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S. A. Ovsienko (Taras Shevchenko National University of Kyiv)

GENERIC REPRESENTATIONS OF FREE BOXES

In this paper we investigate some open sets in the variety of representations of a free box in a fixed dimension. We introduce on the category of representations of the free normal box $\mathcal{A}$ the functor $\mathrm{Ext}^1_{\mathcal{A}}$ in the possible elementary way without a transition to some another category which is useful in the box method applications. Using the standard homological methods and the bilinear forms methods, we prove that the isomorphism classes of some open sets in the variety of representations are parameterized by some open set in an affine space, and obtain the number of sub generic representations.

В статті досліджуються деякі відкриті множини багатовиду зображень вільного бокса в фіксованій розмірності. На категорії зображень вільного нормального бокса $\mathcal{A}$ вводиться функтор $\mathrm{Ext}^1_{\mathcal{A}}$ елементарним чином без переходу до якоїсь іншої категорії, так, щоб це було зручно використовувати у застосуваннях. Використовуючи стандартні гомологічні методи та методи білінійних форм, ми доводимо, що класи ізоморфізму деяких відкритих множин багатовиду зображень бокса параметризуються деякою відкритою множиною в афінному просторі, і отримуємо число загальних зображень.

Introduction. In this paper we investigate some open sets in the variety of representations of a free box in a fixed dimension. We generalize well known result concerning a decomposition of the considered representations in terms of the non symmetrical bilinear form [12] from the case of the finite dimensional algebras [5], [6] on the case of the free boxes [14]. We prove that the isomorphism classes of some open sets in the variety of representations are parameterized by some open set in an affine space and prove a corollary on the number of sub generic representations, analogous to [4] (by generic representation X we mean the following one: in the variety of representations of corresponding dimension the set of all representations isomorphic to X is a Zarisky open set). Remark, that in the free box situation work both the standard homological methods and box reduction technique together with inspired of it bilinear forms methods as well.

The part 1 contains the general definitions of a free box $\mathcal{A}$, the representations category of it $R(\mathcal{A})$ e.t.c. To work with the family of the boxes, depending on the points of some variety we introduce in the part 2 the notion of a scalar representation over some commutative algebra Λ and investigate in this case a change of a basic algebra.

In the part 3 we introduce on the category of representations $R(\mathcal{A})$ of the free normal $\Bbbk-$ box $\mathcal{A}$ (where $\Bbbk$ is a field) the functor $\mathrm{Ext}^1_{\mathcal{A}}$. We do it in the possible elementary way in the category $R(\mathcal{A})$ without a transition to some another category. It seems, that this way to define $\mathrm{Ext}^1_{\mathcal{A}}$ is the usual in the box method applications. After the introducing of notion of extension in $R(\mathcal{A})$ we show that $\mathrm{Ext}^1_{\mathcal{A}}$ classifies the congruence classes of these. We bring also some statements in order to show that so defined $\mathrm{Ext}^1_{\mathcal{A}}$ has reasonable properties. Thereafter we are able to prove the result on the decomposition of representations from an open set of the variety of the representations of the fixed dimension (Proposition 3) and this proof is similar, in fact, with given in [13] for finite dimensional associative algebras.

1. General definitions

1.1. If S is some finite set, then by $|S|$ we denote the number of elements in S. We fix an algebraically closed field $\Bbbk$. If $\mathcal{V}$ is a $\Bbbk$-vector space, then by $|\mathcal{V}|_{\Bbbk} = \dim_{\Bbbk} \mathcal{V}$ or by $|\mathcal{V}|$ we will denote the dimension of it.

1.2. Let Λ be a commutative finite generated algebra over $\Bbbk$ and A be a $\Lambda-$category. If $\alpha, \beta \in \mathrm{Ob}A$, then the set of morphisms from α to β we will denote by $\mathrm{Hom}_A(\alpha, \beta)$, $A(\alpha, \beta)$, or by $(\alpha, \beta)_A$, in case of the category of modules ($\Bbbk -$ mod, $\Lambda -$ mod etc.) we will write $(X, Y)_{\Bbbk}$ instead $\mathrm{Hom}_{\Bbbk}(X, Y)$ etc. We recall, that "A is $\Lambda-$ category" means, that for every α, $\beta \in \mathrm{Ob}A$, $\mathrm{Hom}_A(\alpha,\ \beta)$ is a $\Lambda-$ bimodule, where the left and right multiplications on $\lambda \in \Lambda$ coincide and the morphisms superposition in A is Λ-bilinear. A box over A is a quadruple $\mathcal{A} = (A, V, \mu, \varepsilon)$, often described as a pair $\mathcal{A} = (A, V)$, where V is an A-bimodule, endowed with the structure of an A-cocategory. This structure is defined by two $A-$bimodule morphisms $\mu : V \to V \otimes_A V$, $\varepsilon : V \to A$, where the comultiplication μ is coassociative and ε satisfies the A-counit axioms up to μ. If we want underline, that the category A is a $\Lambda-$ category, we say, that $\mathcal{A}$ is a $\Lambda-$ box. We assume usually, that the category A is reduced. This means, that all idempotents in A are trivial and A doesn't contain isomorphic objects. If these assumptions will be wrong, it will be especially noted. For example, we will consider the category $\mathrm{add}(A)$, that is the fully additive closure of A.

1.3. A representations of $\mathcal{A}$ is a Λ-linear functor $X : A \to \Lambda -$ mod. The representations of $\mathcal{A}$ form a category $R_{\Lambda}(\mathcal{A})$, where the Λ-module of morphisms from the representation X to the representation Y is equal to the Λ -module of A-bimodule morphisms $\mathrm{Hom}_{A-A}(V, \mathrm{Hom}_{\Lambda}(X, Y))$. A morphism f from X to Y we will as usually denote by $f : X \to Y$. We use also some another presentation of the category $R(\mathcal{A})$, hence sometimes the corresponding to f A-bimodule morphism we denote by $b(f) : V \to \mathrm{Hom}_{\Lambda}(X, Y)$ (so $b(f)(v) = f(v)$, $v \in V$). The multiplication of the morphisms $f_1 : X \to Y$ and $f_2 : Y \to Z$ in $R_{\Lambda}(\mathcal{A})$ $f_2 f_1 : X \to Z$ is defined in such a way, that $b(f_2 f_1)$ coincides with a superposition

$$V \xrightarrow{\mu} V \otimes_A V \xrightarrow{b(f_2)\otimes b(f_1)} (Y, Z)_{\Lambda} \otimes_A (X, Y)_{\Lambda} \xrightarrow{m} (X, Z)_{\Lambda}$$

where m is the morphisms superposition in $\Lambda -$ mod. The space of the morphisms from X to Y in $R(\mathcal{A})$ we will denote by $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ or $(X, Y)_{\mathcal{A}}$.

1.4. We need another but equivalent definition of the notions of the morphism and the superposition in the category $R(\mathcal{A})$, [1]. By using the conjugated associativity we get an isomorphism

$$c : \mathrm{Hom}_{R(\mathcal{A})}(X, Y)(\simeq \mathrm{Hom}_{A-A}(V, (X, Y)_{\Lambda})) \simeq \mathrm{Hom}_A(V \otimes_A X, Y),$$

so to $f : X \to Y$ corresponds to an A-module morphism $c(f) : V \otimes_A X \to Y$ and in this terms for $f_1 : X \to Y$ and $f_2 : Y \to Z$ in $R_{\Lambda}(\mathcal{A})$ $c(f_2 f_1)$ is defined as the superposition $c(f_2)(1_V \otimes c(f_1))(\mu \otimes 1_X) : V \otimes_A X \to Z$.

If $\mathcal{A} = (A, V)$, $\mathcal{A}' = (A', V')$ are two boxes over Λ, then a morphism of boxes $F : \mathcal{A} \to \mathcal{A}'$ is a pair (F_0, F_1), where $F_0 : A \to A'$ is a Λ-functor, $F_1 : V \to V'$ is an A-bimodule morphism with the A-structure on V' induced by F_0 and the following holds : $\pi^A_{A'}(F_1 \otimes_A F_1)\mu_{\mathcal{A}} = \mu_{\mathcal{A}'} F_1$, $F_0 \varepsilon_{\mathcal{A}} = \varepsilon_{\mathcal{A}'} F_1$, where $\mu_{\mathcal{A}}$, $\mu_{\mathcal{A}'}$ ($\varepsilon_{\mathcal{A}}$, $\varepsilon_{\mathcal{A}'}$) are the comultiplications (counits) in $\mathcal{A}$ and $\mathcal{A}'$, $\pi^A_{A'} : V' \otimes_A V' \to V' \otimes_{A'} V'$ is the canonical projection. The morphism $F : \mathcal{A} \to \mathcal{A}'$ induces a natural functor $F^* : R_\Lambda(\mathcal{A}') \to R_\Lambda(\mathcal{A})$.

An important partial case is following. If $\mathcal{A} = (A, V)$ is a box and $F : A \to A'$ is a Λ-functor, then we can construct a box $\mathcal{A}^F = (A', V^F)$, $\quad V^F = A' \otimes_A V \otimes_A A'$ with a box morphism, that is denoted also by F $F : \mathcal{A} \to \mathcal{A}^F$, $F = (F_0, F_1)$, where $F_0 = F : A \to A'$ and $F_1 : V \to V^F$ for $v \in V(\alpha, \beta), \alpha, \beta \in \mathrm{Ob}A$ is defined as $F_1(v) = 1_{F(\beta)} \otimes v \otimes 1_{F(\alpha)}$.

Proposition 1 ([3]). *The functor $F^* : R_\Lambda(\mathcal{A}^F) \to R_\Lambda(\mathcal{A})$, induced by F is full and faithful and its image consists all representations of $\mathcal{A}$ $M : A \to \Lambda - mod$, which can be factorized through F.*

1.5. A box $\mathcal{A} = (A, V)$ is called *free*, provided A is a free category and the kernel of $\varepsilon : V \to A$ $\bar{V}$ (called the kernel of box $\mathcal{A}$, [14]) is a free A-bimodule. We will consider a free normal box, that is described usually in terms of the bigraph and differential. A bigraph $S = (S_0,\ S_1)$ contains the set of vertexes (or points) S_0 and the set of arrows S_1 separated in two parts S_1^0, S_1^1. The arrows from S_1^0 (S_1^1) are called and displayed as the *solid* (*dotted*) arrows. On S_1 is defined a function of degree deg : $S_1 \to \{0, 1\}$, deg $(S_1^0) = \{0\}$, $\deg(S_1^1) = \{1\}$ and two functions $q, s : S_1 \to S_0$ of the source and the stink of an arrow. The denotation $a : \alpha \to \beta, a \in S_1, \alpha, \beta \in S_0$ is equivalent to $q(a) = \alpha, s(a) = \beta$. By $S_1(\alpha, \beta)$ ($S_1^0(\alpha, \beta), S_1^1(\alpha, \beta)$) we denote the set of all arrows (solid, dotted arrows), leading from α to β. So called completed bigraph $\hat{S}$ of S we get from S by adding to S_1 the set of the loops $\Sigma = \{e_\alpha\}, \alpha \in S_0, \deg(e_\alpha) = 1$. We will suppose, that the set $\hat{S}_1$ is finite.

By $Q = Q(S)$ we denote the graph, formed by all solid arrows from $S, Q = (Q_0, Q_1), Q_0 = S_0, Q_1 = S_1^0$. By Λ^{S_0} we denote the semi simple category over Λ with the set of object S_0, and by A_Λ (or A, if Λ is fixed) we denote the free category over Λ $\Lambda[Q]$, generated by Q. By $\mathcal{U}$ ($\hat{\mathcal{U}}$) we denote the free graded category over Λ, generated by S ($\hat{S}$) provided the degree on S_1 ($\hat{S}_1$) coincides with introduced above and $\deg(\Lambda^{S_0}) = \{0\}$. In this situation we can consider the decomposition in the graded components $\mathcal{U} = \oplus_{i=0}^{\infty} \mathcal{U}_i$, ($\hat{\mathcal{U}} = \oplus_{i=o}^{\infty} \hat{\mathcal{U}}_i$).

The *kernel* of the constructed normal free box $\mathcal{A} = (A, V)$ $\bar{V}$ and free A-bimodules $\mathcal{P}_{-1}, \mathcal{P}_0, \mathcal{P}_\Sigma$ we define as the graded free bimodules over A, generated by sets of generators, every from each can be identified with a subset in $\hat{S}$. They are endowed with the degrees $1, 0, 1, 1$ and coincides with S_1^1, S_1^0, $\hat{S}_1^1$, Σ in the cases $\bar{V}$ and the free A-bimodules $\mathcal{P}_{-1}$, $\mathcal{P}_0$, $\mathcal{P}_\Sigma$ correspondingly. In the cases $\mathcal{P}_{-1}, \mathcal{P}_0, \mathcal{P}_\Sigma$ the generator, corresponding $x \in S_1$ we will denote by $[x]$. If $f \in \hat{\mathcal{U}}_1 (f \in \bar{V})$ and $f = f_1 \varphi f_2$, $f_1,\ f_2 \in A$, $\varphi \in \hat{S}_1^1$ ($\varphi \in S_1^1$), then we define $[f] = f_1[\varphi] f_2$. Being prolonged by Λ-linearity $[\quad]$ defines an isomorphism $[\quad] : \hat{\mathcal{U}}_1 \to \mathcal{P}_0$ of A-bimodules. The isomorphism $[\quad]$ induces the canonical inclusion $\sigma : \bar{V} \hookrightarrow \mathcal{P}_0, \sigma : \varphi \mapsto [\varphi], \varphi \in S_1^1$.

In this situation the following equivalent data are used for a description of the free normal box $\mathcal{A} = (A, V)$, [8], [14]:

a) a differential $d : \hat{\mathcal{U}} \to \hat{\mathcal{U}}$, $\deg d = 1, d^2 = 0$, such that Leibniz formula holds, $d(e_\alpha) = e_\alpha^2, \alpha \in S_0$ and $\delta(x) = d(x) - e_\alpha x + (-1)^{degx} x e_\beta \in \mathcal{U}$ for any $x \in S_1, x : \beta \to \alpha, \alpha, \beta \in S_0$;

b) a differential $\delta : \mathcal{U} \to \mathcal{U}$, such that $\deg \delta = 1, \delta^2 = 0$ and Leibniz formula holds;

c) the restriction of the differential δ on A and $\bar{V}, \bar{\delta} : A \to \bar{V}, \bar{\delta} : \bar{V} \to \bar{V} \otimes_A \bar{V}$, defining on $\mathcal{U}$ the differential δ, satisfying b).

We construct an A-bimodule homomorphism $\partial : \mathcal{P}_{-1} \to \mathcal{P}_0$. It is enough to define ∂ on the generators of $\mathcal{P}_{-1}$, so we set $\partial([x]) = [d(x)], x \in S_1^0$. Consider the following commutative diagram with exact columns and rows

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & 0 & \longrightarrow & \bar{V} & \xrightarrow{1_{\bar{V}}} & \bar{V} & \to & 0 \\
 & & \downarrow & & \downarrow \subset \sigma & & \downarrow \subset i & & \\
0 & \to & \mathcal{P}_{-1} & \xrightarrow{\partial} & \mathcal{P}_0 & \xrightarrow{\pi} & V & \to & 0 \\
 & & \| & & \downarrow \subset \pi_\Sigma & & \downarrow \subset \varepsilon & & \\
0 & \to & \mathcal{P}_{-1} & \xrightarrow{\partial_A} & \mathcal{P}_\Sigma & \xrightarrow{\pi_A} & A & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The lower row is a free A-bimodule resolution of A, $\partial_A([x]) = [e_\alpha]x - x[e_\beta]$, $x : \beta \to \alpha, x \in S_1^0$, $\pi_A([e_\alpha]) = 1_\alpha, \alpha \in S_0$, π is the canonical projection, $\pi_\Sigma([x]) = 0$, $x \in S_1^1$, $\pi_\Sigma([e_\alpha]) = [e_\alpha]$, $\alpha \in \mathrm{Ob} S_0$, ε is induced by π_Σ and i by σ. So we can define the free box $\mathcal{A} = (A, V)$ with the counit $\varepsilon : V \to A$ and the comultiplication $\mu : V \to V \otimes_A V$ defined from the following diagram

$$\begin{array}{ccc}
\mathcal{P}_0 & \xrightarrow{\pi} & V \\
\subset \mu_{\mathcal{P}_0} \downarrow & & \subset \mu \downarrow \\
\mathcal{P}_0 \otimes_A \mathcal{P}_0 & \xrightarrow{\pi^{\otimes 2}} & V \otimes_A V
\end{array}$$

where $\mu_{\mathcal{P}_0}([\varphi]) = ([\quad] \otimes [\quad])d(\varphi), \varphi \in \hat{S}_1^1$. From the Leibniz formula for d and the equality $d^2 = 0$ follows, that μ is an A-bilinear coassociative comultiplication. In this case for two representations X, $Y \in R(\mathcal{A})$ we can consider a morphism $f : X \to Y$ as an $A-$ bimodule morphism $f : \mathcal{P}_0 \to (X, Y)_{\Bbbk}$, such that $f([d(x)]) = 0$ for all $x \in S_1^0$. If $f : X \to Y$, $g : Y \to Z$, then for every $\varphi \in S_1^1$, $\varphi : \alpha \to \beta$ holds $b(gf)(\varphi) = b(g)(e_\beta)b(f)(\varphi) + b(g)(\varphi)b(f)(e_\alpha) + m(b(g) \otimes b(f))(\delta(\varphi))$, where m is the superposition in the category of the vector spaces.

1.6. We will suppose, that the box $\mathcal{A} = (A, V)$ is triangular ([14], [2]). This means, that exists a filtration $S_1 = S_1^{(N)} \supset S_1^{(N-1)} \supset \ldots \supset S_1^{(1)} \supset S_1^{(0)} = \emptyset$, that is called the *triangular filtration*, and the following holds: if $\mathcal{U}_i \subset \mathcal{U}$ is the free graded category, generated by $S_1^{(i)}$, then $\delta(\mathcal{U}_i) \subset \mathcal{U}_{i-1}, i = 1, ..., N$, in particulary $\delta(\mathcal{U}_1) = 0$. Such a system of free generators of the free category $\mathcal{U}$ we will call triangular. A function of the triangular hight $h : S_1 \to \mathbb{Z}$ is defined by following: if $x \in S_1^{(i)} \backslash S_1^{(i-1)}$, then $h(x) = i$. The triangular box we consider together with a triangular filtration and the number N we will call the triangular hight of the box $\mathcal{A}$. The category of all representations of a triangular box over the field $\Bbbk$ is fully additive [8], [14].

1.7. A box $\mathcal{A} = (A, V)$ is called *elementary*, if the category A is semi simple. An elementary box is automatically free. If $\Lambda = \Bbbk$, then following evident lemma holds.

Lemma 1. *Let A be a semi simple category over $\Bbbk$ and $\mathcal{A} = (A, V)$ be an elementary (not necessary triangular) box. Then $\mathcal{A}$ is normal and the set of generators* *of it admits the triangular filtration if and only if the family of* $\Bbbk$*-dual vector spaces* $\{DV(\alpha,\beta)\}_{\alpha,\beta\in S_0}$ *forms a finite dimensional local (with the local endomorphism rings) category* DV*, where the multiplication in* DV $m : DV(\beta,\gamma)\otimes_A DV(\alpha,\beta)\to DV(\alpha,\gamma)_{\alpha,\beta,\gamma\in S_0}$ *is dual to the comultiplication* $\mu : V(\alpha,\gamma)\to V(\beta,\gamma)\otimes_A V(\alpha,\beta)$.

1.8. The space of the dimensions of the box $\mathcal{A}$ (and the bigraph S) is a $\mathbb{R}$ - vectorspace $\mathcal{L}=\mathcal{L}_{\mathcal{A}}$ with the basis $\{v_\alpha\}_{\alpha\in S_0}$. In case $\Lambda=\Bbbk$ the dimension of a representation $V\in R_{\Bbbk}(\mathcal{A})$ is defined as a vector $\dim V\in\mathcal{L}_A$ with the coordinates $(\dim V)_\alpha=\dim_{\Bbbk} V(\alpha)$, $\alpha\in S_0$. Corresponding to the box $\mathcal{A}$ non symmetrical bilinear form $<,>$ $(=<,>_{\mathcal{A}}=<,>_S) : \mathcal{L}\times\mathcal{L}\to\mathbb{R}$ is defined as $< v_\alpha, v_\beta >= \mid \hat{S}^1_1(\alpha,\beta)\mid - \mid \hat{S}^0_1(\alpha,\beta)\mid, \alpha,\beta\in S_0$, [11]. In this situation we denote by $(,)(=(,)_{\mathcal{A}}=(,)_S)$ the symmetrical bilinear form, corresponding $<,>$ and f $(=f_{\mathcal{A}}=f_S)$ the corresponding quadratic form. A vector $x\in\mathcal{L}$ is called sincere in $\gamma\in S_0$, if $x_\gamma\neq 0$; by supp$x$ we denote the set of $\gamma\in S_0$, such that $x_\gamma\neq 0$; x is called sincere, if supp$x=S_0$.

2. Box, depending on parameters

2.1. The usually assumption in the box method applications is $\Lambda=\Bbbk$. In order to consider a family of free box structures depending on some parameters, but with the same bigraph, we introduce the notion of a scalar representation in case when Λ is a commutative finite generated algebra over the field $\Bbbk$.

2.2. If $\chi:\Lambda\to\Bbbk$ is a $\Bbbk$-homomorphism, then by U_χ we denote the 1− dimensional $\Bbbk$-space with the Λ-module structure, induced by χ. By $\chi-\mathrm{mod}$ we denote the fully additive subcategory in $\Lambda-\mathrm{mod}$, generated by U_χ. A representation $F:A\to\Lambda-\mathrm{mod}$ we call χ-scalar, (or scalar, if χ isn't important) when the following equivalent statements hold:

1) there exists $F_\chi : A\to\chi-\mathrm{mod}$, such that $F=i_\chi F_\chi$, where $i_\chi:\chi-\mathrm{mod}\hookrightarrow\Lambda-\mathrm{mod}$ is the canonical inclusion;

2) if $A_\chi=A\otimes_\Lambda\Bbbk$, where Λ acts on $\Bbbk$ with χ and $\pi_\chi:A\to A_\chi$ is the canonical functor, then F factorizes through π_χ;

3) for every $\lambda\in\Lambda$ and $a\in A$ $F(\lambda a)=\chi(\lambda)F(a)=F(a\lambda)$.

By $R(\mathcal{A})$ we denote the category of the scalar representations for all $\chi:\Lambda\to\Bbbk$ (if $\Lambda=\Bbbk$, then $R(\mathcal{A})=R_\Lambda(\mathcal{A})$). The full subcategory, formed by all χ-scalar representations we denote by $R_\chi(\mathcal{A})$. If $X\in R_\chi(\mathcal{A})$, $X'\in R_{\chi'}(\mathcal{A})$, $\chi\neq\chi'$, then $\mathrm{Hom}_{R(\mathcal{A})}(X,X')=0$.

2.3. If $\mathcal{A}=(A,V)$ and $\mathcal{A}'=(A',V')$ are $\Lambda-$ box and $\Lambda'-$ box correspondingly and $\varphi:\Lambda\to\Lambda'$ is a $\Bbbk$-homomorphism, that endows A' and V' with the Λ-bimodule structure, then a morphism of boxes, associated with φ is a triple $(\varphi,F_0,F_1):(A,V)\to(A',V')$ of φ, a Λ-functor $F_0:A\to A'$ and an A-bimodule morphism $F_1:V\to V'$, commuting with the counits and the comultuplications. In the case $\varphi=1_\Lambda$ we omit it and consider the morphism as a pair (F_0,F_1).

2.4. The necessary example of a box morphism associated with φ is the following. Let $\varphi:\Lambda\to\Lambda'$ be a $\Bbbk-$homomorhism and $\mathcal{A}=(A,V)$ is a box, $F=i_\varphi:A\to A\otimes_\Lambda\Lambda'$ be induced by φ canonical functor $A'=A\otimes_\Lambda\Lambda'$. We can construct as in Proposition1 a $\Lambda-$ box $\mathcal{A}^{F,\varphi}=(A',V^{F,\varphi},\ \mu_{\mathcal{A}^{F,\varphi}},\ \varepsilon_{\mathcal{A}^{F,\varphi}})$, where $V^{F,\varphi}=A'\otimes_A V\otimes_A A'$ together with the canonical morphism of $\Lambda-$boxes $F^\varphi:\mathcal{A}\to\mathcal{A}^{F,\varphi}$. The $\Lambda-$ bimodule $V^{F,\varphi}$ has the obvious structure of $\Lambda'-$bimodule, but the box $\mathcal{A}^{F,\varphi}$ isn't a $\Lambda'-$box, since in general for $v\in V^{F,\varphi}$, $\lambda'\in\Lambda'$ the equation $\lambda'v=v\lambda'$ isn't true. We consider in $V^{F,\varphi}$ an $A'-$subbimodule I, generated by $(\lambda'\otimes v\otimes 1-1\otimes v\otimes\lambda')$ for all $\lambda'\in$

Λ', $v \in V$ and set $A^\varphi = A'$, $V^\varphi = V^{F,\varphi}/I$. Since the comultiplication $\mu_{\mathcal{A}^{F,\varphi}}$ is $\Lambda'-$bilinear, $\mu_{\mathcal{A}^{F,\varphi}}(I) \subset I \otimes_A V^{F,\varphi} + V^{F,\varphi} \otimes_A I$ and $\varepsilon_{\mathcal{A}^{F,\varphi}}(I) = 0$, we get an induced comultiplication $\mu_\varphi : V^\varphi \to V^\varphi \otimes_{A'} V^\varphi$ and a counit $\varepsilon_\varphi : V^\varphi \to A^\varphi$. They endow $\mathcal{A}^\varphi = (A^\varphi, V^\varphi)$ with the structure of $\Lambda'-$box together with an associated with φ box morphism $F_\varphi : \mathcal{A} \to \mathcal{A}^\varphi$, where $F_0 = F$, F_1 $(= F_1^\varphi : V \to V^{F,\varphi} \to V^\varphi)$ is the superposition

$$F_1 : v \mapsto 1_\beta \otimes v \otimes 1_\alpha \mapsto 1_\beta \otimes v \otimes 1_\alpha + I, v : \alpha \to \beta, \alpha, \beta \in ObA.$$

The induced functor $F_\varphi^* : R(\mathcal{A}^\varphi) \to R(\mathcal{A})$ in general isn't an equivalence on its image (as in Proposition 1), since $I \neq 0$.

Remark 1. *We remark also, that $I = 0$ in the important partial cases of φ : either a projection on a factor algebra or a localization.*

If we fix $\chi : \Lambda' \to \mathbb{k}$, then for every pair of representations $V, W \in R_\chi(\mathcal{A}^\varphi) \subset R(\mathcal{A}^\varphi)$ and $f : V \to W$ holds $b(f)(I) = 0$. From Proposition 1 and the preceding construction follows

Lemma 2. *1) Let $\varphi : \Lambda \to \Lambda'$ be a $\mathbb{k}-$ homomorphism, $\mathcal{A} = (A, V)$ be a box over A. Then induced by $F_\varphi = (\varphi, F_0 = i_\varphi, F_1^\varphi)$ functor between the categories of the scalar representations $F^* : R(\mathcal{A}^\varphi) \to R(\mathcal{A})$ for every $\chi : \Lambda' \to \mathbb{k}$ induced the functor $F_\chi^* : R_\chi(\mathcal{A}^\varphi) \to R_{\chi\varphi}(\mathcal{A})$, that is an equivalence on its image.*
2) For $\chi : \Lambda \to \mathbb{k}$ we denote by $\mathcal{A}_\chi$ the box $\mathcal{A}^\varphi$ in the case $\varphi = \chi$ and induced by χ morphism of boxes $F_\chi : \mathcal{A} \to \mathcal{A}_\chi$ by π_χ. Then induced functor $\pi_\chi^* : R(\mathcal{A}_\chi) \to R(\mathcal{A})$ accomplishes an equivalence $R(\mathcal{A}_\chi)$ on $R_\chi(\mathcal{A})$.

2.5. The dimension of a χ-scalar representation F is an indexed by S_0 integer vector $\dim F : S_0 \to \mathbb{Z}$, such that for $\alpha \in S_0$ $(\dim F)(\alpha)$ (or $(\dim F)_\alpha$) is equal to the length of a compositional series (or, that the same, the $\mathbb{k}$-dimension) of the Λ-module $F(\alpha)$. In the case $\Lambda = \mathbb{k}$ this definition coincides with usual. A scalar representation F we call finite dimensional, if $F(\alpha)$ is a finite dimensional $\mathbb{k}$-space for any $\alpha \in S_0$. The notions of a bilinear symmetrical form e.t.c in the case of the free box are defined by the bigraph and coincides with introduced earlier.

2.6. As above, we assume that $\mathcal{A}$ is free, normal and triangular. By $\mathcal{X} = \mathcal{X}(\Lambda)$ $=$ Specm Λ we denote an algebraic variety, formed by all the $\mathbb{k}$-points $\chi : \Lambda \to \mathbb{k}$. All the scalar representations of $\mathcal{A}$, having a fixed dimension $x \in \mathcal{L}_\mathcal{A}$ can be parameterized with the points of a variety $\Pi_x^\Lambda(\mathcal{A}) = \mathcal{X} \times \prod\limits_{a \in S_1^0} \mathrm{Mat}_\mathbb{k}(x_{q(a)} \times x_{s(a)})$. If $\mathcal{A}$ is fixed we write Π_x^Λ instead $\Pi_x^\Lambda(\mathcal{A})$. We denote by $|x| = \sum\limits_{a \in S_1^0} x_{q(a)} x_{s(a)}$, so $\dim_\mathbb{k} \Pi_x^\Lambda = \dim_\mathbb{k} \mathcal{X} + |x|$ (the dimensions as algebraic varieties). By $f_\mathcal{A}^\Lambda(x)$ we denote a quadratic polynomial $f_\mathcal{A}^\Lambda(x) = -\dim_\mathbb{k} \mathcal{X} + f_\mathcal{A}(x)$. Any morphism $F : \mathcal{A} \to \mathcal{A}'$ between a $\Lambda-$ box and a $\Lambda'-$ box associated with some $\varphi : \Lambda \to \Lambda'$, induces a linear map of the spaces of dimensions $l(F^*) : \mathcal{L}_{\mathcal{A}'} \to \mathcal{L}_\mathcal{A}$ such that for $V \in R(\mathcal{A}')$ $\dim F^*(V)$ $= l(F^*)(\dim V)$. Analogously, it is defined for any dimension $x \in \mathcal{L}_{\mathcal{A}'}$ the morphism of the varieties $\Pi(F^*) : \Pi_x^{\Lambda'}(\mathcal{A}') \to \Pi_{l(F^*)}^\Lambda(\mathcal{A})$. The functor $\pi_\chi^* : R(\mathcal{A}_\chi) \to R(\mathcal{A})$ (Lemma 2) induced a morphism of varieties of representations $\pi_\chi : \Pi_x^\mathbb{k}(\mathcal{A}_\chi) \to \Pi_x^\Lambda(\mathcal{A})$, $\pi_\chi : (1_\mathbb{k}, p) \mapsto (\chi, p)$.

By $\mathcal{I}_x^{\mathcal{A}}$ we denote the set of the isomorphism classes of representations of the box $\mathcal{A}$ in the dimension x. Then in the situation above $\Pi(F^*)$ induces the map $\mathcal{I}_F : \mathcal{I}_x^{\mathcal{A}'} \to \mathcal{I}_{l(F^*)(x)}^{\mathcal{A}}$.

If $S \subset \Pi_x^{\Lambda}(\mathcal{A})$ is some set, then the isoclosure $\bar{S}^i \subset \Pi_x^{\Lambda}(\mathcal{A})$ we call the set of all representations $Y \in \Pi_x^{\Lambda}(\mathcal{A})$, such that exists $X(= X(Y)) \in S$, isomorphic to Y. S is called isodence, if $\bar{S}^i$ contains an open in Zarisky topology subset. Analogously we define the isoclosure of a set S of objects in some category C and in this situation $\bar{S}^i$ is a full subcategory in C.

Denote by G_x the algebraic variety $\prod\limits_{\alpha\in S_0} GL_{\Bbbk}(x_\alpha) \times \prod\limits_{a\in S_1^1} \mathrm{Mat}_{\Bbbk}(x_{q(a)} \times x_{s(a)})$. The next lemma follows from [14].

Lemma 3. *Let $\mathcal{A}$ be a free, normal and triangular $\Lambda-$ box. There is defined a regular morphism of algebraic varietes $u : G_x \times \Pi_x^{\Lambda} \to \Pi_x^{\Lambda}$, such that for $x \in \Pi_x^{\Lambda}$ $u^{-1}(x) = \bar{x}^i$. If $\mathcal{X} = \mathcal{X}(\Lambda)$ is irreducible, then if $S \subset \Pi_x^{\Lambda}$ is isodence and $\mathcal{U} \subset S$ is Zarisky open, then $\mathcal{U}$ is also isodence.*

2.7. The following lemma obviously follows from Lemma 1.

Lemma 4. *Let A be a semi simple category over Λ, $\mathcal{A} = (A, V)$ be an (elementary) triangular box and $x \in \mathcal{L}_{\mathcal{A}}$, $x > 0$ be some dimension. Then the set of all the isomorphism classes in the dimension x is in the natural bijection with $\mathcal{X} =$ Specm Λ.*

3. Functor $\mathrm{Ext}^1_{\mathcal{A}}$ in category of representations of free box

3.1. In this section we suppose $\Lambda = \Bbbk$. If $\mathcal{A} = (A, V)$ is some box with a free kernel, then we define $\mathrm{Ext}^i_{\mathcal{A}}(X, Y) = \mathrm{Ext}^i_{A-A}(V, (X, Y)_{\Bbbk}), i \geq 0, X, Y \in R(\mathcal{A})$ (we denote Hom by Ext^0. We consider this definition more detailed in the case of the free box.

Let $0 \to \mathcal{P}_{-1} \xrightarrow{\partial} \mathcal{P}_0 \xrightarrow{\pi} V \to 0$ be the above constructed A-bimodule resolution of V, $X, Y \in R(\mathcal{A})$ be two representation. Applying the functor $\mathrm{Hom}_{A-A}(?, (X, Y)_{\Bbbk})$ to this resolution we get the complex

$$0 \to \mathrm{Hom}_{A-A}(\mathcal{P}_0, (X, Y)_{\Bbbk}) \overset{\partial^*(X,Y)}{\longrightarrow} \mathrm{Hom}_{A-A}(\mathcal{P}_{-1}, (X, Y)_{\Bbbk}) \to 0$$

The homology of this complex we denote by $H_i^{\mathcal{A}}(X, Y)$, so $H_i^{\mathcal{A}}(X, Y) = 0$, $i \neq 0, -1$. By definition $H_0^{\mathcal{A}}(X, Y) = \mathrm{Ker}\ \partial^*(X, Y) = \mathrm{Hom}_{\mathcal{A}}(X, Y) = (X, Y)_{\mathcal{A}}$, that we will denote by $(X, Y)^0_{\mathcal{A}}$. As we defined $\mathrm{Ext}^1_{\mathcal{A}}(X, Y) = H^{\mathcal{A}}_{-1}(X, Y) = \mathrm{Coker}\ \partial^*(X, Y) = (X, Y)^1_{\mathcal{A}}$. The standard homological algebra shows, that $\mathrm{Ext}^i_{\mathcal{A}}(X, Y) = \mathrm{Ext}^i_{A-A}(V, (X, Y)_{\Bbbk}) = \mathrm{Ext}^i_{A}(V \otimes_A X, Y), i \geq 0$. Denote by (φ) the free A-bimodule, generated by $\varphi : \alpha \to \beta$, obviously $\mathrm{Hom}_{A-A}((\varphi), (X, Y)_{\Bbbk}) \simeq \mathrm{Hom}_{\Bbbk}(X(\alpha), Y(\beta))$. From this remark and the equality

$$\sum_{i=-1}^{0} (-1)^i \mid \mathrm{Hom}_{A-A}(\mathcal{P}_i, (X, Y)_{\Bbbk}) \mid = \sum_{i=0}^{1} (-1)^i \mid (X, Y)^i_{\mathcal{A}} \mid$$

we get, analogously [12], the following statement:

Proposition 2. $< \dim X, \dim Y >_{\mathcal{A}} = \mid (X, Y)^0_{\mathcal{A}} \mid - \mid (X, Y)^1_{\mathcal{A}} \mid$.

We try shortly to show, that so defined $\mathrm{Ext}^1_{\mathcal{A}}$ has enough good properties, though the category $R(\mathcal{A})$ is non abelian. We remain, that for every morphism $f : X \to Y$

from $R(\mathcal{A})$ exists $p : Y \to Z$, such that for every $g : Y \to T$ $gf = 0$ exists $h : Z \to T$ and $g = hp$ holds, $c(p) = \mathrm{Coker}((1_V \otimes c(f))(\mu \otimes 1_X))$. But in order to formulate $\mathrm{Ext}^1_{\mathcal{A}}$ properties we use some more restricted class of morphisms.

If we denote by A the box over A $(A,\ A,\ 1_A,\ 1_A)$, then $R(A)$ is canonically equivalent to $A-\mathrm{mod}$. The counit morphism $\varepsilon : V \to A$ induces the morphism of the boxes $\Omega : \mathcal{A} = (A, V) \to A = (A, A)$, such that $\Omega_0 = 1_A$, $\Omega_1 = \varepsilon$. Then induced by Ω functor $\Omega^* : R(A) \to R(\mathcal{A})$ is bijective on the objects. The morphism f, that belongs to $\Omega^*(R(A))$ or, equivalently, such that $b(f)(\bar{V}) = 0$ we will call quiver like. The sequence

$$0 \to \Omega^*(X) \xrightarrow{\Omega^*(\sigma)} \Omega^*(Y) \xrightarrow{\Omega^*(\pi)} \Omega^*(Z) \to 0$$

in $R(\mathcal{A})$ we will call q-exact if and only if

$$0 \to X \xrightarrow{\sigma} Y \xrightarrow{\pi} Z \to 0$$

is an exact sequence in the abelian category $R(A)$.

The morphism $f : X \to Y$ in $R(\mathcal{A})$ we call a proper monomorphism (epimorphism) if for any $\alpha \in S_0$ $b(f)(e_\alpha) : X(\alpha) \to Y(\alpha)$ is a monomorphism (epimorphism). Then with the standard induction by the triangular hight (see [14]) we prove the following lemma.

Lemma 5. *1) For any proper epimorphism (monomorphism) in $R(\mathcal{A})$ $l : F \to G$ exists an isomorphism $f : F' \to F$ ($g : G \to G'$), such that $l' = lf : F' \to G$ is a quiver like epimorphism ($l'' = gl : F \to G'$ is a quiver like monomorphism) in $R(\mathcal{A})$.*

2) Let $0 \to X \xrightarrow{i} Y \xrightarrow{p} Z \to 0$ be a sequence in $R(\mathcal{A})$, such that $pi = 0$ and for every $\alpha \in ObA$ the sequence $0 \to X(\alpha) \xrightarrow{b(i)(e_\alpha)} Y(\alpha) \xrightarrow{b(p)(e_\alpha)} Z(\alpha) \to 0$ of $\Bbbk-$ vector spaces is exact. Then there exists an isomorphism $F : Y \to Y'$, such that the sequence $0 \to X \xrightarrow{\sigma} Y' \xrightarrow{\pi} Z \to 0$ where $\sigma = Fi$, $\pi = pF^{-1}$ is q-exact sequence.

3) In the assumption of 2) i (p) is in the category $R(\mathcal{A})$ a categorical kernel (cokernel) of p (i).

Proof. In the statement 1) we consider the case of the epimorphism, the case of the monomorphism treats in the dual way. To avoid the boring denotations in this part for $f \in R(\mathcal{A})$ instead of $b(f)$ we will write f. First we make the following remark, analogous to [14]:

Remark 2. *Let F be some representation from $R(\mathcal{A})$, $\{F'_\alpha\}_{\alpha \in S_0}$ be a family of $\Bbbk-$ vectorspaces and is given a family of $\Bbbk-$linear maps $\{f_\tau : F'_{q(\tau)} \to F(s(\tau))\}$, $\tau \in \hat{S}^1_1$, such that $f_{e_\alpha} : F'_\alpha \to F(\alpha)$ is an epimorphism for any $\alpha \in S_0$. Then there exists a representation $F' \in R(\mathcal{A})$ and a morphism $f : F' \to F$, such that $f(\tau) = f_\tau$ for all $\tau \in \hat{S}^1_1$. If all f_{e_α}, $\alpha \in S_0$ are isomorphisms, then $F' \in R(\mathcal{A})$ is uniquely determined and f is an isomorphism in the category $R(\mathcal{A})$.*

To construct the representation F' we must set all the morphisms $F'(a) : F'_\beta \to F'_\alpha$, $a \in S^0_1$, $a : \beta \to \alpha$. We can assume, as in [14], that $F'(b)$ is settled for all $b \in S^0_1$, $h(a) > h(b)$. Then from the condition of the first chapter, $0 = f(d(a)) = f(e_\alpha a - ae_\beta$

$+\ \delta(a))$ follows, that for constructed F' and f holds $f_{e_\alpha}F'(a) = F(a)f_{e_\beta} - f(\delta(b))$. Since $\mathcal{A}$ is triangular, the right side is defined and since f_{e_α} is an epimorphism, we can find from last equation the $\Bbbk-$ morphism $F'(a)$. The case f_{e_α}, $\alpha \in S_0$ are isomorphisms is treated in the same way.

Denote by N the triangular hight of the box $\mathcal{A}$. For any morphism $t \in R(\mathcal{A})$ by $n(t)$ we denote the minimal value of $h(\varphi)$ for such $\varphi \in S_1^1$ that $t(\varphi) \neq 0$ and $N + 1$ if $f(S_1^1) = 0$. To prove 1) we consider an isomorphism $f : F' \to F$ such that the value of $n(t)$ for $t = fl$ is the maximal possible. If $n(t) \neq N + 1$, then the set of all $\varphi \in S_1^1$, such that $h(\varphi) = n(t)$ S' isn't empty. Following the preceding remark there exists a representation F'', such that $F''(\alpha) = F'(\alpha)$ for all $\alpha \in S_0$ and an isomorphism $f' : F'' \to F'$ such that $f'(e_\alpha) = 1_{F'(\alpha)}$, $\alpha \in S_0$, $f'(\varphi) = -t(\varphi)$ for $\varphi \in S'$, $f'(\psi) = 0$ for all other $\psi \in S_1^1$. Then for any $\varphi \in S'$ $\varphi : \alpha \to \beta$ holds $(f't)(\varphi) = f'(e_\beta)\ t(\varphi) + f'(\varphi)\ t(e_\alpha) + m(f' \otimes t)(\delta(\varphi)) = t(\varphi) - t(\varphi) = 0$ and, obviously $(f't)(\psi) = 0$ for ψ, such that $h(\psi) < h(\varphi)$, so $n(f't) = n((f'f)l) > n(fl) = n(t)$, that is in the contrary with the minimality of the value $n(t)$.

In order to prove 2) we make first an obvious remark. Let $Y, Y' \in R(\mathcal{A})$ be such, that $Y(\alpha) = Y'(\alpha)$ for all $\alpha \in S_0$, $f : Y \to Y'$ be an isomorphism, such that $f(e_\alpha) = 1_{Y(\alpha)}$ for all $\alpha \in S_0$ and $\varphi \in S_1^1$ be such, that $h(\varphi) = n(f)$. Then $f^{-1}(\varphi) = -f(\varphi)$. Denote $h_{i,p} = min(n(i), n(p))$ and $S_{i,p}$ be the set of all $\varphi \in S_1^1$, $h(\varphi) = h_{i,p}$ and at least one from the operators $i(\varphi)$, $p(\varphi)$ is nonzero. Suppose, that $S_{i,p}$ is nonempty and consider $\varphi \in S_{i,p}$, $\varphi : \alpha \to \beta$. We will construct $Y' \in R(\mathcal{A})$ and $f : Y \to Y'$ with the conditions $f(e_\alpha) = 1_{Y(\alpha)}$, $f(\psi) = 0$ for $\psi \in S_1^1$, $\psi \neq \varphi$ and for $i' = fi$, $p' = pf^{-1}$ holds $i'(\varphi) = 0$, $p'(\varphi) = 0$. Since for $\psi \neq \varphi$, $h(\psi) \geq h(\varphi)$ holds $i'(\psi) = i(\psi)$, $p'(\psi) = p(\psi)$, we conclude, that either $h_{i,p} < h_{i',p'}$ or $h_{i,p} = h_{i',p'}$ and $S_{i',p'} \subset S_{i,p} \setminus \{\varphi\}$, so, iterating this construction, we prove 2). Rewrite the condition $i'(\varphi) = 0 : fi(\varphi) = f(e_\beta)\ i(\varphi) + f(\varphi)\ i(e_\alpha) = i(\varphi) + f(\varphi)\ i(e_\alpha) = 0$. Analogously, $p'(\varphi) = pf^{-1}(\varphi) = -p(e_\beta)\ f(\varphi) + p(\varphi)$. The condition $(pi)(\varphi) = 0$ is equivalent to the commutativity of diagram

$$\begin{array}{ccccc} & X(\alpha) & \xrightarrow{i(e_\alpha)} & Y(\alpha) & \\ {\scriptstyle i(\varphi)} & \downarrow & & \downarrow & {\scriptstyle p(\varphi)} \\ & Y(\beta) & \xrightarrow{-p(e_\beta)} & Z(\beta) & \end{array}$$

But if in this diagram (in the vector spaces category) the upper arrow is a monomorphism and the lower arrow is an epimorphism, then there exists $h : Y(\alpha) \to Y(\beta)$, such that $i(\varphi) = hi(e_\alpha)$, $p(\varphi) = -p(e_\beta)h$, so we can set $f(\varphi) = h$.

We prove 3) in the case of a kernel. Following 2), we can assume, that i and p are quiver like. Let $g : Y' \to Y$ be such that $pg = 0$. For every $\varphi \in S_1^1$, $\varphi : \alpha \to \beta$ as in 2) we calculate $0 = (pg)(\varphi) = p(e_\beta)\ g(\varphi) + p(\varphi)\ g(e_\alpha) + m(p \otimes g)(\delta(\varphi)) = p(e_\beta)\ g(\varphi)$, so $\mathrm{Im} g(\varphi) \subset \mathrm{Im} i(e_\beta) = \mathrm{Ker}\ p(e_\beta)$. Then $g(\varphi)$, for all $\varphi \in \hat{S}_1^1$ defines the unique $f : Y' \to X$, such that $g = if$ by setting $f(\varphi) : Y'(q(\varphi)) \to X(s(\varphi))$ as $f(\varphi)(y') = g(\varphi)(y')$, $y' \in Y'(q(\varphi))$.

3.2. In the case of Lemma 5, 2) the sequence $E : 0 \to X \xrightarrow{i} Y \xrightarrow{p} Z \to 0$ we will call (short) exact in $R(\mathcal{A})$ and by $\mathcal{E}_\mathcal{A}(Z, X)$ ($\mathcal{E}^q_\mathcal{A}(Z, X)$) we denote the set of all exact (q-exact) sequences with the first term X and the last term Z. If

$$0 \to X \xrightarrow{\sigma} Y' \xrightarrow{\pi} Z \to 0$$

is also an exact sequence, then we say that E is congruent to E', $(E \sim E')$ provided there exists an isomorphism $f : Y \to Y'$ such that $fi = i'$, $p'f = p$. By $E_{\mathcal{A}}(Z, X)$ ($E^q_{\mathcal{A}}(Z, X)$) we denote the set of the congruence classes of exact sequences from $\mathcal{E}_{\mathcal{A}}(Z, X)$ ($\mathcal{E}^q_{\mathcal{A}}(Z, X)$). Following Lemma 5 the canonical inclusion $i_{\mathcal{E}} : \mathcal{E}_A(Z, X) \hookrightarrow \mathcal{E}(Z, X)$ induces the bijection $i_{\mathcal{A}} : E^q_{\mathcal{A}}(Z, X) \simeq E_{\mathcal{A}}(Z, X)$. Moreover $\Omega^* : R(A) \to R(\mathcal{A})$ gives us the canonical identification $i_\Omega : \mathcal{E}_A(Z, X) \to \mathcal{E}^q_{\mathcal{A}}(Z, X)$ and induces the projection $\pi_\Omega : E_A(Z, X) \to E_{\mathcal{A}}(Z, X)$.

The introduced notions allow to formulate some statements of the category $R(\mathcal{A})$, analogous to the case of the modules category, for example there exists the vector space structure on the set $E_{\mathcal{A}}(X, Y)$.

Lemma 6. *Let $\sigma : X \to Y$ ($\pi : Y \to Z$) be a proper monomorphism (a proper epimorphism).*
1) For every $f : X \to T$ ($g : S \to Z$) exists the push-out (pull-back) in the category $R(\mathcal{A})$

$$\begin{array}{ccccccccc} X & \xrightarrow{\sigma} & Y & & & Y' & \xrightarrow{\pi'} & S & \\ \subset f\downarrow & & \subset f'\downarrow & (1) & & \subset g'\downarrow & & \subset g\downarrow & (2) \\ T & \xrightarrow{\sigma'} & Y' & & & Y & \xrightarrow{\pi} & Z & \end{array}$$

where $\sigma' : T \to Y'$ is a proper monomorphism ($\pi' : Y' \to S$ is a proper epimorphism).
2) The diagram (1) ((2)) *is universal if and only if for every $\alpha \in ObA$ the diagram* (1α) ((2α)) *in the category of the vector spaces is universal*

$$\begin{array}{ccccccccc} X(\alpha) & \xrightarrow{\sigma(e_\alpha)} & Y(\alpha) & & & Y'(\alpha) & \xrightarrow{\pi'(e_\alpha)} & S(\alpha) & \\ \subset f(e_\alpha)\downarrow & & \subset f'(e_\alpha)\downarrow & (1\alpha) & & \subset g'(e_\alpha)\downarrow & & \subset g(e_\alpha)\downarrow & (2\alpha) \\ T(\alpha) & \xrightarrow{\sigma'(e_\alpha)} & Y'(\alpha) & & & Y(\alpha) & \xrightarrow{\pi(e_\alpha)} & Z(\alpha) & \end{array}$$

3) The diagram (1) ((2)) can be included in the commutative diagram with exact rows 1′ (2′)*, where π, π' is cokernel morphisms (σ, σ' is the kernel morphisms)*

$$\begin{array}{cccccccc} 0\to & X & \xrightarrow{\sigma} & Y & \xrightarrow{\pi} & Z & \to 0 & \\ & \subset f\downarrow & & \subset f'\downarrow & & \| & & (1) \\ 0\to & X & \xrightarrow{\sigma'} & Y' & \xrightarrow{\pi'} & Z & \to 0 & \end{array}$$

$$\left(\begin{array}{cccccccc} 0\to & X & \xrightarrow{\sigma'} & Y' & \xrightarrow{\pi'} & Z & \to 0 & \\ & \| & & \subset g'\downarrow & & \subset g\downarrow & & (1) \\ 0\to & X & \xrightarrow{\sigma} & Y & \xrightarrow{\pi} & Z & \to 0 & \end{array}\right)$$

Proof. We consider the proper monomorphism $\begin{matrix}\sigma \\ -f\end{matrix} : X \to Y \oplus T$ and an isomorphism $s : Y \oplus T \to Q$ such that the superposition $i = s\begin{pmatrix}\sigma \\ -f\end{pmatrix}$ is quiver like. We set $Y' = Q/\mathrm{Im}i$ in the category $A-$mod and consider the canonical projection $p : Q \to Y'$ as a morphism in $\mathcal{R}(\mathcal{A})$. After setting $(f', g') = ps : Y \oplus T \to Y'$ we apply Lemma 5 ,3), that proves 1). 2) follows from Lemma 5, 2), 3) follows from 2) and 5. The case of pull-back treated in the same way.

This defines the maps $(Z', Z)_{\mathcal{A}} \times \mathcal{E}_{\mathcal{A}}(Z, X) \to \mathcal{E}_{\mathcal{A}}(Z', X)$, $\mathcal{E}_{\mathcal{A}}(Z, X) \times (X, X')_{\mathcal{A}} \to \mathcal{E}_{\mathcal{A}}(Z, X')$ that keeps the congruence relations, hence are defined the actions $(Z'$,

$Z)_{\mathcal{A}} \times E_{\mathcal{A}}(Z, X) \to E_{\mathcal{A}}(Z', X)$, $E_{\mathcal{A}}(Z, X) \times (X, X')_{\mathcal{A}} \to E_{\mathcal{A}}(Z, X')$ with usual associativity conditions. As in the category of modules in the category $R(\mathcal{A})$ for E, $E' \in \mathcal{E}_A(Z, X)$ is defined $E \oplus E' \in \mathcal{E}_A(Z \oplus Z, X \oplus X)$ and for $\Delta_Z : Z \to Z \oplus Z$, $z \mapsto (z, z)$, $z \in Z$, $\nabla_X : X \oplus X \to X$, $(x_1, x_2) \mapsto x_1 + x_2$, $x_1, x_2 \in X$ is defined $\nabla_Z(E \oplus E')\Delta_X \in \mathcal{E}_{\mathcal{A}}(Z, X)$. If $\overline{E}$ and $\overline{E'}$ are the corresponding classes in $E_{\mathcal{A}}(Z, X)$, then the sum $\overline{E} + \overline{E'}$ is defined as a class $\overline{\nabla_Z(E \oplus E')\Delta_X} \in E_{\mathcal{A}}(Z, X)$. Analogously is defined the multiplication of the class of $\overline{E}$ on the $\lambda \in \Bbbk$. The above constructed $\pi_\Omega : E_A(Z, X) \to E_{\mathcal{A}}(Z, X)$ is obviously a $\Bbbk-$ homomorphism.

3.3.

Lemma 7. *There exists the $\Bbbk-$ vector space isomorphism $J : E_{\mathcal{A}}(Z, X) \to Ext^1_{\mathcal{A}}(Z, X)$.*

Proof. We recall some properties of the morphisms in the categories of representations of boxes. If $f : X \to Y$ and $f = \Omega^*(g)$, then $c(f) : V \otimes_A X \to Y$ is the superposition $gc(1_X)$. If we identify $A \otimes_A Y$ with Y, then $c(f) = \varepsilon \otimes f$. From it follows, that if $f' : Y \to Z$ is such, that $f' = \Omega^*(g')$, then $c(f'f)$ is equal $g'c(f)$ or $c(f')(1_V \otimes g)$. The following property from [14] needs the normality and triangularity of the box : a morphism $f : X \to Y$ is an isomorphism if and only if for every $\alpha \in S_0$ $b(f)(e_\alpha) : M(\alpha) \to N(\alpha)$ is a $\Bbbk$-isomorphism or equivalently: for every $\alpha \in S_0$ $c(f)$ maps $e_\alpha \otimes M(\alpha)$ isomorphically on $N(\alpha)$.

Let $E : 0 \to X \xrightarrow{\sigma} Y \xrightarrow{\pi} Z \to 0$ be an exact sequence in $R(\mathcal{A})$. Following to Lemma 5, we can assume up to congruence in category $R(\mathcal{A})$, that E is $q-$exact and can be considered as an exact sequence in $A -$ mod. The above constructed exact sequence of A-bimodules $0 \to \bar{V} \xrightarrow{i} V \xrightarrow{\varepsilon} A \to 0$ splits as a right A-module sequence, so by the tensor multiplication with Z we get the exact sequence of left A-modules $0 \to \bar{V} \otimes_A Z \xrightarrow{i \otimes 1_Z} V \otimes_A Z \xrightarrow{\varepsilon \otimes 1_Z} Z \to 0$, where $\bar{V} \otimes Z$ is a projective. The corresponding long exact sequence in $R(A)$ has a form

$$0 \longrightarrow (Z, X)^0_A \longrightarrow (V \otimes_A Z, X)^0_A \longrightarrow (\bar{V} \otimes Z, X)^0_A \longrightarrow$$

$$(Z, X)^1_A \xrightarrow{I} (V \otimes_A Z, X)^1_A (\simeq (Z, X)^1_{\mathcal{A}}) \longrightarrow 0$$

In the category $A -$ mod is defined the isomorphism $J_A : E_A(Z, X) \to \mathrm{Ext}^1_A(Z, X)$ and $J = J_{\mathcal{A}}$ we define as a homomorphism, making the diagram commutative

$$\begin{array}{ccccc} \mathcal{E}_A(A, X) & \overset{\pi_A}{\to} & E_A(Z, X) & \overset{J_A}{\to} & \mathrm{Ext}^1_A(Z, X) \\ \| & & \subset \pi_\Omega \downarrow & & \subset I \downarrow \\ \mathcal{E}^q_{\mathcal{A}}(A, X) & \overset{\pi_{\mathcal{A}}}{\to} & E^q_{\mathcal{A}}(Z, X) & \overset{J_{\mathcal{A}}}{\to} & \mathrm{Ext}^1_{\mathcal{A}}(Z, X) \end{array}$$

where π_A and $\pi_{\mathcal{A}}$ are the canonical projections and $E^q_{\mathcal{A}}$ is identified with $E_{\mathcal{A}}$ by $i_{\mathcal{A}}$. Correctness of this correspondence and the lemma will be proved, when we show, that $E_A \in (Z, X)^1_A$ belongs to Ker I if and only if the corresponding exact sequence in $R(\mathcal{A})$

$$E : 0 \to X \xrightarrow{\sigma} Y \xrightarrow{\pi} Z \to 0$$

can be included in the following commutative diagram in $R(\mathcal{A})$

$$\begin{array}{ccccccc} 0\to & X & \overset{\begin{pmatrix}\subset 1_X\\ \subset 0\end{pmatrix}}{\longrightarrow} & X\oplus Z & \overset{(0,1_Z)}{\longrightarrow} & Z & \to 0 \\ & \| & & \downarrow\subset\varphi & & \| & \\ 0\to & X & \overset{\sigma}{\longrightarrow} & Y & \overset{\pi}{\longrightarrow} & Z & \to 0 \end{array} \qquad (1)$$

where φ is an isomorphism in $R(\mathcal{A})$.

If $E_A \in \operatorname{Ker} I$, then in $A-\text{mod}$ exists the following commutative diagram in the category $R(\mathcal{A})$

$$\begin{array}{ccccccc} 0\to & X & \overset{\begin{pmatrix}\subset 1_X\\ \subset 0\end{pmatrix}}{\longrightarrow} & X\oplus(V\otimes_A Z) & \overset{(0,1_{V\otimes_A Z})}{\longrightarrow} & V\otimes_A Z & \to 0 \\ & \| & & \downarrow\ \subset(f,g) & & \downarrow\ \subset c(1_Z) & \\ 0\to & X & \overset{\sigma}{\longrightarrow} & Y & \overset{\pi}{\longrightarrow} & Z & \to 0 \end{array} \qquad (2)$$

The corresponding $\varphi : X\oplus Z \to Y$ in the category $R(\mathcal{A})$ we define by setting $c(\varphi) = (fc(1_X), g) : (V\otimes_A X)\oplus(V\otimes_A Z) \to Y$. Since $c(1_Z)$ ($fc(1_X)$) being restricted on $e_\alpha\otimes Z(\alpha)$ $(e_\alpha\otimes X(\alpha))$ induces a $\Bbbk-$isomorphism on $Z(\alpha)$(on $X(\alpha)$), we infer that $c(\varphi)$ restricted on $(e_\alpha\otimes X(\alpha))\oplus(e_\alpha\otimes Z(\alpha))$ induces a $\Bbbk-$isomorphism on $Y(\alpha)\simeq X(\alpha)\oplus Z(\alpha)$ for any $\alpha\in S_0$, so φ is an isomorphism in $R(\mathcal{A})$.

From the commutativity of the diagram (2) we get $f=\sigma$, $(0,\varepsilon\otimes 1_Z) = (\pi f, \pi g)$. By definitions the inclusion for $\begin{pmatrix}1_X\\ 0\end{pmatrix} : X\to X\oplus Z$

$$c(\varphi\begin{pmatrix}1_X\\ 0\end{pmatrix}) = (f(\varepsilon\otimes 1_X), g)\begin{pmatrix}1_V\otimes\varepsilon\otimes 1_X\\ 0\end{pmatrix}(\mu\otimes 1_X) =$$

$$(f(\varepsilon\otimes 1_X))((1_V\otimes\varepsilon\otimes 1_X)(\mu\otimes 1_X)) = f(\varepsilon\otimes 1_X)(1_V\otimes 1_X) = \sigma(\varepsilon\otimes 1_X) = c(\sigma)$$

Analogously $c(\pi\varphi) = (\varepsilon\otimes\pi)1_V\otimes(f(\varepsilon\otimes 1_X), 1_V\otimes g)\begin{pmatrix}\mu\otimes 1_X & 0\\ 0 & \mu\otimes 1_Z\\ . & \end{pmatrix}$ But $(\varepsilon\otimes\pi)(1_V\otimes f(\varepsilon\otimes 1_X),\, 1_V\otimes g) = (\varepsilon\otimes\pi f(\varepsilon\otimes 1_X),\, \varepsilon\otimes\pi g) = (0,\, \varepsilon\otimes 1_Z)$ and thereafter $c(\pi\varphi) = (0,\, \varepsilon\mu\otimes 1_Z) = c(0, 1_Z)$. The diagram (1) with the constructed diagram φ is commutative and it proves the part "only if".

On other hand, if exists φ, making the diagram (1) commutative, then we consider $c(\varphi) = (p,q) : (V\otimes_A X)\oplus(V\otimes_A Z)\to Y$ and the diagram in the category $A-\text{mod}$, connecting E_A and $I(E_A)$:

$$\begin{array}{ccccccc} 0\to & X & \overset{\sigma'}{\longrightarrow} & Y' & \overset{\pi'}{\longrightarrow} & V\otimes_A Z & \to 0 \\ & \| & & \downarrow & & \downarrow\ \subset\varepsilon\otimes 1_Z & \\ 0\to & X & \overset{\sigma}{\longrightarrow} & Y & \overset{\pi}{\longrightarrow} & Z & \to 0 \end{array} \qquad (3)$$

From the commutativity of the diagram (1) in the category $A-\text{mod}$ $\pi\varphi\begin{pmatrix}0\\ 1_Z\end{pmatrix} = 1_Z$, we have in the category $R(A)$ $\varepsilon\otimes 1_Z = c(1_Z) = \pi c(\varphi\begin{pmatrix}0\\ 1_Z\end{pmatrix}) =$

$\pi c(\varphi)(1_V \otimes \begin{pmatrix} 0 \\ 1_Z \end{pmatrix}) = \pi q$. But the upper exact sequence in (3) is the pull-back of the exact sequence below with respect to $\varepsilon \otimes 1_Z$. From the definition of the pull-back and the equality $\varepsilon \otimes 1_Z = \pi q$ follows, that the sequence $I(E_A)$ splits in the category $A-$mod.

3.4. In this part we bring some (unnecessary for the further proofs) remarks in order to show, that so defined $\mathrm{Ext}^1_{\mathcal{A}}$ has the common features with Ext^1_{A} over a category (or an associative algebra).

Let A be finite dimensional and $P \in R(\mathcal{A})$ be such that $\mathrm{Ext}^1_{\mathcal{A}}(P, X) = 0$. Since in $R(A)$ exists an exact sequence $0 \to K \xrightarrow{i} A^n \longrightarrow P \to 0$ for some $n \geq 1$, P is a direct summand of A^n and vice versa. Since V is a right projective and $\mathrm{Ext}^i_{\mathcal{A}}(X, Y) = \mathrm{Ext}^i_{A}(V \otimes X, Y)$ we trivially get, that $\mathrm{Ext}^i_{\mathcal{A}}(X, ?)$ and $\mathrm{Ext}^i_{\mathcal{A}}(?, Y)$ induce the standard long exact sequences, connected with the short exact sequence in $R(\mathcal{A})$.

The structure of the cocategory on the bimodule V induces the structure of $A(\infty)-$ cocategory over A on the $\mathbb{Z}-$ graded $A-$ bimodule $\mathcal{P} = \{\mathcal{P}_i\}_{i\in\mathbb{Z}}$, $\mathcal{P}_i = 0$, $i \neq -1, 0$, $\mathcal{P}_0$, $\mathcal{P}_{-1}$ as (see [15]). It induces the $A(\infty)-$ category structure over the semisimple category with the set of the objects $R(\mathcal{A})$ on the $\mathbb{Z}-$ graded bimodule $\{\mathrm{Hom}_{A-A}(\mathcal{P}_i, (V, W)_{\Bbbk})\}_{i\in\mathbb{Z}}$, $V, W \in R(\mathcal{A})$. On the next hand it defines on the $\{\mathrm{Ext}^i_{A}(V, W)\}_{i\in\mathbb{Z}}$, $V, W \in R(\mathcal{A})$ the structure of $A(\infty)-$ category, [7]. In particulary, it defines on $\{\mathrm{Ext}^i_{A}(V, W)\}_{i\in\mathbb{Z}}$, $V, W \in R(\mathcal{A})$ the structure of a category, that is accorded with the structure, given by the standard Joneda multiplication.

The preceding definitions makes possible the following formulation.

Proposition 3. *Let $\mathcal{A} = (A, V)$ be a triangular free box over a finite generated domain Λ. Then for every dimension $x \in \mathcal{L}(= \mathcal{L}_A)$ exist subsets $\mathcal{U} \subset \mathcal{U}_{x1} \subset \mathcal{U}_{x2} \subset \Pi^{\Lambda}_x$, where $\mathcal{U}$ is Zarisky open such that if for every $(\chi, U) \in \mathcal{U}_{x2}$, $\chi : \Lambda \to \Bbbk$, $U \in R(\mathcal{A}_\chi)$, $U = U_1^{\Bbbk_1} \oplus \ldots \oplus U_t^{\Bbbk_t}$ is a decomposition of U in the indecomposables, then*

1) *if $(\chi, U) \in \mathcal{U}_{x1}$, then $(\Bbbk_1, \ldots, \Bbbk_t)$ up to order doesn't depend on (χ, U);*

2) *$Ext^1_{\mathcal{A}_\chi}(U_i, U_j) = 0$ for $i \neq j$, $Ext^1_{\mathcal{A}_\chi}(U_i, U_i) = 0$ if $\Bbbk_i \geq 2, i, j = 1, \ldots, t$;*

3) *if $x_i = \dim\ U_i$, then $< x_i, x_j > \geq 0$ for $i \neq j$ and $< x_i, x_i > \geq 0$, if $\Bbbk_i \geq 2$, $i, j = 1, \ldots, t$.*

We use the following Lemma.

Lemma 8 ([13]). *1) Let $\Lambda = \Bbbk$. If $0 \to X \xrightarrow{\sigma} Y \xrightarrow{\pi} Z \to 0$ is a non split exact sequence in $R(\mathcal{A})$, then $|(Y, Y)^0_{\mathcal{A}}| < |(X \oplus Z, X \oplus Z)^0_{\mathcal{A}}|$.*

2) If $X = X_1 \oplus X_2 \in R(\mathcal{A})$ and for any $Y \in R(\mathcal{A})$ with $\dim X = \dim Y$ holds $| (X, X)^0_{\mathcal{A}} | \leq | (Y, Y)^0_{\mathcal{A}} |$, then $(X_1, X_2)^1_{\mathcal{A}} = 0$.

Since σ and π are mutual Coker and Ker correspondingly, the proof of this lemma coincides with given in [13].

The proof of Proposition 3 in situation of finite dimensional associative algebras is good known, so here we bring only a short sketch of the proof of this statement for free normal boxes.

Let $\Lambda(\Pi^{\Lambda}_x)$ be a coordinate ring of Π^{Λ}_x, $L_x = \prod_{a\in\hat{S}^1_1} \mathrm{Mat}_{\Bbbk}(x_{q(a)} \times y_{s(a)})$, and $\Lambda(L_x) = \Lambda[l_{ij}]$ be a coordinate ring of L_x. The stabilizer of every representation $X \in R(\mathcal{A})$ can be computed as Ker $\partial^*(X,X)$. This means, there exists a system of linear equations in the unknown (l_{ij}) and coefficients from $\Lambda(\Pi_x^\Lambda)$, such that for every specialization of the coefficients $\Phi : \Lambda(\Pi_x^\Lambda) \to \Bbbk$ the solution of obtained linear system is a stabilizer of corresponding point $F \in \Pi_x^\Lambda$. Therefore the set of all representations with a stabilizer of the minimal possible dimension is open in Π_x^Λ. Moreover, the set of representation $R_{(\Bbbk_1,\dots,\Bbbk_t)} \subset \Pi_x^\Lambda$, $x \in \mathcal{L}_\mathcal{A}$, having a decomposition in the indecomposable $U_1^{\Bbbk_1} \oplus \ldots \oplus U_t^{\Bbbk_t}$ is constructive for any $(\Bbbk_1, \dots, \Bbbk_t)$. Hence, using Lemma 8, we obtain the proposition 3, 2). Finally, the item 3) of Proposition 3 follows from Proposition 2.

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