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# Spherical Model of the Stark Effect in External Scalar and Vector Fields

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**Abstract**

The Bohr-Sommerfeld quantization rule and the Gamow formula for the width of quasistationary level are generalized by taking into account the relativistic effects, spin and Lorentz structure of interaction potentials. The relativistic quasi-classical theory of ionization of the Coulomb system ($V_C = -\xi/r$) by radial-constant long-range scalar ($S_{l.r.} = (1-\lambda)(\sigma r + V_0)$) and vector ($V_{l.r.} = \lambda(\sigma r + V_0)$) fields is constructed. In the limiting cases the approximated analytical expressions for the position $E_r$ and width $\Gamma$ of below-barrier resonances are obtained. The strong dependence of the width $\Gamma$ of below-barrier resonances on both the bound level energy and the mixing constant $\lambda$ is detected. The simple analytical formulae for asymptotic coefficients of the Dirac radial wave functions at zero and infinity are also obtained.

## Introduction

Wide range of problems from various fields of physics (elementary particle physics, nuclear physics, physics of atomic collisions, etc.) is related to representations of formation and decay of non-stable (quasistationary) states of quantum systems [1]. Properties of such states are of interest for investigation of ionization of atoms, ions and semiconductors under the influence of constant and homogeneous electric and magnetic fields [2], for description of cluster decays of atomic nuclei [3] and effects of the spontaneous creation of positrons [4, 5], in consideration of a vacuum shell of supercritical atom [4]-[6], and from the point of view of studying the Dirac equation in the strong external fields as well.

The relativistic theory of decaying (quasistationary) states is elaborated quite well for the cases when components of an interaction potential of a fermion with external fields belong to the vector type, i.e. are Lorentz-vector $A_\mu$ components [2, 5]-[7]. However, from the point of view of new problems of the strong interaction theory it is interesting to explore the more general case when a spin-1/2 particle interacts with the scalar and vector fields simultaneously. As is known now there is a reason to think that such interactions exist between quarks in hadrons.

## Position of quasistationary states

Separating the angular variables in the Dirac equation with the spherically symmetric vector $V(r)$ and scalar $S(r)$ interaction potentials, we obtain the system of first-order ordinary differential equations for the radial wave functions $F$ and $G$ ($c = 1$):

$$\left.\begin{aligned} \hbar\frac{dF}{dr} + \frac{\tilde{k}}{r}F - \left[(E - V(r)) + (m + S(r))\right]G = 0, \\ \hbar\frac{dG}{dr} - \frac{\tilde{k}}{r}G + \left[(E - V(r)) - (m + S(r))\right]F = 0. \end{aligned}\right\} \tag{1}$$

Here $\hbar$ is the Planck constant, $\tilde{k} = \hbar k$, $k = \mp(j + 1/2)$ is the motion integral of the Dirac particle in a central field, $j\hbar$ is the total angular moment; the definition and normalization of the function $F$ and $G$ are the same as in the papers [8, 9].

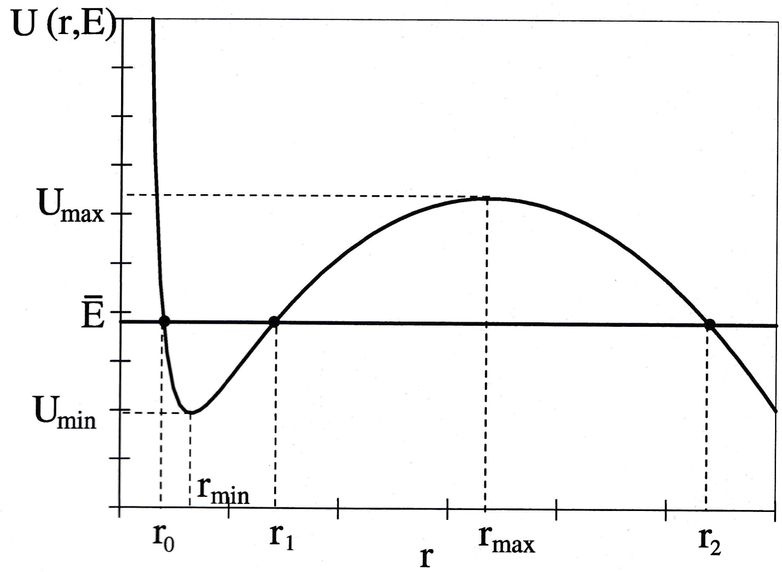


Figure 1: Barrier-type effective potential $U(r, E)$; $r_0$, $r_1$ and $r_2$ are roots of the equation $p^2(r) = 0$.

For the description of the phenomena related to the formation and decay of quasistationary states, we consider the class of potentials $V(r)$ and $S(r)$ for which the effective potential (see (2)) of the squared Dirac equation possesses a barrier (of the type shown in Fig. 1).

The systematic study of the theory of the semiclassical approximation (as $\hbar \to 0$) for the Dirac equation with a scalar.vector interaction was started in [9]. Formal asymptotic expansions in powers of $\hbar$ in initial Dirac system (1) for the radial functions $F(r)$ and $G(r)$ result in a chain of matrix differential equations, which can be solved consecutively using the known technique of left and right eigenvectors of the homogeneous system. For the effective potential (EP) of the barrier type (see Fig. 1)

$$U\,(r,E) = \frac{E}{m}V + S + \frac{S^2 - V^2}{2m} + \frac{k^2}{2m\,r^2} \tag{2}$$

semiclassical expressions were obtained for the wave functions in the classically forbidden and permitted bands and also the quantization condition determining the real part of level energy $E_r = E_{n_r k}$ in the mixture of the scalar and vector potentials:

$$\int\limits_{r_0}^{r_1} \left(p + \frac{k\,w}{p\,r}\right) dr = \left(n_r + \frac{1}{2}\right)\pi, \qquad w = \frac{1}{2}\left(\frac{V' - S'}{m + S + E - V} - \frac{1}{r}\right). \tag{3}$$

Here, $n_r = 0, 1, 2, \ldots$ is the radial quantum number, and

$$p(r) = \left[(E - V(r))^2 - (m + S(r))^2 - (k/r)^2\right]^{1/2} \tag{4}$$

is the semiclassical momentum for the radial motion of the particle in the potential well $r_0 < r < r_1$, where $r_0$ and $r_1$ are the turning points, i.e., the roots of the equation $p^2(r) = 0$.

Now we explore one concrete example of the vector and scalar potentials

$$V(r) = -\frac{\xi}{r} + \lambda v(r), \quad S(r) = (1 - \lambda)v(r), \quad v(r) = \sigma r + V_0, \tag{5}$$

in whose mixture at $1/2 < \lambda \leq 1$ and any $\sigma \neq 0$ quasistationary states of the Fermi particles exist; here $V_0$ is a real constant, $\xi$ is the Coulomb coupling constant, and $\lambda$ is the coefficient of mixing between the vector and scalar parts of the long-range potential $v(r)$. The potential $v(r)$ includes contributions of Lorentz-vector $V_{l.r.}(r) = \lambda v(r)$ and Lorentz-scalar $S_{l.r.}(r) = (1 - \lambda)v(r)$ components, of which (at $1/2 < \lambda \leq 1$) the first one dominates in all range of $r$, $0 < r < \infty$.

In the quantization condition (3) we integrate over the range of $r$ where $\bar{E} - U(r, E_r) > 0$. For the potentials (5) considered by us at $1/2 < \lambda \leq 1$ this means that $r_0 = c$ and $r_1 = b$ where $c < r < b$ and $r > a$ are classically allowed regions, and $b < r < a$ is below-barrier region, in which $p^2(r) < 0$; at $r > a$ the particle goes to infinity. The considered situation is schematically shown in Fig. 1 where black dots indicate the position of turning points.

Using the technique of evaluation of quantization integrals from Sec. 4 of Ref. [8],we have obtained the transcendental equation

$$\frac{2\sqrt{2\lambda-1}}{\sqrt{(a-c)(b-d)}}\left\{\frac{|\sigma|(c-d)^2}{\bar{\Re}}\left[\bar{N}_1F(\bar{\chi}) + \bar{N}_2E(\bar{\chi}) + \bar{N}_3\Pi(\bar{\nu},\bar{\chi}) + \bar{N}_4\Pi\left(\frac{d}{c}\bar{\nu},\bar{\chi}\right)\right] + \right.$$
$$\left. + \frac{k}{2(2\lambda-1)|\sigma|}\left[(c-d)\left(\bar{N}_5\Pi(\bar{\nu}_+,\bar{\chi}) + \bar{N}_6\Pi(\bar{\nu}_-,\bar{\chi})\right) + \bar{N}_7F(\bar{\chi})\right]\right\} = \left(n_r + \frac{1}{2}\right)\pi, \qquad (6)$$

defining (in the quasi-classical approximation) the real part $E_r = E_{n_rk}$ of complex energy of quasistationary states at $U_{min} < \bar{E}_r < U_{max}$. Here $F(\chi)$, $E(\chi)$ č $\Pi(\nu,\chi)$ are the complete elliptic integrals of the respective first, second, and third kind; the quantities $\bar{\nu}$, $\bar{\nu}_\pm$, $\bar{\chi}$, $\bar{\Re}$, $\bar{\aleph}$, $\bar{N}_j$ $(j = 1, 2, ..., 7)$ are obtained from respective expressions (A.1)–(A.7) for $\nu$, $\nu_\pm$, $\chi$, $\Re$, $\aleph$, $N_j$ (recently found by us in Ref. [8]) by making the simultaneous replacements $a \to b$, $b \to c$, $c \to d$ and $d \to a$.

Finding an "exact" solution of equation (20) in the general case is, of course, impossible, but the situation is simplified in two special cases: $\sigma/\xi\widetilde{m}^2 \ll 1$, $\widetilde{E} < \widetilde{m}$ (where $\widetilde{E}_r = E_r - \lambda V_0$ and $\widetilde{m} = m + (1-\lambda)V_0$) and $\sigma\gamma/\widetilde{E}^2 \ll 1$, $\widetilde{E} > \widetilde{m}$.

**Case A**. If $\widetilde{E}_r < \widetilde{m}$, $U_{min} < \bar{E}_r < m$ (see Fig. 1, where $U_{min} = U(r_{min}, E_r)$, $r_{min} \approx \gamma^2/\widetilde{E}_r\xi$, $\gamma^2 = k^2 - \xi^2$) and condition $\sigma/\xi\widetilde{m}^2 \ll 1$ is satisfied, the pair of classical turning points $a$ and $d$ is rather distant from pair of points $c$ and $b$ (see. Ref. [8]):

$$a, b \approx \frac{\widetilde{E}_r\xi \pm \theta}{\mu^2}\left[1 - \frac{\widetilde{E}_r\xi \pm \theta}{\mu^4}\left(\eta_1 \pm \frac{\widetilde{m}\xi\eta_2}{\mu}\right)\sigma\right], \qquad (7)$$

$$c \approx -\frac{\widetilde{m} - \widetilde{E}_r}{\sigma} - \frac{\xi}{\widetilde{m} - \widetilde{E}_r}, \qquad d \approx -\frac{\widetilde{m} + \widetilde{E}_r}{\sigma(1-2\lambda)} + \frac{\xi}{\widetilde{m} + \widetilde{E}_r}. \qquad (8)$$

Hereafter, we use the notation

$$\theta = \sqrt{(\widetilde{E}_rk)^2 - (\widetilde{m}\gamma)^2}, \quad \mu = \sqrt{\widetilde{m}^2 - \widetilde{E}_r^2}, \quad \eta_1 = (1-\lambda)\widetilde{m} + \lambda\widetilde{E}_r, \quad \eta_2 = \lambda\widetilde{m} + (1-\lambda)\widetilde{E}_r.$$

In this case the derivation of asymptotic expansions of quantization integrals (3) in a small parameter $\sigma/\xi\widetilde{m}^2$ is carried out just as in item A of Sec. 4 of Ref. [8] and gives the former expression for the level energy

$$E_r = \widetilde{E}_0 + \lambda V_0 + \frac{\sigma}{2\xi\widetilde{m}^2}\left[\left(\frac{\xi^2\widetilde{m}^2}{\mu_0^2} - k^2\right)\eta_{10} + \left(\frac{2\xi^2\widetilde{m}\widetilde{E}_0}{\mu_0^2} - k\right)\eta_{20}\right] + O\left(\left(\frac{\sigma}{\xi\widetilde{m}^2}\right)^2\right), \qquad (9)$$

$$\widetilde{E}_0 = \widetilde{m}\left[1 + \frac{\xi^2}{(n_r' + \gamma)^2}\right]^{-1/2}, \quad n_r' = n_r + (1 + \operatorname{sgn} k)/2, \qquad (10)$$

and the quantities $\mu_0$, $\eta_{10}$ and $\eta_{20}$ are obtained from $\mu$, $\eta_1$ and $\eta_2$ by replacing $\widetilde{E}_r$ by $\widetilde{E}_0$.

The influence of weak radial-constant scalar and electric fields on the system of Coulomb levels has been analyzed before (see, for example Refs. [8, 9]). The analysis was carried out both on the basis of quasi-classical formulae (9) and by the numerical solution of the transcendental equation (20). In particular, the calculations have shown that the sign change in $\sigma$ $(\sigma \to -\sigma)$ leads to small changes of energy spectrum, if $|\sigma| \ll 0.2$ GeV$^2$.

**Case B**. Let us now consider $\widetilde{m} < \widetilde{E}_r$ and $m < \bar{E}_r < U_{max}$ (see Fig. 1, where $U_{max} = U(r_{max}, E_r)$ and $r_{max} \approx \eta_1[(2\lambda-1)\sigma]^{-1}$). Here the ratio $\sigma\gamma/\widetilde{E}_r^2$ plays a role of the small parameter. In this case the quasistationary states in the composite field (5) exist only at the positive values of parameter $\sigma$.

Under the requirements formulated above, we have obtained

$$a \approx \frac{\widetilde{E}_r + \widetilde{m}}{\sigma(2\lambda-1)} + \frac{\xi}{\widetilde{E}_r + \widetilde{m}}, \qquad b \approx \frac{\widetilde{E}_r - \widetilde{m}}{\sigma} + \frac{\xi}{\widetilde{E}_r - \widetilde{m}}, \qquad c, d \approx \frac{-\widetilde{E}_r\xi \pm \theta}{\widetilde{E}_r^2 - \widetilde{m}^2}. \tag{11}$$

It is seen that turning points $c$ and $b$ are rather distant from one another ($a, b \gg c, |d|$).

Further we shall give only the recipe of evaluation of the quantization integrals $J_{1,2}$. Just as in the item B of Sec. 4 of Ref. [8], the integration range in 3 we divide into two domains $c \le r \le \widetilde{r}$ and $\widetilde{r} \le r \le b$ by introducing the dividing point $\widetilde{r}$ satisfying the requirement $c \ll \widetilde{r} \ll b$. In the first domain $c \le r \le \widetilde{r}$ we calculate the integral 3 by expanding the quasimomentum $p(r)$ in a power series in the parameters $r/a \ll 1$ and $r/b \ll 1$. In the second domain $\widetilde{r} \le r \le b$ the expansion of $p(r)$ we carry out in the small quantities $c/r \ll 1$ and $|d|/r \ll 1$.

When we add the asymptotic expansions of integrals over $c \le r \le \widetilde{r}$ and $\widetilde{r} \le r \le b$ the final result will not contain the quantity $\widetilde{r}$. So, we have obtained the transcendental equation for the level energy $E_r$ which we solve by the method of consecutive iterations. Thus, we arrive at the expression for the energy (within $O(\sigma\gamma/\widetilde{E}_r^2)$)

$$E_r = \zeta^{-1}\left\{B + \left(B^2 + \zeta\left[2\sigma(1-2\lambda)\left(\xi\ln\frac{\sigma|k|(1-\lambda)}{4\widetilde{E}^{(0)\,2}} + 3\xi + \lambda\xi\widetilde{A} + \pi N\right) + \lambda\widetilde{m}^2(1-\lambda\widetilde{A})\right]\right)^{1/2}\right\} + \lambda V_0, \tag{12}$$

where

$$\widetilde{E}^{(0)} = E^{(0)} - \lambda V_0, \quad \widetilde{A} = (2\lambda-1)^{-1/2}\log\left[\left(1+\lambda+\sqrt{2\lambda-1}\right)(1-\lambda)^{-1}\right],$$

$$\zeta = (1-\lambda)^2 A - \lambda - 2\sigma\xi(1-2\lambda)/(\widetilde{E}^{(0)})^2,\ B = (1-\lambda)(1-\lambda\widetilde{A})\widetilde{m} - 4\sigma\xi(1-2\lambda)/\widetilde{E}^{(0)},$$

$$N = n_r + \frac{1}{2} + \frac{\operatorname{sgn} k}{4} + \frac{1}{\pi}\left(\gamma\arccos\left(-\frac{\xi}{|k|}\right) - \xi\right), \widetilde{\eta} = \frac{1}{\sqrt{2\lambda-1}}\ln\left[\left(\eta_1 + \sqrt{(2\lambda-1)(\widetilde{E}_r^2 - \widetilde{m}^2)}\right)\eta_2^{-1}\right], \tag{13}$$

and $\widetilde{E}^{(0)} = E^{(0)} - \lambda V_0$, $E^{(0)}$ is the zeroth approximation of energy, by the choice of which the quantity $E_{n_r k}$ depends very weakly and in most cases one can take $E^{(0)} \approx E_r(\xi = 0)$. Accuracy of the calculation of $E_r$ by the means of formulae (12) is fully appropriate (see Sec. 4 of Ref. [8]), and usually there is no point to make the result more precise for practical purposes.

## Width of quasistationary states

In the consideration above the quasistationary character of the Stark spectrum was ignored. Thereupon it is necessary to remind that for $1/2 < \lambda \le 1$ and any value of $\sigma \ne 0$, $U(r, E_r)$ has the shape of a potential with a barrier, owing to what there are quasistationary states with complex energy $E = E_r - i\Gamma/2$ instead of discrete levels.

The probability of tunnel transition of a particle from the bound state (with energy $E_r$) into the continuum state is defined by the imaginary part (i.e. by the width $\Gamma$) of complex energy of quasistationary states [9]:

$$\Gamma = -2\,\mathrm{Im}[G^*(r)F(r)]_{r\to\infty}.$$

Having calculated the flux of the particles outgoing to infinity, we find the following expression for the level width $\Gamma$

$$\Gamma = \frac{1}{T}\exp\left[-2\,\Omega\right], \tag{14}$$

$$T = 2\int_c^b \frac{E_r - V}{p}\,dr, \quad \Omega = \int_b^a \left(q - \frac{kw}{qr}\right)dr. \tag{15}$$

The obtained quasi-classical formula (14) is the relativistic generalization of the well-known Gamow formula for the width of a quasistationary level. The nontrivial moment of such a generalisation is the modification of expression for the period of oscillations $T$ and the occurrence of the additional factor in the preexponent of expression (14) that depends on a sign of the quantum number $k$ and is caused by the spin-orbit coupling in the mixture of the scalar $S(r)$ and vector $V(r)$ potentials.

Thus, in the quasi-classical approximation the problem is reduced to evaluation of two characteristic phase integrals $T$ and $\Omega$. Using the technique of evaluation of quantization integrals from Sec. 4 of Ref. [8], we have obtained the transcendental equationwe give only the final result

$$T = \frac{4}{|\sigma|\sqrt{(a-c)(b-d)(2\lambda-1)}}\left\{d\widetilde{E}_r+\xi-\lambda\sigma\left(d^2-\frac{(c-d)^2}{2(1-\bar{\nu})}\right)F(\bar{\chi})\right.$$
$$\left.+\frac{\lambda\sigma\bar{\nu}(c-d)^2}{2\bar{\Re}}E(\bar{\chi})+(c-d)\left[\widetilde{E}-\lambda\sigma\left(2d+\frac{(c-d)\bar{\aleph}}{\bar{\Re}}\right)\right]\Pi(\bar{\nu},\bar{\chi})\right\}, \tag{16}$$

$$\Omega = \frac{2\sqrt{2\lambda-1}}{\sqrt{(a-c)(b-d)}}\left\{-\frac{|\sigma|(b-c)^2}{\Re}\left[N_1F(\chi)+N_2E(\chi)+N_3\Pi(\nu,\chi)+N_4\right.\right.$$
$$\left.\left.\times\Pi\left(\frac{c}{b}\nu,\chi\right)\right]+\frac{k}{2(2\lambda-1)|\sigma|}\left[(b-c)\left(N_5\Pi(\nu_+,\chi)+N_6\Pi(\nu_-,\chi)\right)+N_7F(\chi)\right]\right\}. \tag{17}$$

The quantities $\nu$, $\nu_\pm$, $\chi$, $\Re$, $\aleph$, $N_j$ č $\bar{\nu}$, $\bar{\nu}_\pm$, $\bar{\chi}$, $\bar{\Re}$, $\bar{\aleph}$, $\bar{N}_j$ $(j=1,2,...,7)$ belonging to (16), (17) are defined in (A.1)–(A.7) and (20).

The derived formulae (16), (17) are valid at large values of modules of phase integrals $T$, $\Omega$ and together with (14) solve the problem of calculation of width $\Gamma$ of Stark below-barrier resonances at $U_{min} < \bar{E}_r < U_{max}$. However, these formulae are rather cumbersome and not too convenient for concrete calculations. With a purpose of deriving an analytical expression for the width of quasistationary level $\Gamma$ the calculations for the cases $\widetilde{E}_r < \widetilde{m}$ and $\widetilde{E}_r > \widetilde{m}$ should be carried out separately.

**Case A**. Let us begin with the simpler (in the sense of calculation) case of quasistationary levels with $\widetilde{E}_r > \widetilde{m}$ $(\bar{E}_r < U_{max})$ when the under requirements $\sigma\gamma \ll \widetilde{E}_r^2$, $\sigma > 0$ the classical turning points $b$ and $a$ are rather distant from the pair of points $d$ and $c$. Asymptotic expansion of the barrier integral $\Omega$ can be constructed by means of the procedure which is very similar to the procedure applied to quantization integrals in the case of purely discrete spectrum in the item A of the previous section. Omitting the details of the calculation, we give only the final formula for the width of the quasistationary level [10]:

$$\Gamma \approx \frac{1}{T}\exp\left[-2\Omega(\widetilde{E}_r,\lambda)\right], \tag{18}$$

$$\Omega(\widetilde{E}_r,\lambda) = \frac{\pi}{2\sqrt{2\lambda-1}}\left(\frac{\eta_2^2}{\sigma(2\lambda-1)}+2\xi\lambda+\frac{2\widetilde{E}_r\xi\sqrt{2\lambda-1}}{\sqrt{\widetilde{E}_r^2-\widetilde{m}^2}}\right), \tag{19}$$

where $\eta_2$ is defined above in (8). Corresponding expansions for the energy $\widetilde{E}_r$ are given by previous formulae (12). Characteristic feature of the considered case is the fact that in the integral (15), which defines a period of radial oscillations $T$, only the range of values of the integration variable $r$, where the Coulomb potential can be considered a perturbation, is essential. By neglecting the Coulomb interaction, we arrive at the expression

$$T \approx \frac{2}{\sigma(1-2\lambda)}\left[-\lambda\sqrt{\widetilde{E}_r^2-\widetilde{m}^2}+(1-\lambda)\widetilde{\eta}\,\eta_2\right]. \tag{20}$$

As can be seen from the Fig. 2 the function $\Omega(E_r\lambda)$ decreases monotonically with increasing the parameter $\lambda$. Therefore, decreasing the relative weight $(1-\lambda)$ of the Lorentz-scalar $S_{l.r.}(r)$ in the long-range part $v(r)$ of the interaction (5) rapidly increases a probability of ionization of quasistationary level.

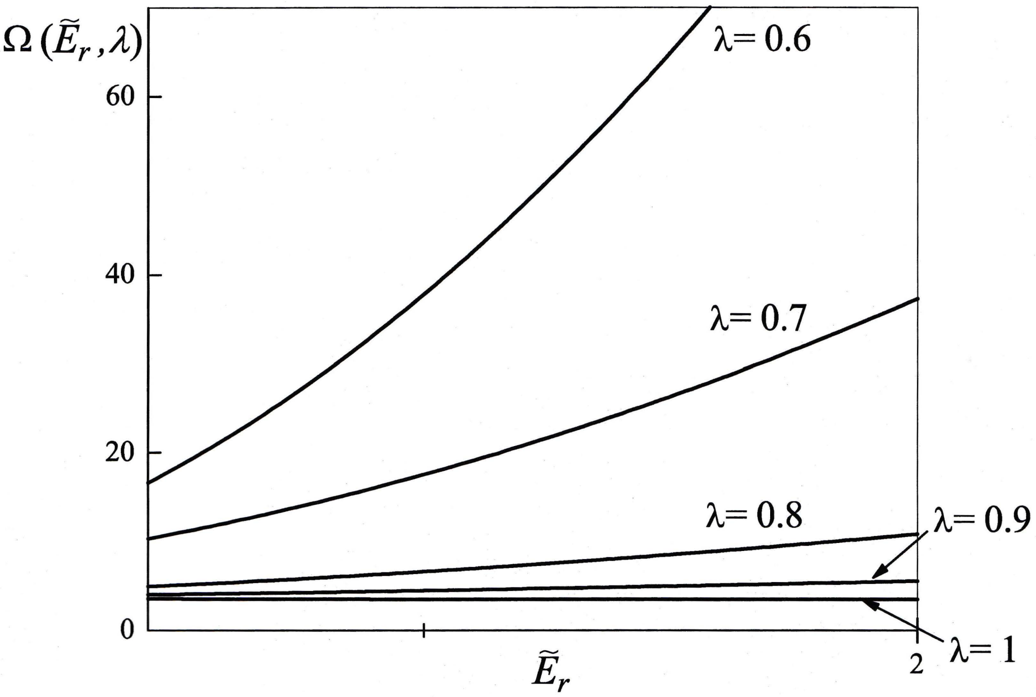


Figure 2: The function $\Omega(E_r, \lambda)$ that defines the dependence of the exponential factor in the ionization probability (18) on the level energy $E_r$, 0.44 GeV$< E_r <$ 2 GeV.

**Case B.** At $\widetilde{E}_r < \widetilde{m}$, $U_{min} < \bar{E}_r$ the asymptotic expansions of $T$ and $\Omega$ in positive powers of small dimensionless parameter $\sigma/\xi\widetilde{m}^2$ are constructed by the same technique, as in the item B of the previous section. By omitting the details, sense of which is clear, we give the asymptotic behavior of the imaginary part of energy of quasistationary state to within terms $O(\sigma/\xi\widetilde{m}^2)$ [10]:

$$\Gamma = 2\mu_0 |A_C|^2 \left(\frac{2\mu_0^2}{|\sigma|\eta_{20}}\right)^{\frac{2\xi\widetilde{E}_0}{\mu_0}} \exp\left\{-\frac{\Phi(\widetilde{E}_0,\lambda)}{|\sigma|} - \frac{2\lambda\mu_0\rho}{2\lambda-1} - \frac{2\,\mathrm{sgn}\,\sigma}{\sqrt{2\lambda-1}}\left[\frac{(1-\lambda)\eta_{20}\rho}{2\lambda-1} + \lambda\xi\right] \arccos\left(-\frac{\eta_{10}}{\eta_{20}}\mathrm{sgn}\,\sigma\right)\right\}, \tag{21}$$

where

$$|A_C| = \left[\frac{(\xi\,\widetilde{m} - k\,\mu_0)\,\mu_0}{2\,\xi\,\widetilde{m}^2\,\Gamma\,(2\gamma + n_r' + 1)\,n_r'!}\right]^{1/2} (2\mu_0)^{\frac{\xi\widetilde{E}_0}{\mu_0}} \tag{22}$$

is the asymptotic coefficient of the normalized wave function in the Coulomb potential, and notations $\widetilde{E}_0$, $\mu_0$, $\eta_{10}$ and $\eta_{20}$ were introduced in (9), (10). The functions $\Phi(\widetilde{E}_0, \lambda)$ and $\rho(\widetilde{E}_0, \lambda)$ from the exponent (21) are given by the formulae

$$\Phi(\widetilde{E}_0,\lambda) = (2\lambda-1)^{-1}\left\{\frac{\eta_{20}^2}{\sqrt{2\lambda-1}} \arccos\left(-\frac{\eta_{10}}{\eta_{20}}\mathrm{sgn}\,\sigma\right) + \eta_{10}\mu_0\,\mathrm{sgn}\,\sigma\right\}, \tag{23}$$

$$\rho(\widetilde{E}_0,\lambda) = \frac{1}{2\xi\widetilde{m}^2}\left[\left(\frac{\xi^2\widetilde{m}^2}{\mu_0^2} - k^2\right)\eta_{10} + \left(\frac{2\xi^2\widetilde{m}\widetilde{E}_0}{\mu_0^2} - k\right)\eta_{20}\right].$$

Comparison of the results of calculations of level energies based on the "exact" quasi-classical formulae (14), (16), and (17) with results of numerical calculations shows that relative error of (21) and (23) does not exceed 2% at $|\sigma| \sim 10^{-6} - 10^{-4}$ GeV$^2$. So the formula (21) is convenient when used for quick estimates of $\Gamma$. As is shown in Fig. 3a ($\sigma < 0$), $\Phi(\widetilde{E}_0, \lambda)$ increases with sinking of the level

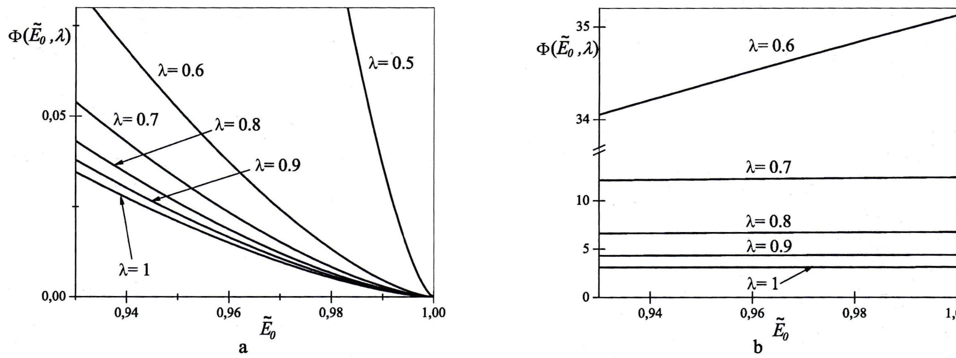


Figure 3: The function $\Phi(\widetilde{E}_0, \lambda)$ that defines the dependence of the exponential factor in the ionization probability (21) on the level energy $E_0$ (in GeV): a) for $\sigma < 0$, b) for $\sigma > 0$.

$\widetilde{E}_0$ and decreases when the mixing coefficient $\lambda$ $(1/2 < \lambda \leq 1)$ increases. In the case of the positive values of $\sigma$ the function $\Phi(\widetilde{E}_0, \lambda)$ decreases monotonically with increasing the parameter $\lambda$ (Fig. 3b), therefore by decreasing the relative weight $(1-\lambda)$ of the Lorentz-scalar $S_{l.r.}(r)$ in the long-range part $v(r)$ of the interaction (5) one rapidly increases the probability of ionization of quasistationary level (at the same value of $\sigma$).

In the case when in addition to the Coulomb field there is only the radial-constant electric field $(\lambda = 1)$, from the formula (21) at $V_0 = 0$, $\sigma < 0$ and $\xi = Z\alpha$ ($Z$ is the nuclear charge, $\alpha \approx 1/137$ is the fine structure constant) we have obtained $(\hbar = m = c = 1)$:

$$\Gamma = 2\mu_0 \left|A_C\right|^2 \left(\frac{2\mu_0^2}{|\sigma|}\right)^{\frac{2E_0 Z\alpha}{\mu_0}} \exp\left[-\frac{\Phi(E_0)}{|\sigma|} + 2Z\alpha \arccos E_0 - 2\rho\sqrt{1-E_0^2}\right], \tag{24}$$

where $E_0$ is the energy of a bound state in the absence $(\sigma = 0)$ of an external long-range field, and

$$\rho = \frac{1}{2Z\alpha}\left[E_0\left(\frac{3Z^2\alpha^2}{1-E_0^2} - k^2\right) - k\right].$$

The function $\Phi(E_0)$ in the exponent is given by the expression

$$\Phi(E_0) = \arccos E_0 - E_0\sqrt{1-E_0^2}, \tag{25}$$

and possesses the obvious property $\Phi(-E_0) = \pi - \Phi(E_0)$.

It is useful to consider the various limiting cases for the quantities appearing in the formula (24):

$$\arccos E_0 = \begin{cases} \left(1-E_0^2\right)^{1/2} + \frac{1}{6}\left(1-E_0^2\right)^{3/2} + ..., & E_0 \to 1, \\ \frac{\pi}{2} - E_0 - \frac{1}{6}E_0^3 + ..., & E_0 \to 0, \\ \pi - \left(1-E_0^2\right)^{1/2} - \frac{1}{6}\left(1-E_0^2\right)^{3/2} + ..., & E_0 \to -1, \end{cases} \tag{26}$$

$$\Phi(E_0) = \begin{cases} \frac{2^{5/2}}{3}(1-E_0)^{3/2}\left[1 - \frac{3}{20}(1-E_0) + ...\right], & E_0 \to 1, \\ \frac{\pi}{2} - 2E_0 + \frac{1}{3}E_0^3 + ..., & E_0 \to 0, \\ \pi - \frac{2^{5/2}}{3}(1+E_0)^{3/2} + ..., & E_0 \to -1. \end{cases} \tag{27}$$

## Conclusions

By means of the formulae obtained above the spectrum of quasistationary levels is described for the accepted hybrid version of SMSE. Such model qualitatively reproduces the following characteristic features of quasistationary states in an mixture of scalar and vector potentials of barrier type (5): 1) very strong (at small $\sigma$) dependence of $\Gamma$ on the binding energy of tunneling fermion and on the mixing coefficient $\lambda$; 2) nonanalytic dependence of shift and width of level on the "force" $\sigma$ of scalar and vector long-range interactions.

## Appendix

$$\nu = \frac{a-b}{a-c}, \quad \nu_\pm = \frac{\lambda_\pm - c}{\lambda_\pm - b}\nu, \quad \lambda_\pm = -\frac{\widetilde{E}+\widetilde{m} \mp \sqrt{(\widetilde{E}+\widetilde{m})^2 - 4\xi\sigma(1-2\lambda)}}{2\sigma\,(1-2\lambda)}, \tag{A.1}$$

$$\chi = \sqrt{\nu\frac{(c-d)}{(b-d)}}, \quad \Re = (1-\nu)\left(\chi^2-\nu\right), \tag{A.2}$$

$$N_1 = \frac{\chi^2\,(b-c)}{4} - \frac{3\aleph\,(b-c)}{8\,(1-\nu)} - \frac{(\chi^2-\nu)}{2}(f+3c) + \frac{\Re}{(b-c)^2}\left(c^3+c^2f+cg+h+l/c\right), \tag{A.3}$$

$$N_2 = -\frac{\nu}{2}\left[f+3c+\frac{3}{4}\frac{(b-c)\,\aleph}{\Re}\right], \tag{A.4}$$

$$N_3 = \frac{1}{2}\left[\frac{3}{4}\frac{(b-c)\,\aleph^2}{\Re} + \frac{2\Re}{(b-c)}\left(3c^2+2cf+g\right) + (b-c)\left(\left(1+\chi^2\right)\nu - 3\chi^2\right) + \aleph\,(f+3c)\right], \tag{A.5}$$

$$N_4 = -\frac{\Re}{(b-c)}\frac{l}{bc}, \quad N_5 = [(b-\lambda_+)(\lambda_+-c)]^{-1}, \quad N_6 = [(b-\lambda_-)(\lambda_--c)]^{-1}, \tag{A.6}$$

$$N_7 = \frac{2}{(\lambda_+-c)(\lambda_--c)}\left(c+\frac{\widetilde{E}+\widetilde{m}}{2(1-2\lambda)\sigma}\right), \quad \aleph = \chi^2\,(3-2\nu)+\nu\,(\nu-2)\,. \tag{A.7}$$

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