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CLOSED EXTENSION TOPOLOGY

The paper contains the results which describe the properties of such general topological construction as closed extension topology. In particular, we prove that this topology is not transitive. We find the base of the least cardinality for the topology and local one for the neighbourhood system of every point. We calculate the interior, the closure, and the sets of isolated and limit points of any set. We also prove that this space is path connected and not metrizable, and investigate its major cardinality characteristics and separation axioms.

У роботі отримано результати, які описують властивості загальної топологічної конструкції — топології замкненого розширення. Зокрема, доведено, що ця топологія не транзитивна, знайдено бази найменшої потужності для топології та системи околів точки, обчислено внутрішність, замикання, мно́жини граничних та ізольованих точок довільної множини. Також доведено лінійну зв'язність і неметризовність цього топологічного простору, досліджено його основні кардинальні інваріанти й аксіоми відокремлюваності.

Closed extension topology is introduced in [1] for the case when its carrier differs from the one of the starting topological space by one point. A particular case of this construction is particular point topology which appears as a closed extension of the discrete topology. The most famous example of particular point topology is Sierpinski space. We generalize this construction to the case of an arbitrary superset of the carrier of the original space.

Let (X, τ) be a topological space and let X^* be a superset of X. Then the family $\tau^* = \{V \subset X^* \mid V \subset X^* \backslash X \text{ or } V = (X^* \backslash X) \cup U,\ U \in \tau\}$ is a topology for X^*, which is called *the closed extension topology of X to X^**. Indeed, if all of the sets $U_\alpha \in \tau^*$, $\alpha \in T$, lies in $X^* \backslash X$, then their union is also contained in $X^* \backslash X$. Otherwise, this union has the form $(X^* \backslash X) \cup U$ for some $U \in \tau$. If the index set T is finite and at least one of U_α lies in $X^* \backslash X$, then the intersection $\bigcap_{\alpha \in T} U_\alpha$ is also contained in $X^* \backslash X$. Otherwise, this intersection has the form $(X^* \backslash X) \cup U$ for some $U \in \tau$.

Thus the open sets of X^* are all subsets of $X^* \backslash X$ and all unions $(X^* \backslash X) \cup U$ where U is open in X. Respectively, closed sets in X^* are all supersets of X and all closed sets in X.

Example 1. *Let X be a topological space and let A be a proper subset of X. Then A-excluded topology for X is the closed extension topology of the indiscrete space A to X.*

Proposition 1. *The closed extension topology τ^* of a topological space (X, τ) to X^* is supremum of topology $\sigma = \{\varnothing\} \cup \{(X^* \backslash X) \cup V,\ V \in \tau\}$ and X-excluded topology for X^*.*

Proof. Topology τ^* contains σ and X-excluded topology for X^*, and so contains their union and supremum. Conversely, each set of τ^* is contained either in σ or in X-excluded topology for X^*, and hence lies in any topology for X^* containing the union of σ and X-excluded topology for X^*.

A map $f : X \to Y$ of topological spaces (X, τ) and (Y, σ) is called *inducing*, if the topology τ is induced by σ and f.

Theorem 1. *Let (X,τ) be a topological space, let X^* be a superset of the set X and let τ^* be the closed extension topology of X to X^*. Then the natural embedding $X \ni x \overset{i}{\mapsto} x \in X^*$ is closed inducing map. In particular, (X,τ) is closed subspace of (X^*,τ^*).*

Proof. Let τ^*_X be the topology for X induced by τ^*. We will show that $\tau^*_X = \tau$. Clearly $\tau \subset \tau^*_X$. Conversely, if $U \in \tau^*_X$, then $U = X \cap V$ for some $V \in \tau^*$. By definition of the closed extension topology the intersection $X \cap V$ either is empty or belongs to τ. Hence X is the subspace of X^*, i. e. the natural embedding i is inducing. Besides, i is closed, since the subspace X is closed in X^*.

Unlike extension topology [2] closed extension topology (like open extension topology [3]) is not transitive. Hence the natural embedding $X \ni x \overset{i}{\mapsto} x \in X^*$ is not quotient, i. e. the closed extension topology is not quotient topology with respect to τ and i.

Example 2. *Consider nested sets $X = \{a\}$, $X^* = \{a,b\}$, $X^{**} = \{a,b,c\}$. Then for the discrete topology τ for X we have $(\tau^*)^* = \{\varnothing, X^{**}, \{c\}, \{b,c\}\} \neq \tau^{**} = \{\varnothing, X^{**}, \{b\}, \{c\}, \{b,c\}\}$.*

Let us describe a base of the least cardinality of the closed extension topology and local one with respect to it.

Proposition 2. *A base of the least cardinality of the closed extension topology of a space X to X^* has the form $\beta^* = \{\{x\}, (X^*\backslash X) \cup U \mid x \in X^*\backslash X, U \in \beta\}$ where β is a base of the least cardinality of the space X.*

Proof. A point $x \in X^*\backslash X$ has the smallest neighbourhood $\{x\}$ which should belong to any base of X^*. So $\{x\} \in \beta^*$, $x \in X^*\backslash X$. Since any open set in X^* is the union of some families of the sets from $\{(X^*\backslash X) \cup U \mid U \in \beta\}$ and $\{\{x\}, x \in X^*\backslash X\}$, the family β^* of open in X^* sets is the base of X^*. Besides, the cardinality of β can not be reduced, and no set from $\{\{x\}, x \in X^*\backslash X\}$ can be removed.

Proposition 3. *Let τ^* be the closed extension topology of a space X to X^*. Then a local base of the least cardinality at any point $x \in X^*$ has the form $\{\{x\}\}$ for $x \in X^*\backslash X$, and $\{(X^*\backslash X) \cup U \mid U \in \beta_x\}$ where β_x is a local base of the least cardinality at point x in X, for $x \in X$.*

Proof. Let β^*_x be a local base of the least cardinality at point $x \in X^*$. If $x \in X^*\backslash X$, then the set $\{x\}$ is the smallest neighbourhood of x, and so $\beta^*_x = \{\{x\}\}$. If $x \in X$, then $\beta^*_x = \{(X^*\backslash X) \cup U \mid U \in \beta_x\}$, since any neighbourhood of x in τ^* is the union of the complement $X^*\backslash X$ and a neighbourhood of x in τ which contains some basic neighbourhood $U \in \beta_x$.

The following Proposition gives an explicit description of the interior, the closure, the sets of isolated and limit points of an arbitrary set of a topological space with the closed extension topology, and necessary and sufficient conditions for density and nowhere density of a given set.

Proposition 4. *Let τ^* be the closed extension topology of a space X to X^* and let $A \subset X^*$. Then:*

1) the interior of A in X^* is equal to $\operatorname{Int}_{X^*} A = (X^*\backslash X) \cup \operatorname{Int}_X(A \cap X)$, where $\operatorname{Int}_X(A \cap X)$ is the interior of the intersection $A \cap X$ in X, when $A \supset X^*\backslash X$, and $\operatorname{Int}_{X^*} A = A\backslash X$ otherwise;

2) the closure of A in X^* has the form $\overline{A}_{X^*} = \overline{A}_X$, where $\overline{A}_X$ is the closure of A in X, when $A \subset X$, and $\overline{A}_{X^*} = A \cup X$ otherwise;

3) the set of isolated points of A in X^* is calculated by formula $\mathrm{I}_{X^*}(A) = \mathrm{I}_X(A)$, where $\mathrm{I}_X(A)$ is the set of isolated points of A in X, when $A \subset X$, and $\mathrm{I}_{X^*}(A) = A \backslash X$ otherwise;

4) the set of limit points of A in X^* is equal to A'_X, where A'_X is the set of limit points of A in X, when $A \subset X$, and $A'_{X^*} = X$ otherwise;

5) if $X^* \backslash X \neq \varnothing$, then A is dense in X^* if and only if $A \supset X^* \backslash X$; otherwise A is dense in X^* if and only if A is dense in X.

6) A is nowhere dense in X^* if and only if $A \subset X$.

Proof. 1) If $A \supset X^* \backslash X$, then the largest open set in X^* contained in A is obviously $(X^* \backslash X) \cup \mathrm{Int}_X(A \cap X)$. Otherwise, no open set in X^* of the form $(X^* \backslash X) \cup U$ where U is open set in X lies in A, and so $\mathrm{Int}_{X^*} A = A \backslash X$.

2) If $A \subset X$, then the largest open set in X^* that does not intersect A is $(X^* \backslash X) \cup \mathrm{Int}_X(X \backslash A)$, and so $\overline{A}_{X^*} = \overline{A}_X$. Otherwise, the largest open set in X^* that does not intersect A is $(X^* \backslash X) \backslash A$, and therefore $\overline{A}_{X^*} = A \cup X$.

3) A point of A is isolated in A if some neighbourhood of this point does not contain distinct from it points of A. Clearly every point x of $A \subset X$ is isolated in A in the space X^* if and only if x is isolated in A in the space X. If $A \not\subset X$, then any point of $A \backslash X$ is isolated in A, but every point x of $A \cap X$ is not, since each neighbourhood of x has at least two point intersection with A.

4) If $A \subset X$, then $A'_{X^*} = \overline{A}_{X^*} \backslash \mathrm{I}_{X^*}(A) = \overline{A}_X \backslash \mathrm{I}_X(A) = A'_X$. Otherwise, $A'_{X^*} = (A \cup X) \backslash (A \backslash X) = X$.

5) According to 2) the set A is dense in X^* if and only if $X^* = \overline{A}_{X^*} = A \cup X$ which is equivalent to inclusion $A \supset X^* \backslash X$.

6) The set A is nowhere dense in X^* if and only if $\varnothing = \mathrm{Int}_{X^*}(\overline{A}_{X^*}) = \overline{A}_{X^*} \backslash X$ which is equivalent to inclusion $A \subset X$.

Now we proceed with the study of topological properties of the closed extension topology.

Theorem 2. *Let τ^* be the closed extension topology of a space X to X^* and $X^* \neq X$. Then X^* is path connected and thus connected.*

Proof. For every points $x \in X$ and $y \in X^* \backslash X$ the map $l : [0,1] \to X^*$ defined by $l(0) = x$ and $l((0,1]) = y$ is a path in X^* joining x to y.

Corollary 1. *Every topological space can be embedded as closed subspace in a path connected (and thus connected) topological space.*

Proof. The desired topological space can be obtained as a space with the closed extension topology for the case when its carrier differs from the one of the starting topological space by one point.

The next fact follows directly from propositions 2 and 3.

Theorem 3. *Let τ^* be the closed extension topology of a space X to X^*. The topological space X^* is first countable if and only if the space X is first countable. The space X^* is second countable if and only if the space X is second countable and the complement $X^* \backslash X$ is at most countable.*

Theorem 4. *Let τ^* be the closed extension topology of a space X to X^*. The topological space X^* is Lindelöf (compact) if and only if the space X is Lindelöf (compact).*

Proof. If X^* is Lindelöf (compact), then X is Lindelöf (compact) as closed subspace of X^* by Theorem 1. Conversely, let $\{V_\alpha, \alpha \in T\}$ be an open covering of X^* and let $T' = \{\alpha \in T \mid V_\alpha = (X^* \backslash X) \cup U_\alpha$ for some open set U_α of the space $X\}$. Then the family $\{U_\alpha, \alpha \in T'\}$ is an open covering of X. Since X is Lindelöf (compact), this covering has at most countable (finite) subcovering $\{U_\alpha, \alpha \in T''\}$. Thus $\{V_\alpha, \alpha \in T''\}$ is at most countable (finite) subcovering of initial covering $\{V_\alpha, \alpha \in T\}$ of the space X^*.

Theorem 5. *Let τ^* be the closed extension topology of a space X to X^* and $X^* \neq X$. The topological space X^* is separable if and only if the complement $X^* \backslash X$ is at most countable. The set $X^* \backslash X$ is at most countable dense set in X^* of the least cardinality.*

Proof. The complement $X^* \backslash X$ is the smallest dense set in X^* vacuously.

Theorem 6. *Let τ^* be the closed extension topology of a space X to X^* and $X^* \neq X$. The topological space X^* is T_0 if and only if the space X is T_0. The space X^* is T_4 if and only if every two nonempty closed sets in X intersect (a stronger condition than T_4 separation axiom). The space X^* is not T_i for $i = 1, 2, 3$. In particular, X^* is not regular, normal and metrizable.*

Proof. Let X satisfy T_0 separation axiom. Then for any two distinct points $x, y \in X$ there is a neighbourhood U of x in X that does not contain y. So $(X^* \backslash X) \cup U$ is a neighbourhood of x in X^* that does not contain y. If $x, y \in X^*$ and $x \in X^* \backslash X$, then the neighbourhood $\{x\}$ of x in X^* does not contain y. Thus X^* is T_0. Conversely, if X^* satisfy T_0 separation axiom, then X is T_0 as a subspace of X^*.

If every two nonempty closed sets in X intersect, then the same holds for X^*. So X^* is T_4. Conversely, let X^* satisfy T_4 separation axiom and let A and B be any two disjoint closed sets in X. Then they are closed sets in X^* having disjoint neighbourhoods in X^*. But every two open sets in X^* with nonempty intersection with X can not be disjoint. So at least one of the sets A or B is empty.

Finally, X^* is not T_1 since for any point $x \in X^* \backslash X$ one point set $\{x\}$ is not closed in X^*. Also X^* is not T_3 because any point $x \in X^* \backslash X$ and closed set X in X^* does not have disjoint neighbourhoods.

References

1. *Lynn Arthur Steen and J. Arthur Seebach, Jr.* Counterexamples in topology. – New York: Dover publications, 1978. – 256 p.
2. *Babych V. M., Filonenko O. M.* Extension Topology // Scientific Bulletin of Uzhhorod University. Series of Mathematics and Informatics. – 2015. – **1 (26)**. – p. 7-11. [in Ukrainian]
3. *Babych V. M., Pyekhtyeryev V. O.* Open Extension Topology // Proc. Intern. Geom. Center. – 2015. – Vol. 8.– No. 2. – p. 20-25. [in Ukrainian]
4. *Engelking R.* General Topology. Revised and completed edition. – Berlin: Heldermann Verlag, 1989. – 540 p.

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