}(A,A) \mid \mathrm{con}_{a_3}\,\xi a_2 \neq 0\}$, $\psi a_2 \approx a_3$ and $h'(\psi) = 1$. If $|X_{23}| > 1$ then there is the correct elementary change of basis leaving X_{12} without change such that $|X'_{23}| = |X_{23}| - 1$. Otherwise we have $X_{23} = \{\varphi_{23}\}$, where, probably, $\varphi_{23} = \varphi_{12}$, and $\varphi_{23}a_2 = \lambda_{23}a_3$.

Consider the set $X_{13} = \{\xi \in \mathsf{R}(A,A) \mid \mathrm{con}_{a_3}\,\xi a_1 \neq 0\}$. Here $h'(\xi) = 2$ and $\xi a_1 \approx a_3$ for any $\xi \in X_{13}$. As above, $X_{13} = \{\varphi_{13}\}$ and $\varphi_{13}a_1 = \lambda_{13}a_3$, $\lambda_{13} \in \Bbbk$. Then the associativity $(\varphi_{23}\varphi_{12})a_1 = \varphi_{23}(\varphi_{12}a_1) = \lambda_{12}\lambda_{23}a_3$ implies $\varphi_{23}\varphi_{12} = \frac{\lambda_{12}\lambda_{23}}{\lambda_{13}}\varphi_{13}$.

Theorem 1. *Let $\mathcal{A} \in \mathcal{C}$ be a bimodule problem with a triangled basis Σ such that $|\Sigma_0^+| = 1$. Then there exists a triangled change of basis from Σ to quasi multiplicative basis Σ'.*

According to the definition 6 quasi multiplicativity of basis it is sufficient to check on the bimodule problems $\mathcal{A}^{(A,B)}$ for all $A, B \in \Sigma_0^+$. Since $\mathcal{A} \in \mathcal{C}$ is a faithful bimodule problem, then the bimodule problems $\mathcal{A}^{(A)}$, $\mathcal{A}^{(B)}$, $\mathcal{A}^{(A,B)}$ are faithful as well by Remark 3.

First of all we consider the full subproblem $\mathcal{A}^{(A)}$ for any $A \in \Sigma_0^+$. Since $\mathrm{ord}\,A \leqslant 3$ for $\mathcal{A} \in \mathcal{C}$, then $|(\Sigma^{(A)})_0^-| \leqslant 3$ and $a \neq a' \in (\Sigma^{(A)})_1^0$ are comparable if and only if $s(a) = s(a')$. We show that there exists a finite superposition of correct in $\mathcal{A}$ elementary changes of $\Sigma_1^1(A,A)$ and $\bigcup\limits_{E\in\Sigma_0^-} \Sigma_1^0(E,A)$ such that the obtained basis $\Sigma^{(A)}$ of $\mathcal{A}^{(A)}$ is quasi multiplicative. Moreover, $|\mathsf{S}(a_1,a_2)| \leqslant 1$ for every non-equal $a_1, a_2 \in (\Sigma^{(A)})_1^0$. If $|(\Sigma^{(A)})_0^-| = 1$ then the proof follows from the lemma 2. Otherwise, if $|(\Sigma^{(A)})_0^-| = |(\Sigma^{(A)})_1^0|$, then $\mathsf{R}' = 0$ and $\Sigma_1^1(A,A) = \varnothing$. In the case $(\Sigma^{(A)})_0^- = \{E_1, E_2\}$ and $|(\Sigma^{(A)})_1^0| = 3$ we have (up to renumbering) $(\Sigma^{(A)})_1^0(E_1,A) = \{a_1,a_2\}$, $(\Sigma^{(A)})_1^0(E_2,A) = \{a\}$. Since $\mathcal{A}$ is faithful, then $\Sigma_1^1(A,A) = \{\alpha\}$, and $a_2 \approx \alpha a_1$. Therefore, any case leads to construction of quasi multiplicative basis.

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