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Tunnel Ionization of Heavy Atom in Parallel Electric and Magnetic Fields

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Abstract

In the framework of the paraxial Fock-Leontovich approximation the three-dimensional version of WKB method is elaborated for solving the Dirac equation with axially symmetrical potentials, which do not permit the complete separation of variables. By means of this approach the relativistic wave functions for the H-like atom in the parallel constant uniform electric and magnetic fields are constructed in the below-barrier and classically allowed ranges. General analytical expression for probability of ionization of the atom in the external electric and magnetic fields is obtained. The comparison of the found formulas with results known previously is carried out.

Introduction

The problem of hydrogen atom in electric and magnetic fields has fundamental meaning for a quantum mechanics and the atomic physics and has many applications (see, for example, [1, 2, 3] and the references therein). Since the twenties [4], properties of an energy spectrum of hydrogen atom and other atoms in external fields were rather intensively studied in the framework of the Schrödinger equation.

At the same time the interior logic of development of study of atomic systems with a high degree of ionization (the multiply charged ions) dictates, obviously, formulation of various qualitatively new problems, similar to those which were previously solved only for neutral or weakly ionized atoms. Essentially relativistic character of motion of electrons in the fields created by multiply charged ions (the characteristic velocity of the electron in H-like ions with nuclear charge Z is $\sim \alpha Zc$; α is the fine structure constant, c is the velocity of light) is the main feature of such ions that distinguishes them from neutral atoms. Thus, the consistent theory of tunnel ionization of such systems should be relativistic because relativistic effects are not small corrections, and fundamentally determine the orders of spectral characteristics.

In order to construct such a theory one should have the solution of the relativistic problem of motion of an electron in the field of nucleus and in the constant uniform electric and magnetic fields. Since the Dirac equation with such superpositional potential does not permit complete separation of variables in any orthogonal system of coordinates, the given problem has no exact analytical solution, and numerical methods are rather onerous.

The relativistic calculations of the linear Stark effect are carried out by means of perturbation theory [5, 6], and quadratic Stark effect was treated by means of RCGF method in the form of the expansion in powers of $Z\alpha$ [7]. However, the most of papers was basically devoted to position of quasi-stationary level, and there are only rare cases of calculation of width Γ in the relativistic case. In our previous paper [8] within quasiclassical approximation the hybrid version of spherically symmetrical model of the Stark effect, taking into account the Lorentz structure of interaction potential, was studied. Rather recently the probability of ionization of s-level, whose binding energy can be of order of the rest energy, in electric and magnetic fields has been calculated by means of generalization of the imaginary time method [9] and of so-called ADK-theory [10]. However, in the general case, widths of quasi-stationary states are not found until now.

Due to such situation in the theory and intensive experimental researches during last years, asymptotic methods of calculation of ionization probability, which are based on clear physical ideas about below-barrier electron transition, are gaining in importance. From this point of view it is worthwhile to use the WKB method (or quasi-classical approximation) which enables to find the approximative analytical solutions of the relativistic problem and to express required ionization probability in terms of quantum penetrability of the potential barrier which separates domains of discrete and continuous spectra. As is known, this method has rather high accuracy even for small quantum numbers. For the first time the three-dimensional version of WKB approximation for the Dirac equation with axially symmetrical potential was elaborated and used for the relativistic two-center problem [11]. Recently, this method was applied to problem of tunnel ionization of H-like multicharged ions in the constant uniform electric field [12].

In the present work we generalize our method by means of introduction into the Dirac equation of the three spatial components of vector-potential A_μ, corresponding to constant uniform magnetic field which is parallel to external electric field. In this case the Dirac equation has an axial symmetry, and one can use the main ideas of the method elaborated previously [12].

Quasi-classical approximation for the Dirac equation with axially symmetrical potentials

For the bispinor Ψ the stationary Dirac equation is ($m_e = e = \hbar = 1$)

$$\left.\begin{array}{l} c\vec{\sigma}\left(\vec{p}-\vec{A}/c\right)\xi = (E-V+c^2)\eta \\ c\vec{\sigma}\left(\vec{p}-\vec{A}/c\right)\eta = (E-V-c^2)\xi \end{array}\right\}, \qquad \Psi = \begin{pmatrix} \xi \\ \eta \end{pmatrix}. \tag{1}$$

Using the substitution $\xi = (W^+)^{1/2}\Phi$, $W^\pm = E - V \pm c^2$, we arrive at the matrix equation

$$\Delta\Phi + \hat{k}^2\Phi = 0,\ \hat{k}^2 = \frac{1}{\hbar^2 c^2}\left[(E-V)^2 - c^4 - \vec{A}^2\right] - \frac{1}{\hbar c}\left[i(\vec{\sigma}\vec{\nabla})(\vec{\sigma}\vec{A}) + 2i\vec{A}\vec{\nabla}\right] - \left[\frac{\vec{\sigma}[\vec{\nabla}V, \vec{A}]}{W^+}\right] - \frac{\Delta V}{2W^+} - \frac{3}{4}\left(\frac{\vec{\nabla}V}{W^+}\right)^2 + \frac{i}{W^+}\vec{\sigma}\left[\vec{\nabla}V, \vec{\nabla}\right]. \tag{2}$$

Let us consider the case when electric field is constant, and magnetic field is constant, uniform and directed along axis z. Besides that both fields are axially symmetrical $V = V(z,\rho)$, $A_x = -\frac{1}{2}Hy$, $A_y = \frac{1}{2}Hx$, $A_z = 0$. Then we can represent the solution of (2) in the form

$$\Phi = \begin{pmatrix} F_1(z,\rho)\exp\left[i\left(m - 1/2\right)\varphi\right] \\ F_2(z,\rho)\exp\left[i\left(m + 1/2\right)\varphi\right] \end{pmatrix}, \tag{3}$$

By substituting (3) into (2), we obtain the matrix differential equation

$$\left(\Delta + \hat{\partial}\right)F = \left(\frac{q^2}{\hbar^2} + \frac{\gamma_1}{\hbar} + \gamma_2\right)F, \qquad F = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}, \tag{4}$$

$$q = \frac{1}{c}\sqrt{c^4 - (E-V)^2 - \frac{H^2\rho^2}{4}},\quad \hat{\partial} = \frac{1}{W^+}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\left(\frac{\partial V}{\partial\rho}\frac{\partial}{\partial z} - \frac{\partial V}{\partial z}\frac{\partial}{\partial\rho}\right), \tag{5}$$

$$\gamma_1 = -\frac{H}{c}\begin{pmatrix} m + 1/2 + \dfrac{\rho}{2W^+}\dfrac{\partial V}{\partial\rho} & -\dfrac{\rho}{2W^+}\dfrac{\partial V}{\partial z} \\ -\dfrac{\rho}{2W^+}\dfrac{\partial V}{\partial z} & m - 1/2 - \dfrac{\rho}{2W^+}\dfrac{\partial V}{\partial\rho} \end{pmatrix},$$

$$\gamma_2 = \begin{pmatrix} a_{m-1/2} & b_{m+1/2} \\ b_{m-1/2} & a_{-m-1/2} \end{pmatrix}, \tag{6}$$

$$a_\mu(z,\rho) = \frac{\mu^2}{\rho^2} + \frac{1}{W^+}\left[\frac{\mu}{\rho}\frac{\partial V}{\partial\rho} + \frac{\Delta V}{2} + \frac{3}{4}\frac{\left(\vec{\nabla}V\right)^2}{W^+}\right],\quad b_\mu(z,\rho) = -\frac{\mu}{\rho W^+}\frac{\partial V}{\partial z}. \tag{7}$$

We seek a solution of equation (4) in the form of a WKB expansion:

$$F = \varphi\exp(\hbar^{-1}S),\quad \varphi = \sum_{n=0}^{\infty}\hbar^n\varphi^{(n)}. \tag{8}$$

Here $\varphi^{(n)}$ is a bispinor (the upper component corresponds to the function F_1, the lower to F_2). Having substituted F, determined by (8), into (4) and equated to zero the coefficients of each power of $\hbar$, we arrive at the hierarchy of equations

$$\left(\vec{\nabla}S\right)^2 - q^2 = 0, \tag{9}$$

$$2\vec{\nabla}S\cdot\vec{\nabla}\varphi^{(0)} + \Delta S\varphi^{(0)} + \hat{\partial}S\varphi^{(0)} = \gamma_1\varphi^{(0)}, \tag{10}$$

$$2\vec{\nabla}S\cdot\vec{\nabla}\varphi^{(n+1)} + \Delta S\varphi^{(n+1)} + \hat{\partial}S\varphi^{(n+1)} + \Delta\varphi^{(n)} + \hat{\partial}\varphi^{(n)} - \gamma_2\varphi^{(n)} = \gamma_1\varphi^{(n+1)}, \tag{11}$$

$n = 0, 1, 2, \ldots$

Consider equation (9) and assume that

$$q^2(z,\rho) = q_0^2(z) + \sum_{k=1}^{\infty} Q_k(z)\rho^{2k}, \quad q_0^2(z) = q^2(z,0), \quad Q_k = \frac{1}{(2k)!}\frac{\partial^{2k} q^2(z,0)}{\partial \rho^{2k}}. \tag{12}$$

Solution of equation (9) can also be represented in the form of an expansion in powers of coordinate the ρ:

$$S(z,\rho) = \sum_{n=0}^{\infty} S_n(z)\rho^{2n}. \tag{13}$$

By inserting (13) into (9) and equating to zero the coefficients of each power of ρ, we obtain

$$(S_0')^2 - q_0^2 = 0, \tag{14}$$
$$2S_0'S_1' + 4S_1^2 - Q_1 = 0, \ldots \tag{15}$$

It is easy to show that the solution of equation (14) is

$$S_0 = \pm \int q_0 dz + C_0, \quad C_0 = \text{const.} \tag{16}$$

Equation (15) is the nonlinear Riccati differential equation and are not solvable analytically in a general case. However, by making the substitution

$$S_1 = \frac{q_0(z)}{2}\left(\frac{1}{2}\frac{q_0'(z)}{q_0(z)} - \frac{\sigma'(z)}{\sigma(z)}\right), \tag{17}$$

one can proceed from (15) to the linear second-order equation

$$\sigma'' + \left[\frac{1}{4}\left(\frac{q_0'}{q_0}\right)^2 - \frac{1}{2}\frac{q_0''}{q_0} - \frac{Q_1}{q_0^2}\right]\sigma = 0. \tag{18}$$

The solutions of the equations (10), (11) are sought in the form

$$\varphi^{(n)}(z,\rho) = \begin{pmatrix} \rho^{|m-1/2|}\sum\limits_{k=0}^{\infty}\varphi_{1k}^{(n)}(z)\rho^{2k} \\ \rho^{|m+1/2|}\sum\limits_{k=0}^{\infty}\varphi_{2k}^{(n)}(z)\rho^{2k} \end{pmatrix}. \tag{19}$$

By substituting (19) into the corresponding equations and equating to zero the coefficients of each power of ρ in the each of the two components, we obtain the system of ordinary first-order differential equations, which is solvable. For $m > 0$ the solutions are expressed as follows:

$$\varphi_{10}^{(0)} = \frac{C_2^+}{\sigma}\left(\frac{\sqrt{q_0}}{\sigma}\right)^{m-1/2}\exp\left[-\frac{(m+1/2)H}{2c}\int\frac{dz}{q_0}\right], \quad \varphi_{20}^{(0)} = 0, \tag{20}$$

For $m < 0$ these solutions are

$$\varphi_{10}^{(0)} = 0, \quad \varphi_{20}^{(0)} = \frac{C_2^-}{\sigma}\left(\frac{\sqrt{q_0}}{\sigma}\right)^{m+1/2} \exp\left[-\frac{(m-1/2)H}{2c}\int\frac{dz}{q_0}\right], \tag{21}$$

The lower component η of Ψ is obtained from the upper one ξ by the operation $\xi \xrightarrow[W^+\to W^-]{} \eta$.

Quasi-classical solutions of the relativistic problem of atom in the parallel constant electric and magnetic fields

If besides the magnetic field the H-like atom is placed in the constant uniform electric field, then an interaction potential is

$$V(z,\rho) = -\frac{Z}{\sqrt{z^2+\rho^2}} - Fz, \tag{22}$$

If we write the quantity $q_0(z)$ as $q_0 = \sqrt{2\,(U_{eff} - E_{eff})}$, then the effective energy $E_{eff} = -\lambda^2/2$ ($\lambda = c\sqrt{1-\varepsilon^2}$, $\varepsilon = E/c^2$) and effective potential

$$U_{eff}(z,\varepsilon) = \varepsilon V_0 - V_0^2/2c^2 \tag{23}$$

correspond to the expressions (4), (12).

When $F \sim H \ll \lambda^4/4Z$ then point z_0 ($2Z/\lambda^2 \ll z_0 \ll \lambda^2/2F$) exist that

$$\Psi \xrightarrow[z\to z_0]{} \Psi_0^{(as)}, \tag{24}$$

$$\Psi_0(\vec{r}) = \begin{pmatrix} f(r)\,\Omega_{jlm}(\vec{n}) \\ ig(r)\,\Omega_{jl'm}(\vec{n}) \end{pmatrix}, \quad l = j \pm 1/2, \quad l' = 2j - l, \quad \vec{n} = \vec{r}/r, \tag{25}$$

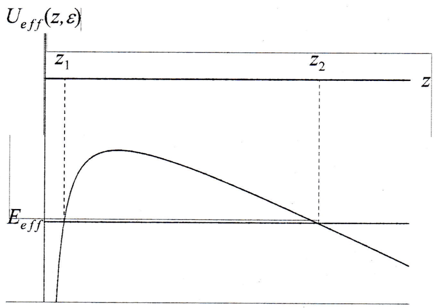

Figure 1: The effective potential $U_{eff}(z,\varepsilon)$.

$$\left.\begin{array}{l} f(r) \\ g(r) \end{array}\right\} = \pm\sqrt{1 \pm \varepsilon_0} A r^{\varepsilon_0 Z/\lambda_0 - 1} e^{-\lambda_0 r}, \quad \varepsilon_0 = E_0/c^2, \quad \lambda_0 = c\sqrt{1-\varepsilon_0^2}. \tag{26}$$

$$A = \lambda_0 \left(2\lambda_0\right)^{\varepsilon_0 Z/\lambda_0} \left(\frac{Z/\lambda_0 - k}{2Z\, n_r!\,\Gamma\left(2\gamma + n_r + 1\right)}\right)^{1/2}. \tag{27}$$

Energy of relativistic H-like atom in weak parallel constant electric and magnetic fields can be calculated within the first order of perturbation theory

$$E = E_0 + \operatorname{sgn} k \frac{3}{4}\sqrt{N^2 - k^2}\frac{(n_r+\gamma)mF}{j(j+1)Z} + \Delta E_H, \tag{28}$$

where $N = \sqrt{n^2 - 2n_r(|k| - \gamma)}$, $E_0 = c^2/\sqrt{1 + [Z\alpha/(n_r+\gamma)]^2}$ is the energy of nonperturbed relativistic H-like atom, ΔE_H is the correction due to the magnetic field.

Let us find the wave function Ψ in the range $z_0 < z < z_2$.

$$S_0(z) = -\int\limits_{z_1}^{z} q_0(x)dx + C_0. \tag{29}$$

From boundary condition (24) follows that

$$S_0 \xrightarrow[z \sim z_0]{} -\lambda_0 z_0 + \frac{\varepsilon_0 Z}{\lambda_0}\ln z_0. \tag{30}$$

In the range $z_0 < z < z_2$, an influence of the Coulomb potential is weak. Therefore,

$$z_1 \approx \frac{Z}{c^2 - E}, \qquad z_2 \approx \frac{c^2 - E}{F} - \frac{Z}{c^2 - E}, \qquad C_0 = \frac{\varepsilon_0 Z}{\lambda_0}\ln\frac{Z}{2\lambda_0^2 e} - Z\alpha \arccos\varepsilon_0, \tag{31}$$

where $e = 2.718...$.

Similarly we find constants

$$C_1^{(+)} = \frac{A\sqrt{\lambda_0}}{c}\frac{(-1)^{|m|+1/2}\operatorname{sgn}k}{2^{|m|-1/2}(|m|-1/2)!}\sqrt{\frac{(j+|m|)!}{4\pi(j-|m|)!}}, \quad C_1^{(-)} = C_1^{(+)}(-1)^{|m|+1/2}\operatorname{sgn}k. \tag{32}$$

and the wave function

$$\Psi^{(\pm)} = \frac{C^{(\pm)}}{\sigma}\left(\frac{\sqrt{q_0}\rho}{\sigma}\right)^{|m|-1/2} \exp\left\{-\int\limits_{z_1}^{z}\left[q_0(x) + \frac{\operatorname{sgn} m(|m| + 1/2H)}{2cq_0(x)}\right]dx + \right.$$

$$\left. + S_1(z)\rho^2 + i(m \mp 1/2)\phi\right\} \begin{pmatrix} \sqrt{c^2 + E_0 - V_0}\,\delta_{m,|m|} \\ \sqrt{c^2 + E_0 - V_0}\,\delta_{-m,|m|} \\ i\sqrt{c^2 - E_0 + V_0}\,\delta_{m,|m|} \\ -i\sqrt{c^2 - E_0 + V_0}\,\delta_{-m,|m|} \end{pmatrix}, \tag{33}$$

where upper (lower) sign corresponds to the case $m > 0$ ($m < 0$),

$$q_0 = \frac{1}{c}\sqrt{c^4 - (E - V_0)^2}, \qquad V_0 = -\frac{Z}{z} - Fz,$$

$$C^{(\pm)} = C_1^{(\pm)}\sqrt{\lambda_0}\left(\frac{Z}{2\lambda_0^2 e}\right)^{\varepsilon_0 Z/\lambda_0} e^{-Z\alpha \arccos \varepsilon_0}. \tag{34}$$

Wave function in the classically allowed domain. Width of below-barrier resonance

Transition through the turning point $z = z_2$ into classically allowed domain is performed within the Zwaan method and reduced to the replacement $q^{(0)} \to -ip^{(0)}$ in the wave function where $p^{(0)} = c^{-1}\sqrt{(E+Fz)^2 - c^4}$:

$$\Psi^{(\pm)} = \frac{B^{(\pm)}}{\sigma}\left(\frac{\sqrt{p_0}\rho}{\sigma}\right)^{|m|-1/2} \exp\left\{ i\int_{z_2}^{z}\left[p_0(x) - \frac{\operatorname{sgn} m(|m|+1/2)H}{2cp_0(x)}\right]dx + \right.$$
$$\left. + S_1(z)\rho^2 + i(m \mp 1/2)\phi\right\}\begin{pmatrix} \sqrt{E - V_0 + c^2}\delta_{m,|m|} \\ \sqrt{E - V_0 + c^2}\delta_{-m,|m|} \\ \sqrt{E - V_0 - c^2}\delta_{m,|m|} \\ -\sqrt{E - V_0 - c^2}\delta_{-m,|m|} \end{pmatrix}, \tag{35}$$

where

$$B^{(\pm)} = C^{(\pm)} e^{-\int_{z_1}^{z_2}\left(q_0(z) + \frac{\operatorname{sgn} m(|m|+1/2)H}{2cq_0(z)}\right)dz + i\pi/4}, \quad S_1 = \frac{ip_0}{2}\left(\frac{\sigma'}{\sigma} - \frac{1}{2}\frac{p_0'}{p_0}\right). \tag{36}$$

The ionization rate is equal to the total probability flux through the plane which is perpendicular to z-axis and located in the domain $z > z_2$:

$$w = c\int_S \Psi^+\vec{\alpha}\Psi d\vec{S} = c\int_0^{2\pi}\int_0^{\infty}(\Psi^+\alpha_z\Psi)\rho d\rho d\phi, \tag{37}$$

Having substituted (35) into (37) and calculating the integral, we obtain:

$$w = \frac{\lambda_0|A|^2}{4^{|m|}(|m|-1/2)!}\frac{(j+|m|)!}{(j-|m|)!}\left(\frac{Z}{2\lambda_0^2 e}\right)^{\frac{2\varepsilon_0 Z}{\lambda_0}}\frac{e^{-2Z\alpha\arccos\varepsilon_0}}{\Theta^{|m|+1/2}} \times$$
$$e^{-2\int_{z_1}^{z_2}\left(q_0(z) + \frac{\operatorname{sgn} m(|m|+1/2)H}{2cq_0(z)}\right)dz}, \tag{38}$$

where $\Theta = \operatorname{Re}\sigma(z)\operatorname{Im}\sigma'(z) - \operatorname{Re}\sigma'(z)\operatorname{Im}\sigma(z)$.

In order to calculate the quantity Θ let us assume that $\sigma(z) = C_1\sigma_1(z) + C_2\sigma_2(z)$ is the solution of equation (18). Then

$$\Theta = (\operatorname{Re}C_1\operatorname{Im}C_2 - \operatorname{Im}C_1\operatorname{Re}C_2)W(\sigma_1,\sigma_2), \tag{39}$$

where $W(\sigma_1, \sigma_2) = \text{const}$ is the Wronskian of Eq. (18). In the vicinity of point z_2

$$\sigma_1(z) = \sqrt{p_0(z)}, \quad \sigma_2(z) = \sqrt{p_0(z)}\int_{z_2}^{z}\frac{dx}{p_0(x)}.$$

So $W(\sigma_1, \sigma_2) = 1$.

Approximative solution of equation

$$\sigma'' + \left[\frac{1}{4}\left(\frac{q_0'}{q_0}\right)^2 - \frac{1}{2}\frac{q_0''}{q_0} - \frac{Q_1}{q_0^2}\right]\sigma = 0, \tag{40}$$

with the boundary condition $\sigma \xrightarrow[z\sim z_0]{} z_0\sqrt{q_0(z_0)}$ is

$$\sigma = \lambda_0\sqrt{q_0}\int\limits_{z_1}^{z}\frac{dx}{q_0(x)} \xrightarrow[z>z_2]{} e^{-i\pi/4}\lambda_0\sqrt{p_0}\left(\int\limits_{z_1}^{z_2}\frac{dz}{q_0(z)} + i\int\limits_{z_2}^{z}\frac{dx}{p_0(x)}\right), \tag{41}$$

and the quantity Θ is

$$\Theta = \lambda_0^2\int\limits_{z_1}^{z_2}\frac{dx}{q_0(x)}.$$

Therefore we yield

$$w = \frac{2\lambda_0|A|^2}{(|m|-1/2)!}\frac{(j+|m|)!}{(j-|m|)!}\left(\frac{Z}{2\lambda_0^2 e}\right)^{\frac{2\varepsilon_0 Z}{\lambda_0}}\frac{e^{-2\int\limits_{z_1}^{z_2}\left(q_0(z)+\frac{\operatorname{sgn} m(|m|+1/2)H}{2cq_0(z)}\right)dz-2Z\alpha\arccos\varepsilon_0}}{\left(4\lambda_0^2\int\limits_{z_1}^{z_2}\frac{dz}{q_0(z)}\right)^{|m|+1/2}}, \tag{42}$$

The integrals in (42) can be approximately calculated in the same way as it was done in one-dimensional case [8]. Omitting details of these calculations we write the final result for the ionization rate:

$$w = \frac{2\lambda_0|A|^2}{(|m|-1/2)!}\frac{(j+|m|)!}{(j-|m|)!}\frac{e^{2Z\alpha\arccos\varepsilon_0}}{(2c\arccos\varepsilon_0)^{|m|+1/2}}\left(\frac{2\lambda_0^2}{F}\right)^{\frac{2\varepsilon_0 Z}{\lambda_0}-|m|-1/2}\times$$

$$\exp\left\{-\frac{c^3\Phi(\varepsilon)}{F} - \frac{H}{F}\arccos\varepsilon_0\right\}, \tag{43}$$

where

$$\Phi(\varepsilon) = \arccos\varepsilon - \varepsilon\sqrt{1-\varepsilon^2}. \tag{44}$$

Special cases

i) Ionization rate of s-level ($j = m = 1/2$). Consider ionization rate of s-level by the constant uniform electric field ($H = 0$):

$$w_s = \frac{\lambda_0|A|^2 e^{2Z\alpha\arccos\varepsilon_0}}{c\arccos\varepsilon_0}\left(\frac{2\lambda_0^2}{F}\right)^{\frac{2\varepsilon_0 Z}{\lambda_0}-1}\exp\left\{-\frac{c^3\Phi(\varepsilon)}{F} - \frac{H}{F}\arccos\varepsilon_0\right\}, \tag{45}$$

that within the factor 2 coincides with the result of [9].

For H-like ions our expression (45) becomes

$$w_{1s_{1/2}} = \frac{2Z^3 e^{2Z\alpha \arccos \varepsilon_0}}{\Gamma(2\varepsilon_0 + 1) c \arccos \varepsilon_0} \left(\frac{4Z^3}{F} \right)^{2\varepsilon_0 - 1} \exp\left\{ -\frac{c^3 \Phi(\varepsilon_0)}{F} \right\}, \tag{46}$$

and coincides with result obtained in [10].

ii) Ionization of s-level of negative ions (e.g. H$^-$, Na$^-$ etc)
Here $Z = 0$ and at $r > a$ the unperturbed radial wave functions is of the form

$$\left. \begin{array}{c} f(r) \\ g(r) \end{array} \right\} = \pm\sqrt{1 \pm \varepsilon_0} A_0 \frac{e^{-\lambda_0 r}}{r}, \tag{47}$$

If $a \ll 1$ then $|A_0|^2 \approx \lambda_0$ and

$$w_{s0} = \frac{F}{2c \arccos \varepsilon_0} \exp\left\{ -\frac{c^3 \Phi(\varepsilon_0)}{F} \right\}. \tag{48}$$

In the nonrelativistic limit formula (48) becomes the known result of paper [14].
iii) Limiting cases without magnetic field ($H = 0$)
In the nonrelativistic limit $c \to \infty$ using the replacements $\lambda \to \kappa$, $\varepsilon \to 1 - \kappa^2/2c^2$, $A \to B/\sqrt{2}$, $|m| \to |m| + 1/2$,

$$\frac{(j + |m|)!}{(j - |m|)!} \to (2l + 1) \frac{(l + |m|)!}{(l - |m|)!},$$

we obtain result of article [13].

For level near lower continuum $\varepsilon \to -1$ exponential factor becomes $\exp(-\pi c^3/F)$ as in the Schwinger's formula [16].

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